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Σ-Associated Primes over Extension Rings

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Abstract In this paper we introduce a concept, called Σ-associated primes, that is a generalization of both associated primes and nilpotent associated primes. We first observe the basic properties of Σ-associated primes and construct typical examples. We next describe all Σ-associated primes of the Ore extension $R[x;\alpha,\delta]$, the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$ and the skew power series ring $R[[x;\alpha]]$, in terms of the Σ-associated primes of R in a very straightforward way. Consequently several known results relating to associated primes and nilpotent associated primes are extended to a more general setting.

Keywords associated prime; Ore extension; Σ-associated prime

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1. Introduction

Throughout this paper all rings R are associative with identity, and all modules are right unital. The set of all nilpotent elements of R is denoted by $\mathrm{nil}(R)$. A ring R is called an NI ring [10] if $\mathrm{nil}(R)$ forms an ideal, and a ring R is reduced if it has no nonzero nilpotent elements. Let U and V be two nonempty subsets of R. We define $U:V=\{x\in R\mid Vx\subseteq U\}$. If V is singleton, say $V=\{m\}$, we use $U:m$ in place of $U:\{m\}$. It is easy to see that if U, V are two right ideals of R, then $U:V$ is an ideal of R and such an ideal is usually called the quotient of U by V.

Given a right R-module N_R, the right annihilator of N_R in R is denoted by $r_R(N_R)=\{a\in R\mid Na=0\}$. We say that N_R is prime if $N_R\neq 0$, and $r_R(N_R)=r_R(N'_R)$ for every nonzero submodule $N'_R\subseteq N_R$ (see [1]). Let M_R be a right R-module. Then an ideal $\wp$ of R is called associated prime of M_R if there exists a prime submodule $N_R\subseteq M_R$ such that $\wp=r_R(N_R)$. The set of associated primes of M_R is denoted by $\mathrm{Ass}(M_R)$ (see [1]). Associated primes are well-known in commutative algebra for their important role in the primary decomposition, and have attracted a lot of attention in recent years. In [4], Brewer and Heinzer used localization

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theory to prove that the associated primes of the polynomial ring $R[x]$ (viewed as a module over itself) over a commutative ring R are all extended; that is, every $\wp \in \mathrm{Ass}(R[x])$ may be expressed as $\wp = \wp_0[x]$, where $\wp_0 = \wp \cap R \in \mathrm{Ass}(R)$. Using results of Shock in [13] on good polynomials, Faith has provided a new proof in [5] of the same result which does not rely on localization or other tools from commutative algebra. In [1], Scott Annin showed that Brewer and Heinzer's result still holds in the more general setting of a polynomial module $M[x]$ over a skew polynomial ring $R[x;\alpha]$, with possibly noncommutative base R. So the properties of associated primes over a commutative ring can be profitably generalized to noncommutative setting as well. For more details and properties of associated primes, we refer to [1–5].

For a set X of a ring R, $N\mathrm{Ann}_R(X) = \{a \in R \mid Xa \subseteq \mathrm{nil}(R)\}$ will stand for the weak annihilator of X in R (see [11]). Due to Ouyang and Birkenmeier [11], a right ideal I of R is a right quasi-prime ideal if $I \nsubseteq \mathrm{nil}(R)$ and $N\mathrm{Ann}_R(I) = N\mathrm{Ann}_R(I')$ for every right ideal $I' \subseteq I$ and $I' \nsubseteq \mathrm{nil}(R)$. Let R be an NI ring. An ideal $\wp$ of R is called a nilpotent associated prime of R if there exists a right quasi-prime ideal I such that $\wp = N\mathrm{Ann}_R(I)$. The set of nilpotent associated primes of R is denoted by $N\mathrm{Ass}(R)$ (see [11]). Note that $\mathrm{Ass}(R)$ and $N\mathrm{Ass}(R)$ are two different sets in general. But if R is a reduced ring, then $\mathrm{Ass}(R) = N\mathrm{Ass}(R)$. In [11], Ouyang and Birkenmeier also showed that the nilpotent associated primes of the Ore extension $R[x;\alpha,\delta]$ can be determined in terms of the nilpotent associated primes of the ring R.

Motivated by the results in [1–5] and [11], in this article, we continue the study of Σ-associated primes of R. We first introduce the notion of Σ-associated primes, which is a generalization of both associated primes and nilpotent associated primes, and investigate its basic properties. We next describe the Σ-associated primes of the Ore extension $R[x;\alpha,\delta]$, the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$ and the skew power series ring $R[[x;\alpha]]$, in terms of the Σ-associated primes of R in a very straightforward way. As a consequence we extend and unify several known results related to associated primes and nilpotent associated primes.

2. Σ-associated primes

In this section we introduce the notion of Σ-associated primes and investigate its basic properties. We begin with the following definition.

Definition 2.1 *Let U be an ideal of R. For a right ideal I of R, we say that I is Σ_U-prime if $I \nsubseteq U$ and $U : I = U : I'$ for every right ideal $I' \subseteq I$ and $I' \nsubseteq U$.*

Let U be an ideal of R. In the following remark, we offer a few basic properties of the Σ_U-prime ideals.

Remark 2.2 Let U be an ideal of R.

(1) If $U = 0$ where 0 denotes the zero ideal of R, then for any right ideal I of R, $U : I = r_R(I)$, and so I is a Σ_0- prime ideal if and only if I is a prime submodule of R_R. Let R be an NI ring and let $U = \mathrm{nil}(R)$. Then $U : I = N\mathrm{Ann}_R(I)$, and so I is a $\Sigma_{\mathrm{nil}(R)}$-prime ideal if and only if I is a right quasi-prime ideal. Hence both prime submodules of R_R and right quasi-prime

ideals of R are special Σ-prime ideals.

(2) Any nonzero ideal of a domain R is a prime submodule of R_R, a right quasi-prime ideal of R as well as a Σ_0-prime ideal of R.

(3) Recall that an ideal P of a ring R is said to be a prime ideal if $P \neq R$ and for right ideals U, V in R, $UV \subseteq P$ implies that $U \subseteq P$ or $V \subseteq P$. Hence if U is a prime ideal, then for all right ideals I and I' with $I \nsubseteq U$, $I' \subseteq I$ and $I' \nsubseteq U$, we have $U : I = U : I' = U$. So if U is a prime ideal, then any ideal $I \nsubseteq U$ is Σ_U-prime.

(4) Let U be an ideal of R and let I and J be right ideals of R with $I \nsubseteq U$ and $J \nsubseteq U$. If $U : I \neq U : J$, then $I \bigoplus J$ is not Σ_U-prime. Hence the direct sums of Σ_U-prime ideals need not be Σ_U-prime. So the direct products of Σ_U-prime ideals also need not be Σ_U-prime.

(5) Let U be an ideal of R and I a right ideal of R with $I \nsubseteq U$. If all the right ideals $J \subsetneqq I$ are contained in U, then I is Σ_U-prime.

Let U be an ideal of R and I a right ideal of R. Suppose that

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

$$I_n = \left\{ \begin{pmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \mid b, b_{ij} \in I \right\},$$

and

$$T_n(U) = \left\{ \begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} \mid x \in U, x_{ij} \in R \right\}.$$

Then under usual matrix operations, I_n is a right ideal of R_n, and $T_n(U)$ is an ideal of R_n.

Proposition 2.3 *Let U be an ideal of R and I a right ideal of R. Then I_n is a $\Sigma_{T_n(U)}$-prime ideal of R_n if and only if I is a Σ_U-prime ideal of R.*

Proof $\Rightarrow$. Suppose that I_n is $\Sigma_{T_n(U)}$-prime and I' is a right ideal of R with $I' \subseteq I$ and $I' \nsubseteq U$. We will show that $U : I = U : I'$. Since $U : I \subseteq U : I'$ is clear, it suffices to show that $U : I' \subseteq U : I$. Since $I' \subseteq I$ and $I' \nsubseteq U$, we have $I'_n \subseteq I_n$ and $I'_n \nsubseteq T_n(U)$. Thus

$T_n(U):I_n=T_n(U):I'_n$ since I_n is $\Sigma_{T_n(U)}$-prime. If $a\in U:I'$, then

$$\begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}\in T_n(U)$$

for each $\begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}\in I'_n$, and so $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}\in T_n(U):I'_n=T_n(U):I_n$. Thus for each $x\in I$, we have $\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}\in T_n(U)$, and so $xa\in U$ for each $x\in I$. Hence $a\in U:I$ and so $U:I=U:I'$. Therefore I is Σ_U-prime.

$\Leftarrow$. Assume that I is Σ_U-prime and V is a right ideal of R_n with $V\subseteq I_n$ and $V\nsubseteq T_n(U)$. We see that $T_n(U):V=T_n(U):I_n$. Clearly, $T_n(U):I_n\subseteq T_n(U):V$. So we only need to show the reverse containment. Consider the set defined as follows:

$$W=\left\{r\in I \mid \begin{pmatrix} r & r_{12} & \cdots & r_{1n} \\ 0 & r & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix}\in V\right\}.$$

Then W is a right ideal of R with $W\subseteq I$. Since $V\nsubseteq T_n(U)$, we have $W\nsubseteq U$. If

$$\begin{pmatrix} r & r_{12} & \cdots & r_{1n} \\ 0 & r & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix}\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}\in T_n(U)$$

for each $\begin{pmatrix} r & r_{12} & \cdots & r_{1n} \\ 0 & r & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix}\in V$, then $ra\in U$ for each $r\in W$ and so $a\in U:W$. Since I is

Σ_U-prime, we have $a \in U : W = U : I$. Hence for each $\begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix} \in I_n$, we have

$$\begin{pmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{pmatrix}\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in T_n(U),$$

and so $T_n(U) : V \subseteq T_n(U) : I_n$. Hence $T_n(U) : V = T_n(U) : I_n$. Therefore, I_n is $\Sigma_{T_n(U)}$-prime.

Based on Proposition 2.3, one may suspect that I_n may be Σ_{U_n}-prime whenever I is Σ_U-prime. But the following example erases the possibility.

Example 2.4 Let R be a domain, $U = 0$ and I a nonzero right ideal of R. Then $U_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ and $I_2 = \left\{\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} \mid a, a_{12} \in I\right\}$. Obviously, I is $\Sigma_{U=0}$-prime since R is a domain. Now we show that I_2 is not Σ_{U_2}-prime. Since $I \neq 0$, we can find $\begin{pmatrix} x & x_{12} \\ 0 & x \end{pmatrix} \in I_2$ with $x \neq 0$. If $\begin{pmatrix} x & x_{12} \\ 0 & x \end{pmatrix}\begin{pmatrix} a & a_{12} \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $xa = 0$ and $xa_{12} + x_{12}a = 0$. From $xa = 0$ and $x \neq 0$, we obtain $a = 0$. From $xa_{12} + x_{12}a = 0$ and $a = 0$, we obtain $a_{12} = 0$. Thus $U_2 : I_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$. Consider the set $V = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in I\right\}$. Then V is a right ideal of R_2 with $V \subseteq I_2$ and $V \nsubseteq U_2$. Let c be a nonzero element of R. Since $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for each $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in V$, we have $U_2 : I_2 \neq U_2 : V$. Therefore I_2 is not Σ_{U_2}-prime.

From Example 2.4, we know that I_n need not be a prime submodule of $(R_n)_{R_n}$ whenever I is a prime submodule of R_R. As to right quasi-prime ideals, we have the following.

Corollary 2.5 *Let R be a domain. Then for any nonzero right ideal I of R, I_n is a right quasi-prime ideal of R_n.*

Proof Let $U = 0$. Then

$$T_n(U) = \left\{\begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid x_{ij} \in R\right\} = \mathrm{nil}(R_n).$$

Since R is a domain and $U = 0$, any nonzero right ideal I of R is $\Sigma_{U=0}$-prime. Thus we complete the proof by Proposition 2.3 and Remark 2.2.

Let R be a ring and $W(R)=\left\{\begin{pmatrix} a & 0 & 0\\ a_{21} & a & a_{23}\\ 0 & 0 & a\end{pmatrix} \mid a, a_{ij}\in R\right\}$. Then $W(R)$ is a 3×3 subring of $M_3(R)$ under usual matrix addition and multiplication. Let U be an ideal and I a right ideal of R. Define $D(U)$ and $W(I)$ as follows:

$$D(U)=\left\{\begin{pmatrix} a & 0 & 0\\ a_{21} & a & a_{23}\\ 0 & 0 & a\end{pmatrix} \mid a\in U, a_{ij}\in R\right\},$$

and

$$W(I)=\left\{\begin{pmatrix} a & 0 & 0\\ a_{21} & a & a_{23}\\ 0 & 0 & a\end{pmatrix} \mid a, a_{ij}\in I\right\}.$$

Then $D(U)$ is an ideal of $W(R)$ and $W(I)$ is a right ideal of $W(R)$. $\square$

Proposition 2.6 *Let U be an ideal and I a right ideal of R. Then $W(I)$ is $\Sigma_{D(U)}$-prime if and only if I is Σ_U-prime.*

Proof By the same method as in the proof of Proposition 2.3, we complete the proof. $\square$

Corollary 2.7 *Let R be a domain. Then for any nonzero right ideal I of R, $W(I)$ is a right quasi-prime ideal of $W(R)$.*

Proof Let $U=0$. Note that $\mathrm{nil}(W(R))=\left\{\begin{pmatrix} 0 & 0 & 0\\ a_{21} & 0 & a_{23}\\ 0 & 0 & 0\end{pmatrix} \mid a_{ij}\in R\right\}=D(U)$. Then we complete the proof by Proposition 2.6 and Remark 2.2. $\square$

Definition 2.8 *Let U be an ideal of R. An ideal $\wp$ of R is called Σ_U-associated prime if there exists a Σ_U-prime ideal I of R such that $\wp=U:I$. The set of Σ_U-associated primes of R is denoted by Σ_U-$\mathrm{Ass}(R)$.*

Clearly, if $\wp$ is Σ_U-associated prime, then $\wp$ is a prime ideal of R. Let $U=0$. Then $\wp$ is Σ_0-associated prime if and only if $\wp$ is an associated prime ideal of R_R. Suppose that R is an NI ring and $U=\mathrm{nil}(R)$. Then $\wp$ is $\Sigma_{U=\mathrm{nil}(R)}$-associated prime if and only if $\wp$ is nilpotent associated prime. Hence both associated primes and nilpotent associated primes are special Σ-associated primes.

Example 2.9 Let R be a domain and let

$$R_n=\left\{\begin{pmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{pmatrix} \mid a, a_{ij}\in R\right\}$$

and

$$U = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid a_{ij} \in R \right\}.$$

Then for any ideal V of R_n with $V \not\subseteq U$, we have $U : V = U$. Hence Σ_U-Ass$(R_n) = \{U\}$.

Example 2.10 Let k be any field, and consider the ring $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ of 2×2 lower triangular matrices over k. We can write down all of the proper nonzero right ideals of R :

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, m_3 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Note that m_1, m_3 and α are ideals of R, and m_3 is a right ideal of R.

(1) Let $U = 0$. Then by [3], we have Σ_0-Ass$(R) =$ Ass$(R_R) = \{m_1\}$.

(2) Let $U = \alpha = \mathrm{nil}(R)$. Then by [11], we have $\Sigma_{\mathrm{nil}(R)}$-Ass$(R) = N$Ass$(R) = \{m_1, m_2\}$.

(3) Let $U = m_1$. Then all the right ideals of R not contained in $U = m_1$ are m_2 and m_3 with $m_2 \supseteq m_3$. Now we show that m_2 is Σ_{m_1}-prime. Clearly, $m_1 \subseteq m_1 : m_2$ since $m_2 m_1 = 0$. Given $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in m_1 : m_2$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in m_1$. Then $a = 0$ and so $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in m_1$. Hence $m_1 : m_2 = m_1$. Similarly, we have $m_1 : m_3 = m_1$. Therefore m_2 is Σ_{m_1}-prime, and Σ_{m_1}-Ass$(R) = \{m_1\}$.

(4) Let $U = m_2$. Then all the right ideals of R not contained in m_2 are m_1. By a routine computations, we have m_1 is Σ_{m_2}-prime, and Σ_{m_2}-Ass$(R) = \{m_2\}$.

3. Σ-associated primes over extension rings

In this section we always denote the Ore extension by $R[x; \alpha, \delta]$, where $\alpha : R \longrightarrow R$ is an endomorphism and $\delta : R \longrightarrow R$ is an α-derivation. Recall that an α-derivation δ is an additive operator on R with the property that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The elements of $R[x; \alpha, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \alpha, \delta]$ is given by the multiplication in R and the condition $xa = \alpha(a)x + \delta(a)$ for all $a \in R$.

For any $0 \leq i \leq j$, $f_i^j \in \mathrm{End}(R, +)$ will denote the map which is the sum of all possible words in α and δ built with i letters α and $j - i$ letters δ.

Using recursive formulas for the f_i^j's and induction, as done in [8], one can show with a routine computation that

$$x^j a = \sum_{i=0}^{j} f_i^j(a) x^i.$$

This formula uniquely determines a general product of polynomials in $R[x; \alpha, \delta]$ and will be used freely in what follows.

Let I be a subset of R. $I[x;\alpha,\delta]$ means the set $\{u_0+u_1x+\cdots+u_nx^n\in R[x;\alpha,\delta]\mid u_i\in I, 0\le i\le n\}$, that is, for any skew polynomial $f(x)=u_0+u_1x+\cdots+u_nx^n\in R[x;\alpha,\delta]$, $f(x)\in I[x;\alpha,\delta]$ if and only if $u_i\in I$ for all $0\le i\le n$.

Let α be an endomorphism and δ an α-derivation of R. Following Hashemi and Moussavi [6], a ring R is said to be α-compatible if for each $a,b\in R, ab=0\Leftrightarrow a\alpha(b)=0$. Moreover, R is called δ-compatible if for each $a,b\in R, ab=0\Rightarrow a\delta(b)=0$. If R is both α-compatible and δ-compatible, then R is said to be (α,δ)-compatible.

Let I be an ideal of R. Due to Hashemi [7], I is said to be α-compatible if for each $a,b\in R, ab\in I\Leftrightarrow a\alpha(b)\in I$. Moreover, I is called δ-compatible if for each $a,b\in R, ab\in I\Rightarrow a\delta(b)\in I$. If I is both α-compatible and δ-compatible, then I is said to be (α,δ)-compatible. Clearly, a ring R is an (α,δ)-compatible ring if and only if 0 is an (α,δ)-compatible ideal.

The following lemma appears in [7].

Lemma 3.1 ([7, Proposition 2.3]) *Let I be an (α,δ)-compatible ideal, and a, $b\in R$.*

(1) If $ab\in I$, then $a\alpha^n(b)\in I$ and $\alpha^n(a)b\in I$ for every positive integer n. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b\in I$ for some positive integer k, then $ab\in I$.

(2) If $ab\in I$, then $\alpha^m\delta^n(b)\in I$ and $\delta^m(a)\alpha^n(b)\in I$ for any nonnegative integers m, n.

Lemma 3.2 *Let I be an (α,δ)-compatible ideal and a, $b\in R$. If $ab\in I$, then $af_i^j(b)\in I$ and $f_i^j(a)b\in I$ for all $0\le i\le j$.*

Proof It follows directly from Lemma 3.1.

Lemma 3.3 *Let U be an (α,δ)-compatible ideal. If $mr\in U$, then $mx^ir\in U[x;\alpha,\delta]$.*

Proof We have $mx^ir=mf_0^i(r)+mf_1^i(r)x+\cdots+mf_{i-1}^i(r)x^{i-1}+m\alpha^i(r)x^i$. Then by Lemma 3.2, we complete the proof. $\Box$

Proposition 3.4 *Let U be an (α,δ)-compatible ideal and I a right ideal of R with $\wp=U:I$. If I is Σ_U-prime, then we have the following.*

(1) $\wp[x;\alpha,\delta]=U[x;\alpha,\delta]:I[x;\alpha,\delta]$.

(2) $I[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-prime.

Proof (1) Let $i(x)=a_0+a_1x+\cdots+a_mx^m\in I[x;\alpha,\delta]$ and $p(x)=b_0+b_1x+\cdots+b_nx^n\in\wp[x;\alpha,\delta]$. Then

$$i(x)p(x)=\Big(\sum_{i=0}^{m}a_ix^i\Big)\Big(\sum_{j=0}^{n}b_jx^j\Big)=\sum_{k=0}^{m+n}\Big(\sum_{s+t=k}\Big(\sum_{i=s}^{m}a_if_s^i(b_t)\Big)\Big)x^k.$$

Since $a_ib_t\in U$ for all $0\le i\le m$ and $0\le t\le n$, we obtain $a_if_s^i(b_t)\in U$ by Lemma 3.2, and so $\sum_{s+t=k}(\sum_{i=s}^{m}a_if_s^i(b_t))\in U$ for all $0\le k\le m+n$. Thus $i(x)p(x)\in U[x;\alpha,\delta]$ and so $U[x;\alpha,\delta]:I[x;\alpha,\delta]\supseteq\wp[x;\alpha,\delta]$.

In order to prove the reverse inclusion, let $f(x)=\sum_{i=0}^{m}a_ix^i\in U[x;\alpha,\delta]:I[x;\alpha,\delta]$. Then for each $r\in I$, we have $rf(x)=\sum_{i=0}^{m}ra_ix^i\in U[x;\alpha,\delta]$ and so $ra_i\in U$ for each $r\in I$ and each $0\le i\le m$. Thus for each $0\le i\le m$, $a_i\in U:I=\wp$ and so $f(x)\in\wp[x;\alpha,\delta]$. Hence

$U[x;\alpha,\delta]:I[x;\alpha,\delta]\subseteq\wp[x;\alpha,\delta]$. Therefore $U[x;\alpha,\delta]:I[x;\alpha,\delta]=\wp[x;\alpha,\delta]$ is proved.

(2) It suffices to show that for every $i(x)\in I[x;\alpha,\delta]$ and $i(x)\notin U[x;\alpha,\delta]$, we have

$$U[x;\alpha,\delta]:(i(x)R[x;\alpha,\delta])=U[x;\alpha,\delta]:I[x;\alpha,\delta]=\wp[x;\alpha,\delta],$$

where $i(x)R[x;\alpha,\delta]$ denotes the right ideal of $R[x;\alpha,\delta]$ generated by $i(x)$.

In the following we use essentially the same method as in the proof of [2, Theorem 2.1] to claim the above statement. Clearly,

$$U[x;\alpha,\delta]:(i(x)R[x;\alpha,\delta])\supseteq U[x;\alpha,\delta]:I[x;\alpha,\delta]=\wp[x;\alpha,\delta].$$

Now assume that the reverse inclusion fails. There would exist an element $g(x)\notin\wp[x;\alpha,\delta]$ such that $i(x)R[x;\alpha,\delta]g(x)\in U[x;\alpha,\delta]$. Choose $g(x)=\sum_{i=0}^{l}a_ix^i$ $(a_l\neq0)$ of smallest possible degree l satisfying these conditions.

Suppose that $a_l\in\wp=U:I$. Then $g'(x)=\sum_{i=0}^{l-1}a_ix^i\notin\wp[x;\alpha,\delta]$, and since $a_lx^l\in\wp[x;\alpha,\delta]\subseteq U[x;\alpha,\delta]:(i(x)R[x;\alpha,\delta])$, we would have $i(x)R[x;\alpha,\delta]g'(x)\in U[x;\alpha,\delta]$. But now the fact that $g'(x)$ has degree less than l contradicts the minimality of l. Thus we may assume that $a_l\notin\wp$. Let $i_k\neq0$ be the leading coefficients of $i(x)\in I[x;\alpha,\delta]$. Since I is Σ_U-prime, $U:(i_kR)=U:I=\wp$ where i_kR is the right ideal of R generated by i_k. Hence there exists $r\in R$ with $i_kra_l\notin U$. By the α-compatibility, $i_k\alpha^k(ra_l)\notin U$. So the leading coefficient of $i(x)rg(x)$ is not contained in U, contradicting the statement that $i(x)R[x;\alpha,\delta]g(x)\subseteq U[x;\alpha,\delta]$. Thus we finish the proof of (2). □

Let U be an ideal of R and $m(x)=m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n\notin U[x;\alpha,\delta]$. If $m_k\notin U$, and $m_i\in U$ for all $i>k$, then we say that the Σ-degree of $m(x)$ is k. To simplify notations, we write $\Sigma\deg(m(x))$ for the Σ-degree of $m(x)$. If $m(x)\in U[x;\alpha,\delta]$, then we define $\Sigma\deg(m(x))=-1$.

Definition 3.5 *Let $m(x)=m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n\notin U[x;\alpha,\delta]$ with $\Sigma\deg(m(x))=k$. We say that $m(x)$ is a Σ-good polynomial if for any $i<k$, $U:m_k\subseteq U:m_i$.*

In the following example, we offer a few natural constructions of Σ-good polynomials.

Example 3.6 Let U be an ideal of R.

(1) Any element not contained in U is a Σ-good polynomial of Σ-degree 0.

(2) An ideal P of R is called a completely prime ideal if $ab\in P$ implies that $a\in P$ or $b\in P$. If U is a completely prime ideal and $m\notin U$, then any skew polynomial with leading coefficient m is a Σ-good polynomial.

(3) Let U be an (α,δ)-compatible completely prime ideal and $m\notin U$. If $b\in R$ with $mx^nb\notin U[x;\alpha,\delta]$, then the skew polynomial mx^nb is a Σ-good polynomial of Σ-degree n and leading coefficient $m\alpha^n(b)$.

(4) Suppose that $m(x)$ is a Σ-good polynomial. Then $m(x)x^i$ is also Σ-good for any $i\geq0$.

Lemma 3.7 *For any $m(x)=m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n\notin U[x;\alpha,\delta]$, if U is (α,δ)-compatible, then there exists $r\in R$ such that $m(x)r$ is a Σ-good polynomial.*

Proof Assume the result is false and let $m(x) = m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n \notin U[x;\alpha,\delta]$ be a counterexample of minimality Σ-degree $k \geq 1$. In particular, $m(x)$ is not a Σ-good polynomial. Hence there exists $i \leq k$ such that $U : m_k \not\subseteq U : m_i$. So we can find $b \in R$ with $m_kb \in U$ and $m_ib \notin U$. Note that the degree k coefficient of $m(x)b$ is $m_k\alpha^k(b)+\sum_{i=k+1}^{n} m_if_k^i(b)$ and $m_k\alpha^k(b) \in U$ due to the (α,δ)-compatibility of U. On the other hand, we have $\Sigma\deg(m(x)) = k$. Thus $m_i \in U$ for all $i > k$, and so $m_if_k^i(b) \in U$ for all $k < i \leq n$. Thus $m_k\alpha^k(b)+\sum_{i=k+1}^{n} m_if_k^i(b) \in U$ and so $m(x)b$ has Σ-degree at most $k-1$. Now we show that $m(x)b \notin U[x;\alpha,\delta]$. Suppose on the contrary that $m(x)b \in U[x;\alpha,\delta]$. Then we have

$$\begin{aligned}m(x)b =&(m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n)b\\ =&\sum_{s=0}^{n} m_sf_0^s(b)+\Big(\sum_{s=1}^{n} m_sf_1^s(b)\Big)x+\cdots+\Big(\sum_{s=k}^{n} m_sf_k^s(b)\Big)x^k+\cdots+\\ &m_n\alpha^n(b)x^n \in U[x;\alpha,\delta].\end{aligned}$$

So we have

$$\sum_{s=i}^{n} m_sf_i^s(b) \in U, \sum_{s=i+1}^{n} m_sf_{i+1}^s(b) \in U, \ldots, \sum_{s=k-1}^{n} m_sf_{k-1}^s(b) \in U.$$

From $\sum_{s=k-1}^{n} m_sf_{k-1}^s(b) \in U$ and the conditions that:

(a) $m_i \in U$ for all $i > k$,

(b) $m_kb \in U$,

(c) U is an (α,δ)-compatible ideal,

we obtain that $m_{k-1}b \in U$. Similarly, we obtain $m_{k-2}b \in U, m_{k-3}b \in U, \ldots, m_ib \in U$. This contradicts the fact that $m_ib \notin U$. Thus $m(x)b \notin U[x;\alpha,\delta]$. By the minimality of k, we know that there exists $c \in R$ with $m(x)bc$ Σ-good, which contradicts the fact that $m(x)$ is a counterexample to the statement. $\Box$

Lemma 3.8 *Let U be an (α,δ)-compatible ideal and $m(x) = m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n$ be a Σ-good polynomial with $\Sigma\deg(m(x)) = k$. Then for any $r \in R$ with $m(x)r \notin U[x;\alpha,\delta]$, we have $m(x)r$ is also a Σ-good polynomial with $\Sigma\deg(m(x)r) = k$.*

Proof We have

$$\begin{aligned}m(x)r &= (m_0+m_1x+\cdots+m_kx^k+\cdots+m_nx^n)r\\ &= \sum_{s=0}^{n} m_sf_0^s(r)+(\sum_{s=1}^{n} m_sf_1^s(r))x+\cdots+(\sum_{s=k}^{n} m_sf_k^s(r))x^k+\cdots+m_n\alpha^n(r)x^n\\ &= \Delta_0+\Delta_1x+\cdots+\Delta_kx^k+\cdots+\Delta_nx^n\end{aligned}$$

where $\Delta_p = \sum_{s=p}^{n} m_sf_p^s(r)$, $p = 0,1,\ldots,n$.

Since $\Sigma\deg(m(x)) = k$, we have $m_j \in U$ for all $j > k$, and so $\Delta_j = \sum_{s=j}^{n} m_sf_j^s(r) \in U$ $(j > k)$, and $\sum_{s=k+1}^{n} m_sf_k^s(r) \in U$. Suppose

$$\Delta_k = \sum_{s=k}^{n} m_sf_k^s(r) = m_k\alpha^k(r)+\sum_{s=k+1}^{n} m_sf_k^s(r) \in U.$$

Then $m_k\alpha^k(r) \in U$, and so by Lemma 3.1, we have $m_k r \in U$. Since $m(x)$ is a Σ-good polynomial with $\Sigma\deg(m(x)) = k$, we have $m_i r \in U$ for all $0 \le i \le k$, and so $m_i r \in U$ for all $0 \le i \le n$. Then it is easy to see that $m(x)r \in U[x;\alpha,\delta]$, contradicting the fact that $m(x)r \notin U[x;\alpha,\delta]$. Thus we obtain $\Sigma\deg(m(x)r) = k$. If $a \in U : \Delta_k$, then $\Delta_k a = m_k\alpha^k(r)a + (\sum_{s=k+1}^n m_s f_k^s(r))a \in U$, and so $m_k\alpha^k(r)a \in U$ since $m_s \in U$ for all $s > k$. Then by Lemma 3.1, we obtain $m_k ra \in U$. Since $m(x)$ is a Σ-good polynomial with $\Sigma\deg(m(x)) = k$, we have $m_i ra \in U$ for all $i < k$, and so $m_i ra \in U$ for all $0 \le i \le n$. Since U is an (α,δ)-compatible ideal, it is easy to see that $\Delta_i a = (\sum_{s=i}^n m_s f_i^s(r))a \in U$. Hence $U : \Delta_k \subseteq U : \Delta_i$ for all $i < k$. Therefore $m(x)r$ is a Σ-good polynomial. $\square$

A ideal I of R is a Σ-ideal if $a^2b \in I$ implies $ab \in I$ for all a, $b \in R$.

Proposition 3.9 *Let U be an (α,δ)-compatible ideal and $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n$ be a Σ-good polynomial with $\Sigma\deg(m(x)) = k$ and $U : (m_kR) = \wp$, where m_kR denotes the right ideal of R generated by m_k. Then we have the following.*

(1) We have $U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta]) = \wp[x;\alpha,\delta]$, where $m(x)R[x;\alpha,\delta]$ denotes the right ideal of $R[x;\alpha,\delta]$ generated by $m(x)$.

(2) If U is a Σ-ideal and $m(x)R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-prime, then m_kR is Σ_U-prime.

Proof (1) We first show that

$$U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta]) \supseteq \wp[x;\alpha,\delta].$$

Let $r(x) = r_0 + r_1x + \cdots + r_sx^s + \cdots + r_tx^t \in R[x;\alpha,\delta]$ with $\Sigma\deg(r(x)) = s$, and $h(x) = h_0 + h_1x + \cdots + h_lx^l + \cdots + h_qx^q \in \wp[x;\alpha,\delta]$ with $\Sigma\deg(h(x)) = l$. In order to show that $m(x)r(x)h(x) \in U[x;\alpha,\delta]$, we need only to show that

$$\Big(\sum_{i=0}^k m_ix^i\Big)\Big(\sum_{j=0}^s r_jx^j\Big)\Big(\sum_{v=0}^l h_vx^v\Big) \in U[x;\alpha,\delta].$$

A typical term of $(\sum_{i=0}^k m_ix^i)(\sum_{j=0}^s r_jx^j)(\sum_{v=0}^l h_vx^v)$ is $m_ix^ir_jx^jh_vx^v$. The coefficients of $m_ix^ir_jx^jh_vx^v$ can be written as sums of monomials in m_i, $f_\alpha^\beta(r_j)$ and $f_\gamma^\delta(h_v)$. Consider each monomial $m_if_\alpha^\beta(r_j)f_\gamma^\delta(h_v)$. Since $h_v \in \wp = U : (m_kR)$, we have $m_kRh_v \subseteq U$. Since $m(x)$ is a Σ-good polynomial with $\Sigma\deg(m(x) = k$, we obtain $m_iRh_v \subseteq U$ for all $i \le k$. Since U is (α,δ)-compatible, by Lemma 3.2, we obtain $m_if_\alpha^\beta(R)f_\gamma^\delta(h_v) \subseteq U$, and so $m_if_\alpha^\beta(r_j)f_\gamma^\delta(h_v) \in U$. Thus $m_ix^ir_jx^jh_vx^v \in U[x;\alpha,\delta]$ and so

$$\Big(\sum_{i=0}^k m_ix^i\Big)\Big(\sum_{j=0}^s r_jx^j\Big)\Big(\sum_{v=0}^l h_vx^v\Big) \in U[x;\alpha,\delta].$$

Hence $U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta]) \supseteq \wp[x;\alpha,\delta]$.

For the reverse inclusion, assume that $g(x) = b_0 + b_1x + \cdots + b_lx^l + \cdots + b_mx^m \in U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta])$ with $\Sigma\deg(g(x)) = l$. Then we have $m(x)R[x;\alpha,\delta]g(x) \subseteq U[x;\alpha,\delta]$. Note that

$m(x)R[x;\alpha,\delta]g(x)\subseteq U[x;\alpha,\delta]$ if and only if

$$\Big(\sum_{i=0}^{k}m_ix^i\Big)R[x;\alpha,\delta]\Big(\sum_{j=0}^{l}b_jx^j\Big)\subseteq U[x;\alpha,\delta]$$

if only if

$$\Big(\sum_{i=0}^{k}m_ix^i\Big)R\Big(\sum_{j=0}^{l}b_jx^j\Big)\subseteq U[x;\alpha,\delta].$$

The leading coefficients of

$$\Big(\sum_{i=0}^{k}m_ix^i\Big)R\Big(\sum_{j=0}^{l}b_jx^j\Big)$$

is $m_k\alpha^k(Rb_l)$. Since U is (α,δ)-compatible, by Lemma 3.1, we obtain $m_kRb_l\subseteq U$, and so $b_l\in\wp=U:(m_kR)$. Since $m(x)$ is a Σ-good polynomial with $\Sigma\deg(m(x))=k$, we obtain $m_iRb_l\subseteq U$ for all $0\le i\le k$. Thus from $(\sum_{i=0}^{k}m_ix^i)R(\sum_{j=0}^{l}b_jx^j)\subseteq U[x;\alpha,\delta]$, we obtain $(\sum_{i=0}^{k}m_ix^i)R(\sum_{j=0}^{l-1}b_jx^j)\subseteq U[x;\alpha,\delta]$. Using the same method as above, we obtain $b_{l-1}\in\wp$. Continuing this procedure yields $b_l\in\wp, b_{l-1}\in\wp,\dots,b_0\in\wp$. Since $b_v\in U$ for all $v>l$, it is easy to see that for all $v>l$, $b_v\in U:(m_kR)=\wp$. Hence for all $0\le j\le m$, $b_j\in\wp$. So $g(x)=b_0+b_1x+\cdots+b_lx^l+\cdots+b_mx^m\in\wp[x;\alpha,\delta]$, which implies that $U[x;\alpha,\delta]:(m(x)R[x;\alpha,\delta])\subseteq\wp[x;\alpha,\delta]$. Therefore $U[x;\alpha,\delta]:(m(x)R[x;\alpha,\delta])=\wp[x;\alpha,\delta]$.

(2) Since $m_k\notin U$, we have $m_kR\notin U$. Assume that a right ideal $Q\subseteq m_kR$, and $Q\not\subseteq U$. Then $U:Q\supseteq U:(m_kR)$. Now we show that $U:Q\subseteq U:(m_kR)=\wp$. Set $W=\{m(x)r\mid r\in Q\}$, and let $WR[x;\alpha,\delta]$ be the right ideal of $R[x;\alpha,\delta]$ generated by W. Clearly, $WR[x;\alpha,\delta]\subseteq m(x)R[x;\alpha,\delta]$. Since $Q\not\subseteq U$, there exists $a\in R$ such that $m_ka\in Q$ and $m_ka\notin U$. If $m_k\cdot m_ka\in U$, then by the condition that U is a Σ-ideal, we have $m_ka\in U$. This contradicts the fact that $m_ka\notin U$. Thus $m_k\cdot m_ka\notin U$. Now we show that $m(x)m_ka\notin U[x;\alpha,\delta]$. Assume on the contrary that $m(x)m_ka\in U[x;\alpha,\delta]$. Since $\Sigma\deg(m(x))=k$, we have $m(x)m_ka\in U[x;\alpha,\delta]$ if and only if $(m_0+m_1x+\cdots+m_kx^k)m_ka\in U[x;\alpha,\delta]$. The leading coefficient of $(m_0+m_1x+\cdots+m_kx^k)m_ka$ is $m_k\alpha^k(m_ka)$. Thus we have $m_k\alpha^k(m_ka)\in U$, and so $m_km_ka\in U$ since U is (α,δ)-compatible. This contradicts the fact that $m_km_ka\notin U$. Hence $m(x)\cdot m_ka\not\subseteq U[x;\alpha,\delta]$, and so $WR[x;\alpha,\delta]\not\subseteq U[x;\alpha,\delta]$. Since $m(x)R[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-prime, we obtain

$$U[x;\alpha,\delta]:(WR[x;\alpha,\delta])=U[x;\alpha,\delta]:(m(x)R[x;\alpha,\delta])=\wp[x;\alpha,\delta].$$

Suppose $q\in U:Q$. Then $rq\in U$ for each $r\in Q$. For any $m(x)rf(x)\in WR[x;\alpha,\delta]$ where $f(x)=a_0+a_1x+\cdots+a_lx^l\in R[x;\alpha,\delta]$ and $r\in Q$. The typical term of $m(x)rf(x)$ is $m_ix^ira_jx^j$. Since $ra_j\in Q$, we have $ra_jq\in U$. Then by Lemma 3.3, we have $ra_jx^iq\in U[x;\alpha,\delta]$ and so $m_ix^ira_jx^jq\in U[x;\alpha,\delta]$. Thus for any

$$\sum m(x)r_if_i(x)\in WR[x;\alpha,\delta],$$

it is easy to see that

$$\Big(\sum m(x)r_if_i(x)\Big)q\in U[x;\alpha,\delta].$$

Hence

$$q \in U[x;\alpha,\delta] : (WR[x;\alpha,\delta]) = U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta]) = \wp[x;\alpha,\delta],$$

and so $q \in \wp = U : (m_kR)$. Thus $U : Q \subseteq U; (m_kR)$, and this implies that $U : Q = U : (m_kR)$. Therefore m_kR is Σ_U-prime. $\square$

With the help of Propositions 3.4 and 3.9, we get the central result of this paper.

Theorem 3.10 *Let U be an (α,δ)-compatible Σ-ideal. Then we have*

$$\Sigma_{U[x;\alpha,\delta]}\text{-Ass}(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in \Sigma_U\text{-Ass}(R)\}.$$

Proof Let $\wp \in \Sigma_U$-Ass(R). By definition, there exists a right ideal $I \not\subseteq U$ with I being Σ_U-prime and $\wp = U : I$. Then by Proposition 3.4, we have $\wp[x;\alpha,\delta] = U[x;\alpha,\delta] : I[x;\alpha,\delta]$ and $I[x;\alpha,\delta]$ is $\Sigma_{U[x;\alpha,\delta]}$-prime. Thus $\wp[x;\alpha,\delta] \in \Sigma_{U[x;\alpha,\delta]}$-Ass$(R[x;\alpha,\delta])$ and so

$$\Sigma_{U[x;\alpha,\delta]}\text{-Ass}(R[x;\alpha,\delta]) \supseteq \{\wp[x;\alpha,\delta] \mid \wp \in \Sigma_U\text{-Ass}(R)\}.$$

Now we prove $\subseteq$ in Theorem 3.10. Let $I \in \Sigma_{U[x;\alpha,\delta]}$-Ass$(R[x;\alpha,\delta])$. By definition, there exists a $\Sigma_{U[x;\alpha,\delta]}$-prime ideal $\pounds$ with $I = U[x;\alpha,\delta] : \pounds$. Pick any $m(x) = m_0 + m_1x + \cdots + m_kx^k + \cdots + m_nx^n \in \pounds$ and $m(x) \notin U[x;\alpha,\delta]$. By Lemma 3.7, we may assume that $m(x)$ is Σ-good, and $\Sigma\deg(m(x)) = k$. Since $\pounds$ is $\Sigma_{U[x;\alpha,\delta]}$-prime, we have

$$I = U[x;\alpha,\delta] : \pounds = U[x;\alpha,\delta] : (m(x)R[x;\alpha,\delta])$$

and $m(x)R[x;\alpha,\delta]$ is also $\Sigma_{U[x;\alpha,\delta]}$-prime. Let $\wp = U : (m_kR)$. Then by Proposition 3.9, we have $I = \wp[x;\alpha,\delta]$, and m_kR is Σ_U-prime. Hence

$$\Sigma_{U[x;\alpha,\delta]}\text{-Ass}(R[x;\alpha,\delta]) \subseteq \{\wp[x;\alpha,\delta] \mid \wp \in \Sigma_U\text{-Ass}(R)\}.$$

Therefore

$$\Sigma_{U[x;\alpha,\delta]}\text{-Ass}(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in \Sigma_U\text{-Ass}(R)\}. \quad \square$$

Corollary 3.11 *Let U be a Σ-ideal of R. Then we have the following;*

(1) If U is α-compatible, then $\Sigma_{U[x;\alpha]}$-Ass$(R[x;\alpha]) = \{\wp[x;\alpha] \mid \wp \in \Sigma_U$-Ass$(R)\}$.

(2) If U is δ-compatible, then $\Sigma_{U[x;\delta]}$-Ass$(R[x;\delta]) = \{\wp[x;\delta] \mid \wp \in \Sigma_U$-Ass$(R)\}$.

Corollary 3.12 *Let R be an (α,δ)-compatible NI-ring.*

$$\Sigma_{\text{nil}(R)[x;\alpha,\delta]}\text{-Ass}(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in \Sigma_{\text{nil}(R)}\text{-Ass}(R)\}.$$

Proof Let $U = \text{nil}(R)$. Then by [10, Lemma 2.4] and [10, Lemma 2.5], it is easy to see that nil(R) is an (α,δ)-compatible Σ-ideal. According to Theorem 3.10, we complete the proof. $\square$

Corollary 3.13 ([11, Theorem 3.1]) *Let R be an (α,δ)-compatible 2-primal ring. Then NAss$(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in NAss(R)\}$.*

Proof Let $U = \text{nil}(R)$. Using the same method as in the proof of Corollary 3.12, we obtain nil(R) is an (α,δ)-compatible Σ-ideal. In view of [11, Corollary 2.2], we have nil$(R)[x;\alpha,\delta] =$ nil$(R[x;\alpha,\delta])$. Then according to Theorem 3.10, we complete the proof. $\square$

Corollary 3.14 *Let R be a 2-primal ring. Then* $N\mathrm{Ass}(R[x]) = \{\wp[x] \mid p \in N\mathrm{Ass}(R)\}$.

Proof It follows from Corollary 3.13.

Note that if R is an (α,δ)-compatible ring and 0 a Σ-ideal, then by Theorem 3.10, we obtain $\mathrm{Ass}(R[x;\alpha,\delta]) = \{\wp[x;\alpha,\delta] \mid \wp \in \mathrm{Ass}(R)\}$. But by using some special nice properties of zero ideal, Annin showed that the condition that 0 is a Σ-ideal is superfluous [2, Theorem 2.1].

Let α be an automorphism of a ring R. The skew Laurent polynomial ring $R[x,x^{-1};\alpha]$ is the ring where elements are the form $\sum_{i=s}^{n} a_i x^i$ where s, $n \in \mathbb{Z}$. The addition is defined as usual and the multiplication by $x^i b = \alpha^i(b)x$ for any $i \in \mathbb{Z}$. Let I be a right ideal of R. $I[x,x^{-1};\alpha]$ means the set $\{\sum_{i=s}^{n} a_i x^i \in R[x,x^{-1};\alpha] \mid a_i \in I \text{ for all } s \le i \le n\}$. For skew Laurent polynomial rings, we can derive results analogous to Theorem 3.10 above. □

Theorem 3.15 *Let $\alpha : R \longrightarrow R$ be an automorphism. If U is an α-compatible Σ-ideal, then*

$$\Sigma_{U[x,x^{-1};\alpha]}\text{-}\mathrm{Ass}(R[x,x^{-1};\alpha]) = \{\wp[x,x^{-1};\alpha] \mid \wp \in \Sigma_U\text{-}\mathrm{Ass}(R)\}.$$

Proof All statements here are proved in essentially the same way as Theorem 3.10, so we will discuss the proof briefly. First we observe that if U is α-compatible, then U is α^i-compatible for all $i \in \mathbb{Z}$. Let I be a right ideal of R with $I \nsubseteq U$ and $\wp = U : I$. By using the same way as in the proof of Proposition 3.4, we can show that if I is Σ_U-prime, then $I[x,x^{-1};\alpha]$ is $\Sigma_{U[x,x^{-1};\alpha]}$-prime and $U[x,x^{-1};\alpha] : I[x,x^{-1};\alpha] = \wp[x,x^{-1};\alpha]$. Thus

$$\Sigma_{U[x,x^{-1};\alpha]}\text{-}\mathrm{Ass}(R[x,x^{-1};\alpha]) \supseteq \{\wp[x,x^{-1};\alpha] \mid \wp \in \Sigma_U\text{-}\mathrm{Ass}(R)\}.$$

Let $m(x) = \sum_{i=s}^{n} m_i x^i$ be a skew Laurent polynomial in $R[x,x^{-1};\alpha]$. We say that $\Sigma\deg(m(x)) = k$ if there exists some $k \in \mathbb{Z}$ such that $m_k \notin U$ and $m_i \in U$ if $i > k$. We say that $m(x) = \sum_{i=s}^{n} m_i x^i$ with $\Sigma\deg(m(x)) = k$ is a Σ-good skew Laurent polynomial if $U : m_k \subseteq U : m_i$ for all $i < k$. Then by using the same way as in the proof of Proposition 3.9, we obtain that

$$\Sigma_{U[x,x^{-1};\alpha]}\text{-}\mathrm{Ass}(R[x,x^{-1};\alpha]) \subseteq \{\wp[x,x^{-1};\alpha] \mid \wp \in \Sigma_U\text{-}\mathrm{Ass}(R)\}.$$

Therefore

$$\Sigma_{U[x,x^{-1};\alpha]}\text{-}\mathrm{Ass}(R[x,x^{-1};\alpha]) = \{\wp[x,x^{-1};\alpha] \mid \wp \in \Sigma_U\text{-}\mathrm{Ass}(R)\}.$$

Corollary 3.16 *We have the following:*

(1) *Let R be an α-compatible NI-ring where α is an automorphism of R. Then*

$$\Sigma_{\mathrm{nil}(R)[x,x^{-1};\alpha]}\text{-}\mathrm{Ass}(R[x,x^{-1};\alpha]) = \{\wp[x,x^{-1};\alpha] \mid \wp \in \Sigma_{\mathrm{nil}(R)}\text{-}\mathrm{Ass}(R)\}.$$

(2) *Let R be an α-compatible 2-primal ring where α is an automorphism of R. Then*

$$N\mathrm{Ass}(R[x,x^{-1};\alpha]) = \{\wp[x,x^{-1};\alpha] \mid \wp \in N\mathrm{Ass}(R)\}.$$

Proof (1) Let $U = \mathrm{nil}(R)$. Using the same method as in the proof of Corollary 3.12, we have $\mathrm{nil}(R)$ is an α-compatible Σ-ideal. Then according to Theorem 3.15, we complete the proof.

(2) Let $U = \mathrm{nil}(R)$. By the proof of (1), we have $\mathrm{nil}(R)$ is an α-compatible Σ-ideal. Analogously to [10, Lemma 2.6], we show that $\mathrm{nil}(R)[x,x^{-1};\alpha] = \mathrm{nil}(R[x,x^{-1};\alpha])$. According to Theorem 3.15, we complete the proof. □

Corollary 3.17 *Let R be a 2-primal ring. Then*

$$N\mathrm{Ass}(R[x,x^{-1}]) = \{\wp[x,x^{-1}] \mid \wp \in N\mathrm{Ass}(R)\}.$$

Proof It follows directly from Corollary 3.16.

Let $\alpha : R \longrightarrow R$ be an endomorphism and U an ideal of R. In the following we investigate the relationship between the Σ-associated primes of the skew power series ring $R[[x;\alpha]]$ and that of the ring R. $\square$

Definition 3.18 *Let $k \in \mathbb{Z}$ and $m(x) = \sum_{i=0}^{\infty} m_i x^i \notin U[[x;\alpha]]$. We say that $m(x)$ is a k-Σ-good power series if $m_k \notin U$, and $U : m_k \subseteq U : m_i$ if $i < k$.*

Definition 3.19 *Let U be an ideal of R and $m(x) = \sum_{i=0}^{\infty} m_i x^i \notin U[[x;\alpha]]$. We say that $m(x)$ is a Σ-good power series if there exists some $k \in \mathbb{Z}$ such that $m_k \notin U$ and $U : m_k \subseteq U : m_i$ if $i \neq k$.*

Proposition 3.20 *Let U be an ideal of R and $m(x) = \sum_{i=0}^{\infty} m_i x^i \notin U[[x;\alpha]]$. If R is a left perfect ring, then there exists $r \in R$ such that $m(x)r$ is a Σ-good power series.*

Proof Note that if $m_k \in U$ for some $k \in \mathbb{Z}$, then $U : m_k = R$, and so for any coefficient m_i of $m(x)$, we have $U : m_i \subseteq U : m_k$. Hence without loss of generality, we may assume that $m_i \notin U$ for all $0 \leq i \leq \infty$. Consider the polynomial $m^1(x) = m_0 + m_1 x$. By Lemma 3.7, there exists $r_1 \in R$ such that $m^1(x)r_1$ is a Σ-good polynomial, and so there exists $r_1 \in R$ such that $m(x)r_1$ is a 1-Σ-good power series. Then inductively, we can find $r_i \in R$ such that $m(x)r_1r_2\cdots r_i$ is i-Σ-good. Consider the descending chain of cycle right modules

$$m(x)R \supseteq m(x)r_1R \supseteq m(x)r_1r_2R \supseteq \cdots$$

Since R is left perfect, this chain stabilizes, say at $m(x)r_1r_2\cdots r_kR$. Let $m'(x) = m(x)r_1r_2\cdots r_k$. Then by analogy with the proof of [1, Theorem 5.2], we can show that $m'(x) = m(x)r_1r_2\cdots r_k$ is a Σ-good power series. $\square$

Theorem 3.21 *Let R be a left perfect ring and U an α-compatible Σ-ideal. Then*

$$\Sigma_{U[[x;\alpha]]}(R[[x;\alpha]]) = \{\wp[[x;\alpha]] \mid \wp \in \Sigma_U\text{-Ass}(R)\}.$$

Proof By analogy with the proof of [1, Theorem 5.1], we can show that

$$\Sigma_{U[[x;\alpha]]}(R[[x;\alpha]]) \supseteq \{\wp[[x;\alpha]] \mid \wp \in \Sigma_U\text{-Ass}(R)\}.$$

Then by analogy with proof of Proposition 3.10, we can see the reverse containment. $\square$

Corollary 3.22 *Let R be an α-compatible left perfect ring. Then we have the following:*

(1) If R is an NI-ring, then

$$\Sigma_{\mathrm{nil}(R)[[x;\alpha]]}\text{-Ass}(R[[x;\alpha]]) = \{\wp[[x;\alpha]] \mid \wp \in \Sigma_{\mathrm{nil}(R)}\text{-Ass}(R)\}.$$

(2) If R is a right noetherian NI ring, then

$$N\mathrm{Ass}(R[[x;\alpha]]) = \{\wp[[x;\alpha]] \mid \wp \in N\mathrm{Ass}(R)\}.$$

Proof (1) Let $U = \mathrm{nil}(R)$. Using the same method as in the proof of Corollary 3.12, we can show that $\mathrm{nil}(R)$ is an α-compatible Σ-ideal. Then we complete the proof by Theorem 3.21.

(2) Let $U = \mathrm{nil}(R)$. By the proof of (1), we obtain that $\mathrm{nil}(R)$ is an α-compatible Σ-ideal. Since R is a right noetherian NI ring, by Levitzki's Theorem [9], $\mathrm{nil}(R)$ is nilpotent. Then by [12, Proposition 2.5], we can show that $\mathrm{nil}(R)[[x;\alpha]] = \mathrm{nil}(R[[x;\alpha]])$. Then by Theorem 3.21 we complete the proof. $\square$

Note that if R is an α-compatible left perfect ring, and 0 is a Σ-ideal, then by Theorem 3.21, we obtain that

$$\mathrm{Ass}(R[[x;\alpha]]) = \{\wp[[x;\alpha]] \mid \wp \in \mathrm{Ass}(R)\}. \tag{$*$}$$

But we must mention that the condition that 0 is a Σ-ideal is superfluous. Annin showed in [1, Theorem 5.2] that if R is an α-compatible left perfect ring, then the equation $(*)$ above is also true.

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