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On Polynomial Dichotomy of Linear Discrete-Time Systems in Banach Spaces

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Abstract This paper deals with two concepts of polynomial dichotomy for linear difference equations which are defined in a Banach space and whose norms can increase not faster than exponentially. Some illustrating examples clarify the relations between these concepts. Our approach is based on the extension of techniques for exponential dichotomy to the case of polynomial dichotomy. The obtained results are generalizations of well-known theorems about the exponential stability and exponential dichotomy.

Keywords linear discrete-time systems; uniform polynomial dichotomy; polynomial dichotomy

MR(2010) Subject Classification 34D05; 34D09

1. Introduction

In the last decades, the asymptotic behavior of solutions of evolution equations both in finite-dimensional and infinite-dimensional spaces has witnessed significant development [1–17]. The notion of (uniform) exponential dichotomy introduced by Perron [10] for differential equations and by Li [5] for difference equations plays a central role in the theory of dynamical systems.

The exponential dichotomy property has made an important progress since the appearance of two remarkable monographs due to Massera and Schäffer [9], respectively, Daleckii and Krein [4]. Later, diverse and important concepts of exponential dichotomy were studied by Sacher and Sell in [15], Barreira and Valls [1,2].

The concept of polynomial asymptotic behavior has been considered in the notable works of Barreira and Valls [3] for evolution operators. Remarkable results were obtained by Megan et al. [6] for Barreira-Valls polynomial stability of evolution operators and for polynomial stability in [7,8]. Moreover, characterizations for uniform and nonuniform polynomial dichotomy of evolution operators have been given in [13], respectively, and [14].

In this paper, we study two polynomial dichotomy concepts for linear discrete-time systems in Banach spaces. The main objective is to give characterizations for polynomial dichotomies of linear difference equations. The obtained results are generalizations of some well-known theorems

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in the case of exponential stability given in [7,11] and in the case of exponential dichotomy in [12]. Some simple examples are included to illustrate the connections between the dichotomy concepts considered in the present paper.

2. Preliminaries

Let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X, will be denoted by $\|\cdot\|$. Let $\Delta=\{(m,n)\in\mathbb{N}^2, m\geq n\}$, $T=\{(m,r,n)\in\mathbb{N}^3, m\geq r\geq n\}$. Let I be the identity operator on X.

In the present paper we consider linear discrete-time system of difference equations

$$x_{n+1}=A(n)x_n,\ n\in\mathbb{N}, \tag{1}$$

where $A:\mathbb{N}\to\mathcal{B}(X)$ is a sequence in $\mathcal{B}(X)$. Then every solution $x=\{x_n\}$ of the system (1) is given by $x_m=\mathcal{A}_m^n x_n$ for all $(m,n)\in\Delta$, where the mapping $\mathcal{A}:\Delta\to\mathcal{B}(X)$ is defined by

$$\mathcal{A}_m^n:=\begin{cases} I, & m=n\\ A(m-1)\cdots A(n), & m>n.\end{cases}$$

It is easy to see that $\mathcal{A}_m^r\mathcal{A}_r^n=\mathcal{A}_m^n$, for all $(m,r,n)\in T$.

For the particular case when (1) is autonomous, i.e., $A(n)=A\in\mathcal{B}(X)$ for all $n\in\mathbb{N}$, then $\mathcal{A}_m^n=A^{m-n}$ for all $(m,n)\in\Delta$.

Definition 2.1 *An application $P:\mathbb{N}\to\mathcal{B}(X)$ is said to be a projection family on X if $P^2(n)=P(n)$, for all $n\in\mathbb{N}$.*

Remark 2.2 If $P:\mathbb{N}\to\mathcal{B}(X)$ is a projection family on X, then the mapping $Q:\mathbb{N}\to\mathcal{B}(X)$, $Q(n)=I-P(n)$ is also a projection family on X, which is called the complementary projection of P. One can easily see that

$$P(n)Q(n)=Q(n)P(n)=0,$$

for every $n\in\mathbb{N}$.

Definition 2.3 *A projection family $P:\mathbb{N}\to\mathcal{B}(X)$ is said to be compatible with the system (1), if*

$$A(n)P(n)=P(n+1)A(n),$$

for all $n\in\mathbb{N}$.

Remark 2.4 If the projection family $P:\mathbb{N}\to\mathcal{B}(X)$ is compatible with the system (1), then its complementary projection $Q:\mathbb{N}\to\mathcal{B}(X)$ is also compatible with the system (1). In addition,

$$\mathcal{A}_m^nP(n)=P(m)\mathcal{A}_m^n \text{ and } \mathcal{A}_m^nQ(n)=Q(m)\mathcal{A}_m^n$$

for all $(m,n)\in\Delta$.

In what follows, for a projection family $P:\mathbb{N}\to\mathcal{B}(X)$ compatible with the system (1), we will denote

$$_P\mathcal{A}_m^n = \mathcal{A}_m^n P(n) \text{ and } _Q\mathcal{A}_m^n = \mathcal{A}_m^n Q(n).$$

We deduced that

$$_P\mathcal{A}_m^r {}_P\mathcal{A}_r^n = {}_P\mathcal{A}_m^n \text{ and } _Q\mathcal{A}_m^r {}_Q\mathcal{A}_r^n = {}_Q\mathcal{A}_m^n$$

for all $(m, r, n) \in T$.

Definition 2.5 *The linear discrete-time system (1) is said to be uniformly exponentially dichotomic if there exist two constants $N \geq 1$ and $v > 0$ such that*

$$e^{v(m-n)} \left(\|_P\mathcal{A}_m^n x\| + \|Q(n)x\|\right) \leq N \left(\|P(n)x\| + \|_Q\mathcal{A}_m^n x\|\right)$$

for all $(m, n, x) \in \Delta \times X$.

Definition 2.6 *The linear discrete-time system (1) is said to be uniformly polynomially dichotomic if there exist two constants $N \geq 1$ and $v > 1$ such that*

$$(\frac{m+1}{n+1})^v \left(\|_P\mathcal{A}_m^n x\| + \|Q(n)x\|\right) \leq N \left(\|P(n)x\| + \|_Q\mathcal{A}_m^n x\|\right)$$

for all $(m, n, x) \in \Delta \times X$.

Remark 2.7 The linear discrete-time system (1) is uniformly polynomially dichotomic if and only if there are $N \geq 1$ and $v > 1$ such that

$$(\frac{m+1}{n+1})^v \left(\|_P\mathcal{A}_m^r x\| + \|_Q\mathcal{A}_n^r x\|\right) \leq N \left(\|_P\mathcal{A}_n^r x\| + \|_Q\mathcal{A}_m^r x\|\right)$$

for all $(m, n, r, x) \in T \times X$.

Remark 2.8 It is obvious that if the linear discrete-time system (1) is uniformly exponentially dichotomic with $v > 1$, then it is uniformly polynomially dichotomic. But the converse statement is not necessarily valid. This fact is illustrated by the following example.

Example 2.9 Let $X = \mathbb{R}^2$ and $A : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1, x_2) = \Big(\frac{n^2+1}{(n+1)^2+1} x_1, \frac{(n+1)^2+1}{n^2+1} x_2\Big)$$

for all $(n, x_1, x_2) \in \mathbb{N} \times \mathbb{R}^2$. Let us consider the projection families $P, Q : \mathbb{N} \to \mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1, x_2) = (x_1, 0), \;\; Q(n)(x_1, x_2) = (0, x_2)$$

for all $n \in \mathbb{N}$ and all $x = (x_1, x_2) \in X$. We have that

$$_P\mathcal{A}_m^n(x_1, x_2) = (\frac{n^2+1}{m^2+1} x_1, 0), \quad _Q\mathcal{A}_m^n(x_1, x_2) = (0, \frac{m^2+1}{n^2+1} x_2).$$

Then

$$(\frac{m+1}{n+1})^2 (\|_P\mathcal{A}_m^n x\| + \|Q(n)x\|) = \frac{(m+1)^2}{m^2+1} \frac{n^2+1}{(n+1)^2} \|P(n)x\| + (\frac{m+1}{n+1})^2 \|Q(n)x\|$$

$$\leq 2\big(\|P(n)x\| + \frac{m^2+1}{n^2+1} \|Q(n)x\|\big) = 2 \left(\|P(n)x\| + \|_Q\mathcal{A}_m^n x\|\right)$$

for all $(m,n,x)\in\Delta\times\mathbb{R}^2$. Thus Definition 2.6 is satisfied for $N=v=2$, hence system (1) is uniformly polynomially dichotomic.

On the other hand, if we suppose that system (1) is uniformly exponentially dichotomic, then there exist $N\geq 1$ and $v>0$ such that $e^{v(m-n)}\cdot\frac{n^2+1}{m^2+1}\leq N$ for all $(m,n)\in\Delta$. In particular, for $n=0$, we obtain $e^{vm}\leq N(m^2+1)$, which is absurd for $m\to\infty$. Hence system (1) is not uniformly exponentially dichotomic.

Definition 2.10 *The linear discrete-time system (1) is said to be polynomially dichotomic if there exist three constants $N\geq 1$, $v>0$ and $\mu\geq 0$ such that*

$$(\frac{m+1}{n+1})^v\,(\|{}_P\mathcal{A}_m^n x\|+\|Q(n)x\|)\leq N(n+1)^\mu\,(\|P(n)x\|+\|{}_Q\mathcal{A}_m^n x\|)\,,$$

for all $(m,n,x)\in\Delta\times X$.

Remark 2.11 The linear discrete-time system (1) is polynomially dichotomic if and only if there are $N\geq 1$, $v>0$ and $\mu\geq 0$ such that

$$(\frac{m+1}{n+1})^v\,(\|{}_P\mathcal{A}_m^r x\|+\|{}_Q\mathcal{A}_n^r x\|)\leq N(n+1)^\mu\,(\|{}_P\mathcal{A}_n^r x\|+\|{}_Q\mathcal{A}_m^r x\|)\,,$$

for all $(m,n,r,x)\in T\times X$.

Remark 2.12 If the linear discrete-time system (1) is uniformly polynomially dichotomic, then it is polynomially dichotomic. But the converse statement is not necessarily valid. This fact is illustrated by the following example.

Example 2.13 Let $X=\mathbb{R}^2$ and $A:\mathbb{N}\to\mathcal{B}(\mathbb{R}^2)$ defined by

$$A(n)(x_1,x_2)=(\frac{\varphi_n}{\varphi_{n+1}}x_1,\frac{\phi_{n+1}}{\phi_n}x_2)$$

for all $(n,x_1,x_2)\in\mathbb{N}\times\mathbb{R}^2$, where the sequences $\varphi,\phi:\mathbb{N}\to\mathbb{R}$ are given by

$$\varphi_n=(n+1)^2(n+1-n\sin\frac{n\pi}{2}),\quad \phi_n=n^2(n+1+n\sin\frac{n\pi}{2}).$$

Let us consider the projection families $P,Q:\mathbb{N}\to\mathcal{B}(\mathbb{R}^2)$ defined by

$$P(n)(x_1,x_2)=(x_1,0),\ Q(n)(x_1,x_2)=(0,x_2)$$

for all $n\in\mathbb{N}$ and all $x=(x_1,x_2)\in X$. We have that

$${}_P\mathcal{A}_m^n(x_1,x_2)=\Big(\frac{(n+1)^2(n+1-n\sin(n\pi/2))}{(m+1)^2(m+1-m\sin(m\pi/2))}x_1,0\Big)$$

and

$${}_Q\mathcal{A}_m^n(x_1,x_2)=\Big(0,\frac{m^2(m+1+m\sin(m\pi/2))}{n^2(n+1+n\sin(n\pi/2))}x_2\Big).$$

Then

$$(\|{}_P\mathcal{A}_m^n x\|+\|Q(n)x\|)\leq(\frac{n+1}{m+1})^2(2n+1)\,\|P(n)x\|+\frac{2(n+1)m^2}{(2n+1)n^2}(\frac{n+1}{m+1})^2\,\|Q(n)x\|$$

$$\leq 2(n+1)(\frac{n+1}{m+1})^2\,(\|P(n)x\|+\|{}_Q\mathcal{A}_m^n x\|)$$

for all $(m,n,x)\in\Delta\times\mathbb{R}^2$. Thus Definition 2.10 is satisfied for $N=v=2$ and $\mu=1$, hence system (1) is polynomially dichotomic.

On the other hand, if we suppose that system (1) is uniformly polynomially dichotomic, then according to Definition 2.6 there exist $N\geq 1$ and $v>1$ such that

$$(\frac{m+1}{n+1})^v \leq N\frac{m^2\,(m+1+m\sin(m\pi/2))}{n^2\,(n+1+n\sin(n\pi/2))}$$

for all $(m,n)\in\Delta$. In particular, for $m=4k+3$ and $n=4k+1,\ k\in\mathbb{N}$, we obtain that

$$(\frac{4k+4}{4k+2})^v \leq N\frac{(4k+3)^2}{(4k+1)^2\,(8k+3)},$$

which is false for $k\to\infty$. Hence system (1) is not uniformly polynomially dichotomic.

3. The main results

Theorem 3.1 *The linear discrete-time system (1) is uniformly polynomially dichotomic if and only if there exist $M\geq 1$, $\alpha>0$, $D\geq 1$ and $\beta>1$ such that*

(i) $\|{}_P\mathcal{A}_m^r x\|\leq M(\frac{m+1}{n+1})^\alpha\,\|{}_P\mathcal{A}_n^r x\|$,

(ii) $\|{}_Q\mathcal{A}_n^r x\|\leq M(\frac{m+1}{n+1})^\alpha\,\|{}_Q\mathcal{A}_m^r x\|$,

(iii) $\sum_{k=n}^{\infty}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}\,\|{}_P\mathcal{A}_k^n x\|+\sum_{k=n}^{m}\frac{(m+1)^\beta}{(k+1)^{\beta+1}}\,\|{}_Q\mathcal{A}_k^n x\|\ \leq\ D\,(\|P(n)x\|+\|{}_Q\mathcal{A}_m^n x\|)$, *for all* $(m,n,r,x)\in T\times X$.

Proof Necessity. By Remark 2.7, it is easy to see that the relations (i) and (ii) hold. Now we prove (iii). Let $(m,n,x)\in\Delta\times X$ and $\beta\in(1,v)$. We have

$$\begin{aligned}
&\sum_{k=n}^{\infty}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}\,\|{}_P\mathcal{A}_k^n x\|+\sum_{k=n}^{m}\frac{(m+1)^\beta}{(k+1)^{\beta+1}}\,\|{}_Q\mathcal{A}_k^n x\|\\
&\leq N\,\|P(n)x\|\sum_{k=n}^{\infty}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}(\frac{n+1}{k+1})^v+N\,\|{}_Q\mathcal{A}_m^n x\|\sum_{k=n}^{m}\frac{(m+1)^\beta}{(k+1)^{\beta+1}}(\frac{k+1}{m+1})^v\\
&\leq\frac{N\,\|P(n)x\|}{(n+1)^{\beta-v}}\sum_{k=n}^{\infty}(k+1)^{\beta-v-1}+\frac{N}{(m+1)^{v-\beta}}\,\|{}_Q\mathcal{A}_m^n x\|\sum_{k=n}^{m}(k+1)^{v-\beta-1}\\
&\leq\frac{N}{v-\beta}\,(\|P(n)x\|+\|{}_Q\mathcal{A}_m^n x\|)\leq D\,(\|P(n)x\|+\|{}_Q\mathcal{A}_m^n x\|)\,,
\end{aligned}$$

where $D=1+N/(v-\beta)$.

Sufficiency. Let $(m,n,x)\in\Delta\times X$. If $m>2n$, we have

$$\begin{aligned}
&(\frac{m+1}{n+1})^\beta\,(\|{}_P\mathcal{A}_m^n x\|+\|Q(n)x\|)\\
&\leq\frac{2}{m+1}\sum_{k=[m/2]}^{n}(\frac{m+1}{n+1})^\beta\,\|{}_P\mathcal{A}_m^n x\|+\sum_{k=n}^{2n}\frac{(m+1)^\beta}{(n+1)^{\beta+1}}\,\|Q(n)x\|\\
&\leq 2M\sum_{k=[m/2]}^{n}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}(\frac{m+1}{k+1})^{\alpha+\beta-1}\,\|{}_P\mathcal{A}_k^n x\|+
\end{aligned}$$

$$
\begin{aligned}
& M\sum_{k=n}^{2n}\frac{(m+1)^{\beta}}{(n+1)^{\beta+1}}\Big(\frac{k+1}{n+1}\Big)^{\alpha}\|_Q\mathcal{A}_k^n x\| \\
& \le 2M\sum_{k=[m/2]}^{n}\frac{(k+1)^{\beta-1}}{(n+1)^{\beta}}\Big(\frac{m+1}{[m/2]+1}\Big)^{\alpha+\beta-1}\|_P\mathcal{A}_k^n x\|+ \\
& M\sum_{k=n}^{2n}\frac{(m+1)^{\beta}}{(k+1)^{\beta+1}}\Big(\frac{k+1}{n+1}\Big)^{\alpha+\beta+1}\|_Q\mathcal{A}_k^n x\| \\
& \le 2^{\alpha+\beta+1}M\sum_{k=n}^{\infty}\frac{(k+1)^{\beta-1}}{(n+1)^{\beta}}\|_P\mathcal{A}_k^n x\|+ \\
& M\sum_{k=n}^{2n}\frac{(m+1)^{\beta}}{(k+1)^{\beta+1}}\Big(\frac{2n+1}{n+1}\Big)^{\alpha+\beta+1}\|_Q\mathcal{A}_k^n x\| \\
& \le 2^{\alpha+\beta+1}MD\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right).
\end{aligned}
$$

On the other hand, if $2n \ge m \ge n$, then

$$
\begin{aligned}
& \Big(\frac{m+1}{n+1}\Big)^{\beta}\left(\|_P\mathcal{A}_m^n x\|+\|Q(n)x\|\right) \le M\Big(\frac{m+1}{n+1}\Big)^{\alpha+\beta}\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right) \\
& \le M2^{\alpha+\beta}\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right).
\end{aligned}
$$

Finally, by Definition 2.6 we conclude that the system is uniformly polynomially dichotomic. □

A sufficient condition for uniform polynomial dichotomy of discrete linear-time systems is presented by the following theorem.

Theorem 3.2 *If there are two constants $D \ge 1$ and $\beta > 1$ such that*

$$
\sum_{k=n}^{m}\Big(\frac{m+1}{k+1}\Big)^{\beta}\left\|_P\mathcal{A}_m^k x\right\|+\sum_{k=n}^{m}\Big(\frac{k+1}{n+1}\Big)^{\beta}\left\|_Q\mathcal{A}_m^k x\right\| \le D\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right)
$$

for all $(m,n,x) \in \Delta\times X$, then the linear discrete-time system (1) is uniformly polynomially dichotomic.

Proof From the hypothesis it results that

$$
\begin{aligned}
& \Big(\frac{m+1}{n+1}\Big)^{\beta}\left(\|_P\mathcal{A}_m^n x\|+\|Q(n)x\|\right) \le \sum_{k=n}^{m}\Big(\frac{m+1}{k+1}\Big)^{\beta}\left\|_P\mathcal{A}_m^k x\right\|+\sum_{k=n}^{m}\Big(\frac{k+1}{n+1}\Big)^{\beta}\left\|_Q\mathcal{A}_m^k x\right\| \\
& \le D\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right)
\end{aligned}
$$

for all $(m,n,x) \in \Delta\times X$. Hence system (1) is uniformly polynomially dichotomic. □

Theorem 3.3 *The linear discrete-time system (1) is polynomially dichotomic if and only if there exist constants α,β with $\alpha > 2\beta \ge 0$ and $K \ge 1$ such that*

$$
\sum_{k=n}^{m}\Big(\frac{m+1}{k+1}\Big)^{\alpha}\Big(\frac{1}{k+1}\left\|_P\mathcal{A}_m^k x\right\|+\|_Q\mathcal{A}_k^n x\|\Big) \le K(m+1)^{\beta}\left(\|P(n)x\|+\|_Q\mathcal{A}_m^n x\|\right) \tag{2}
$$

for all $(m,n,x) \in \Delta\times X$.

Proof Necessity. If the linear discrete-time system (1) is polynomially dichotomic, then from

Remark 2.11 it follows that there are $N \geq 1$, $v > 0$ and $\mu \geq 0$ such that for all $\alpha > 0$ with $0 \leq \mu < \alpha \leq v \leq v + \mu < \alpha + 1$ we have that

$$\begin{aligned}
&\sum_{k=n}^{m} (\frac{m+1}{k+1})^{\alpha} (\frac{1}{k+1} \left\| {}_P\mathcal{A}_m^k x \right\| + \|{}_Q\mathcal{A}_k^n x\|) \\
&\leq N \|P(n)x\| \sum_{k=n}^{m} (\frac{m+1}{k+1})^{\alpha} \frac{(k+1)^{v+\mu-1}}{(m+1)^{v}} + N \|{}_Q\mathcal{A}_m^n x\| \sum_{k=n}^{m} \frac{(k+1)^{v+\mu-\alpha}}{(m+1)^{v-\alpha}} \\
&\leq N \|P(n)x\| (m+1)^{\alpha-v} \sum_{k=n}^{m} (k+1)^{v+\mu-\alpha-1} + N(m+1)^{\mu} \|{}_Q\mathcal{A}_m^n x\| \sum_{k=n}^{m} (\frac{k+1}{m+1})^{v-\alpha} \\
&\leq \frac{N}{v+\mu-\alpha}(m+1)^{\mu} \|P(n)x\| + N(m+1)^{\mu} \frac{m-n+1}{m+1} \|{}_Q\mathcal{A}_m^n x\| \\
&\leq K(m+1)^{\beta} (\|P(n)x\| + \|{}_Q\mathcal{A}_m^n x\|)
\end{aligned}$$

for all $(m, n, x) \in \Delta \times X$, where $K = \max\{N(v+\mu-\alpha)^{-1}, N\}$ and $\beta = \mu$.

Sufficiency. According to the relation (2), if we consider $k = n$, then

$$(\frac{m+1}{n+1})^{\alpha} \big(\frac{1}{n+1} \|{}_P\mathcal{A}_m^n x\| + \|Q(n)x\|\big) \leq K(m+1)^{\beta} (\|P(n)x\| + \|{}_Q\mathcal{A}_m^n x\|)$$

for all $(m, n, x) \in \Delta \times X$. From this and the hypothesis it follows that

$$\begin{aligned}
&(\frac{m+1}{n+1})^{\alpha-2\beta} (\|{}_P\mathcal{A}_m^n x\| + \|Q(n)x\|) \\
&\leq (\frac{m+1}{n+1})^{\alpha-\beta} \|{}_P\mathcal{A}_m^n x\| + (\frac{m+1}{n+1})^{\alpha-2\beta} \|Q(n)x\| \\
&\leq K(n+1)^{\beta+1} \|P(n)x\| + (\frac{m+1}{n+1})^{\alpha-2\beta} \frac{1}{m-n+1} \sum_{k=n}^{m} \|Q(n)x\| \\
&\leq K(n+1)^{\beta+1} \|P(n)x\| + K(\frac{m+1}{n+1})^{\alpha-2\beta} \sum_{k=n}^{m} \frac{(n+1)^{\alpha}}{(k+1)^{\alpha-\beta}} \|{}_Q\mathcal{A}_k^n x\| \\
&= K(n+1)^{\beta+1} \|P(n)x\| + K(\frac{n+1}{m+1})^{2\beta} \sum_{k=n}^{m} (k+1)^{\beta} (\frac{m+1}{k+1})^{\alpha} \|{}_Q\mathcal{A}_k^n x\| \\
&\leq K(n+1)^{\beta+1} \|P(n)x\| + K\frac{(n+1)^{2\beta}}{(m+1)^{\beta}} \sum_{k=n}^{m} (\frac{m+1}{k+1})^{\alpha} \|{}_Q\mathcal{A}_k^n x\| \\
&\leq K(n+1)^{\beta+1} \|P(n)x\| + K^2(n+1)^{2\beta} \|{}_Q\mathcal{A}_m^n x\| \\
&\leq K^2(n+1)^{2\beta+1} (\|P(n)x\| + \|{}_Q\mathcal{A}_m^n x\|)
\end{aligned}$$

for all $(m, n, x) \in \Delta \times X$. Thus Definition 2.10 is satisfied for $v = \alpha - 2\beta$, $\mu = 2\beta + 1$, and $N = K^2$. Hence the linear discrete-time system (1) is polynomially dichotomic. □

Next, we will present a sufficient condition for polynomial dichotomy of discrete linear-time systems.

Theorem 3.4 *If there are constants $\alpha > 0$, $\beta \geq 0$ and $K \geq 1$ such that*

$$\sum_{k=n}^{m} (\frac{k+1}{n+1})^{\alpha} \left(\|{}_P\mathcal{A}_k^n x\| + \left\|{}_Q\mathcal{A}_m^k x\right\|\right) \leq K(n+1)^{\beta} (\|P(n)x\| + \|{}_Q\mathcal{A}_m^n x\|)$$

for all $(m,n,x) \in \Delta \times X$, then the linear discrete-time system (1) is polynomially dichotomic.

Proof It is a simple exercise, for $k = m$. $\square$

Remark 3.5 The preceding theorems are variants for the case of polynomial dichotomy property of well-known theorems due to Popa et al. [11,12] for exponential stability and exponential dichotomy. They can also be considered as the variants for polynomial stability of theorems proved by Megan et al. [7], for the case of polynomial stability of variational nonautonomous difference equations.

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