

Multi-Symplectic Geometry and Explicit Multi-symplectic Method for Solving Zhiber-Shabat Equation

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Abstract In the paper, we derive a multi-symplectic Fourier pseudospectral method for Zhiber-Shabat equation. The Zhiber-Shabat equation, which describes many important physical phenomena, has been investigated widely in last several decades. The multi-symplectic geometry and multi-symplectic Fourier pseudospectral method for the Zhiber-Shabat equation is presented. The numerical experiments are given, showing that the multi-symplectic Fourier pseudospectral method is an efficient algorithm with excellent long-time numerical behaviors.

Keywords the Zhiber-Shabat equation; multi-symplectic theory; Fourier pseudospectral method; local conservation laws

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1. Introduction

In the paper, we consider the following Zhiber-Shabat equation

$$u_{xt} + \alpha e^{mu} + \beta e^{-mu} + \gamma e^{-2mu} = 0, \quad (1.1)$$

where α, β, γ are three non-zero real numbers, m is a positive integer. The equation (1.1) contains Liouville equation, Sinh-Gordon equation and Dodd-Bullough-Mikhailov equation.

When $\beta \neq 0, \gamma = 0$, equation (1.1) becomes Sinh-Gordon equation. When $\beta = 0, \gamma = 0$, equation (1.1) becomes Liouville equation. When $\beta = 0, \gamma \neq 0$, equation (1.1) becomes Dodd-Bullough-Mikhailov equation. Recently, Wazwaz [1] obtained some solitary wave and periodic wave solutions for special Dodd-Bullough-Mikhailov equation by using the tanh method. Fan and Hon [2] obtained some exact explicit parametric representations of the traveling solutions for the generalized Dodd-Bullough-Mikhailov equation by using the proposed extended tanh method. Tang [3] obtained some explicit parametric representations of the traveling solutions for special Dodd-Bullough-Mikhailov equation by using the method of bifurcation theory of dynamical systems. Rui [4] obtained some explicit parametric representations of the traveling solutions for equation (1.1) by using the method of bifurcation theory of dynamical systems. However, to our knowledge, no much work has been done to construct the numerical solutions for equation (1.1) till now.

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In the paper we consider numerical method to study Zhiber-Shabat equation (1.1). However, since the common numerical methods are not conservative, all the qualitative behavior such as norm conservation of the system has been lost in the discretization. In recent years, there has been an increasing emphasis on constructing numerical methods to preserve certain invariant quantities in the continuous dynamical systems. Feng [5] proposed a new approach to computing Hamiltonian systems from the view point of symplectic geometry in 1984. The disadvantage of this approach is that it is global. To overcome this limitation, Bridge and Reich [6,7] presented a multi-symplectic integrator based on a multi-symplectic structure of some Hamiltonian PDEs.

An outline of the paper is as follows. In Section 2, we present the multi-symplectic geometry for the Zhiber-Shabat equation. In Section 3, the multi-symplectic Hamilton formulations for equation (1.1) are established and three local conservation laws are obtained. Section 4 involves the construction of multi-symplectic Fourier pseudospectral method and error estimates of energy conservation law. In Section 5, numerical experiments are given.

2. Multi-symplectic geometry for the Zhiber-Shabat equation

In the section, we consider the multi-symplectic geometry for Zhiber-Shabat equation (1.1). The covariant configuration space for Zhiber-Shabat equation is $X \times U$. We define the first-order prolongation of $X \times U$ as

$$U^{(1)} = X \times U \times U_1,$$

where $X = (x, t)$ represents the space of independent variables, $U = (u)$ represents the space of dependent variables, $U_1 = (u_x, u_t)$ represents the space consisting of first-order partial derivatives.

Let $\varphi : X \rightarrow U$ be a section and we denote first prolongation of φ by

$$pr^1\varphi = (x, t, u, u_x, u_t).$$

The Lagrangian density for the Zhiber-Shabat equation (1.1) is

$$\mathcal{L}(pr^1(\varphi)) = L(pr^1(\varphi))dx \wedge dt, \quad (2.1)$$

where

$$L(pr^1(\varphi)) = -\frac{1}{2}u_t u_x + \frac{\alpha}{m}e^{mu} - \frac{\beta}{m}e^{-mu} - \frac{\gamma}{2m}e^{-2mu}. \quad (2.2)$$

We define the action functional by

$$S(\varphi) = \int_M \mathcal{L}(pr^1(\varphi)), \quad M \text{ is an open set of } X. \quad (2.3)$$

Let V be a vector field on $X \times U$ with the form

$$V = \tau(x, t) \frac{\partial}{\partial t} + \xi(x, t) \frac{\partial}{\partial x} + \alpha(x, t, v) \frac{\partial}{\partial v}.$$

The flow $\exp(\lambda V)$ of the vector V is a one-parameter transformation group of $X \times U$ and transforms a section $\varphi : M \rightarrow U$ to a family of sections $\tilde{\varphi} : \tilde{M} \rightarrow U$, which depend on the parameter λ .

By direct calculation, we can obtain the variation of the action functional (2.3) as follows

$$\begin{aligned} \delta S &= \frac{d}{d\lambda} \Big|_{\lambda=0} S(\tilde{\varphi}) = \frac{d}{d\lambda} \Big|_{\lambda=0} \int_{\tilde{M}} \left[-\frac{1}{2} \tilde{u}_t \tilde{u}_x + \frac{\alpha}{m} e^{m\tilde{u}} - \frac{\beta}{m} e^{-m\tilde{u}} - \frac{\gamma}{2m} e^{-2m\tilde{u}} \right] d\tilde{x} \wedge d\tilde{t} \\ &= \int_M I dx \wedge dt + \int_{\partial M} B, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} I &= (D_t(-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu}) + D_x(-\frac{1}{2} u_t^2)) V^t + \\ &\quad (D_t(-\frac{1}{2} u_x^2) + D_x(-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\beta}{2m} e^{-2mu})) V^x + \\ &\quad (u_{xt} + \alpha e^{mu} + \beta e^{-mu} + \gamma e^{-2mu}) V^u, \end{aligned} \tag{2.5}$$

$$\begin{aligned} B &= ((-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu}) dx + \frac{1}{2} u_t^2 dt) V^t + \\ &\quad (-\frac{1}{2} u_x^2 dx - (\frac{1}{2} u_t^2 + (-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu})) dt) V^x + \\ &\quad (-\frac{1}{2} u_t dt + \frac{1}{2} u_x dx) V^u. \end{aligned} \tag{2.6}$$

Chen [8] have proved that the variation τ yields the local energy conservation law

$$D_t(-\frac{1}{2} u_x^2) + D_x(-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\beta}{2m} e^{-2mu}) = 0, \tag{2.7}$$

the variation ξ yields the local momentum conservation law

$$D_t(-\frac{1}{2} u_x^2) + D_x(-\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\beta}{2m} e^{-2mu}) = 0, \tag{2.8}$$

the variation α yields the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt} \left(\frac{\partial L}{\partial u_t} \right) - \frac{d}{dx} \left(\frac{\partial L}{\partial u_x} \right) = u_{xt} + \alpha e^{mu} + \beta e^{-mu} + \gamma e^{-2mu} = 0. \tag{2.9}$$

If we define the Cartan form

$$\Theta_{\mathcal{L}} = -\frac{1}{2} u_t du \wedge dt + \frac{1}{2} u_x du \wedge dx + \left(\frac{1}{2} u_t u_x + \frac{\alpha}{m} e^{mu} - \frac{\beta}{m} e^{-2mu} - \frac{\gamma}{2m} e^{-2mu} \right) dx \wedge dt, \tag{2.10}$$

then multi-symplectic form is $\Omega_{\mathcal{L}} = d\Theta_{\mathcal{L}}$, and the multi-symplectic form formula is

$$\int_{\partial M} (pr^1 \varphi)^* (pr^1 V] \Omega_{\mathcal{L}} = 0.$$

3. Multi-symplectic structure for the Zhiber-Shabat equation

With the multi-symplectic theory [9–11], many partial differential equations can be written as multi-symplectic system

$$Mz_t + Kz_x = \nabla_z S(z), \quad z \in R^d, \quad (x, t) \in R^2, \tag{3.1}$$

where $M, K \in R^{d \times d}$ are the skew-symmetric matrices, $S : R^n \rightarrow R$ is a smooth function, $\nabla_z S(z)$ denotes the gradient of the function $S = S(z)$ with respect to variable z .

The multi-symplectic system (3.1) has multi-symplectic conservation law (MSCL)

$$\frac{\partial}{\partial t} w + \frac{\partial}{\partial x} k = 0, \tag{3.2}$$

where

$$w = dz \wedge M_+ dz, \quad k = dz \wedge K_+ dz. \tag{3.3}$$

The multi-symplectic system (3.1) satisfies the local energy conservation law (LECL)

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} F = 0, \tag{3.4}$$

where $E = S(z) + z_x^T K_+ z$, $F = -z_t^T K_+ z$, and local momentum conservation law (LMCL)

$$\frac{\partial}{\partial t} I + \frac{\partial}{\partial x} G = 0, \tag{3.5}$$

where

$$I = -z_x^T M_+ z, \quad G = S(z) + z_t^T M_+ z.$$

M_+ and K_+ satisfy

$$M = M_+ - M_+^T, \quad K = K_+ - K_+^T.$$

By introducing a pair of conjugate momenta $u_t = \psi$, $u_x = \varphi$, the Zhiber-Shabat equation (1.1) can be written as the first-order PDEs

$$\begin{cases} \frac{1}{2}\varphi_t + \frac{1}{2}\psi_x = -\alpha e^{mu} - \beta e^{-mu} - \gamma e^{-2mu}, \\ -\frac{1}{2}u_x = -\frac{1}{2}\varphi, \\ -\frac{1}{2}u_t = -\frac{1}{2}\psi. \end{cases} \tag{3.6}$$

If we define the state variable $z = (u, \psi, \varphi)^T$, the PDEs (3.6) is equivalent to the multi-symplectic system (3.1), where the skew-symmetric matrices are

$$M = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the smooth Hamiltonian is

$$S(z) = -\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu} + \frac{1}{2} \psi \varphi. \tag{3.7}$$

The system (3.6) satisfies the multi-symplectic conservation law (3.2), where

$$w = dz \wedge M_+ dz = \frac{1}{2} du \wedge d\varphi, \quad k = dz \wedge K_+ dz = \frac{1}{2} du \wedge d\psi.$$

The system (3.6) has a local energy conservation law (3.4), where

$$\begin{aligned} E = S(z) + z_x^T K_+ z &= -\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu} + \frac{1}{2} \psi \varphi - \frac{1}{4} \psi u_x + \frac{1}{4} u \psi_x, \\ F = -z_t^T K_+ z &= -\frac{1}{4} \psi u_t + \frac{1}{4} u \psi_t. \end{aligned}$$

The system (3.6) also has local momentum conservation law (3.5), where

$$\begin{aligned} I = -z_x^T M_+ z &= -\frac{1}{4} \varphi u_x + \frac{1}{4} u \varphi_x, \\ G = S(z) + z_t^T M_+ z &= -\frac{\alpha}{m} e^{mu} + \frac{\beta}{m} e^{-mu} + \frac{\gamma}{2m} e^{-2mu} + \frac{1}{2} \psi \varphi - \frac{1}{4} \varphi u_t + \frac{1}{4} u \varphi_t. \end{aligned}$$

4. Multi-symplectic Fourier pseudospectral method for the Zhiber-Shabat equation

In this section, we will apply Fourier pseudospectral method to the Zhiber-Shabat equation (1.1). Bridges and Reich [11] proposed multi-symplectic spectral discretization on Fourier space. Based on their theory, many authors proposed multi-symplectic Fourier pseudospectral method for Hamiltonian PDEs with periodic boundary conditions.

The discretization of the multi-symplectic system (3.1) and the multi-symplectic conservation law (3.2) can be solved numerically by

$$M\partial_t^{j,n} z_j^n + K\partial_x^{j,n} z_j^n = \nabla_z S(z_j^n), \tag{4.1}$$

$$\partial_t^{j,n} w_j^n + \partial_x^{j,n} k_j^n = 0, \tag{4.2}$$

where

$$w_j^n = \frac{1}{2} dz_j^n \wedge M dz_j^n, \quad k_j^n = \frac{1}{2} dz_j^n \wedge K dz_j^n,$$

$\partial_t^{j,n}$ and $\partial_x^{j,n}$ are discretizations of the partial derivatives ∂_t and ∂_x , respectively.

Definition 4.1 *The numerical scheme (4.1) of multi-symplectic system (3.1) is said to be multi-symplectic if equation (4.2) is a discrete conservation law of equation (3.1).*

We consider Zhiber-Shabat equation (1.1) with the periodic boundary condition $u(L_1, t) = u(L_2, t)$. We approximate $u(x, t)$ by $I_N u(x, t)$ which interpolates $u(x, t)$ at the following set of collocation points $x_j = L_1 + \frac{L}{N} j, j = 0, 1, 2, \dots, N - 1$, where N is an even number, L is the period. We approximate $u(x, t)$ by $I_N u(x, t)$

$$I_N u(x, t) = \sum_{j=0}^{N-1} u_j g_j(x), \tag{4.3}$$

where $u_j = u(x_j, t), g_j(x_k) = \delta_j^k$. $g_j(x)$ is a trigonometric polynomial of degree $\frac{N}{2}$ given explicitly by

$$g_j(x) = \frac{1}{N} \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{C_l} e^{il\mu(x-x_j)}, \tag{4.4}$$

where $C_l = 1(|l| \neq \frac{N}{2}), C_{\frac{N}{2}} = C_{-\frac{N}{2}} = 2, \mu = \frac{2\pi}{L}$. Substituting equation (4.4) into the expression of $I_N u(x, t)$ gives

$$I_N u(x, t) = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{C_l} e^{il\mu x} \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-il\mu x_j}, \tag{4.5}$$

$$u_i = I_N u(x_i, t) = \sum_{l=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{C_l} e^{il\mu x_i} \frac{1}{N} \sum_{j=0}^{N-1} u_j e^{-il\mu x_j}. \tag{4.6}$$

For u, v , we can define the bilinear form

$$(u, v)_N = h \sum_{j=0}^{N-1} u(x_j, t)v(x_j, t),$$

$$(I_N u(x_i, t), v(x_j, t))_N = (u(x_i, t), v(x_j, t))_N.$$

The crucial step here is to obtain values for the derivative $\frac{d^k}{dx^k} I_N(x, t)$ at the collocation points x_i in terms of the values u_i . The resulting expression at the points x_i is

$$\frac{d^k}{dx^k} I_N(x_i, t) = \frac{1}{N} \sum_{j=0}^{N-1} u_j \frac{d^k g_j(x_i)}{dx^k} = (D_k u)_i,$$

where D_k represents Fourier pseudospectral differential matrix with the elements

$$(D_1)_{i,j} \begin{cases} \frac{1}{2} \mu (-1)^{i+j} \cot(\mu \frac{x_i - x_j}{2}), & i \neq j, \\ 0, & i = j, \end{cases}$$

$$(D_2)_{i,j} \begin{cases} \frac{1}{2} \mu^2 (-1)^{i+j+1} \frac{1}{\sin^2(\mu \frac{x_i - x_j}{2})}, & i \neq j, \\ -\mu^2 \frac{N^2+2}{12}, & i = j. \end{cases}$$

Definition 4.2 Let $u = (u_0, u_1, u_2, \dots, u_{N-1})^T$ and $v = (v_0, v_1, v_2, \dots, v_{N-1})^T$. We define the Hadamard product of vectors by

$$u \circ v = (u_0 v_0, u_1 v_1, u_2 v_2, \dots, u_{N-1} v_{N-1})^T.$$

Theorem 4.3 Let $u = (u_0, u_1, u_2, \dots, u_{N-1})^T$ and $v = (v_0, v_1, v_2, \dots, v_{N-1})^T$, where $u_i = u(x_i, t), v_i = v(x_i, t), i = 0, 1, 2, \dots, N - 1$. We can obtain

$$D_N(u \circ v) = D_N u \circ v + D_N v \circ u.$$

From the above analysis, we can obtain a semi-discrete system for (3.6)

$$\begin{cases} \frac{1}{2} \frac{d\varphi_j}{dt} + \frac{1}{2} (D_1 \psi)_j = -\alpha e^{m u_j} - \beta e^{-m u_j} - \gamma e^{-2m u_j}, \\ -\frac{1}{2} (D_1 u)_j = -\frac{1}{2} \varphi_j, \\ -\frac{1}{2} \frac{d u_j}{dt} = -\frac{1}{2} \psi_j, \end{cases} \tag{4.7}$$

where $\varphi_j = \varphi(x_j, t), \psi_j = \psi(x_j, t), u_j = u(x_j, t)$.

Theorem 4.4 The scheme (4.7) satisfies semi-discrete multi-symplectic conservation laws,

$$\frac{d}{dt} w_i + \sum_{j=0}^{N-1} (D_1)_{i,j} k_{i,j} = 0, \quad i = 1, 2, \dots, N - 1, \tag{4.8}$$

and satisfies the total symplecticity in time $\frac{d}{dt} \sum_{i=0}^{N-1} w_i = 0$, where

$$\begin{cases} w_i = \frac{1}{2} du_i \wedge d\varphi_i, \\ k_{i,j} = \frac{1}{2} du_i \wedge d\psi_j. \end{cases}$$

In order to derive the algorithms conveniently, we give some operators definition. Define the finite difference operator

$$D_t u^j = \frac{u^{j+1} - u^j}{\Delta t}, \tag{4.9}$$

and averaging operator

$$A_t w^j = \frac{u^{j+1} + u^j}{2}. \quad (4.10)$$

From the above analysis, applying the midpoint symplectic integration in time, we can obtain a fully discrete system for (3.6)

$$\begin{cases} \frac{1}{2} D_t \varphi_j^n + \frac{1}{2} (D_1 A_t \psi^n)_j = -\alpha e^{m A_t u_j^n} - \beta e^{-m A_t u_j^n} - \gamma e^{-2m A_t u_j^n}, \\ -\frac{1}{2} (D_1 A_t u^n)_j = -\frac{1}{2} A_t \varphi_j^n, \\ -\frac{1}{2} D_t u_j^n = -\frac{1}{2} A_t \psi_j^n. \end{cases} \quad (4.11)$$

Theorem 4.5 *The scheme (4.11) satisfies fully discrete multi-symplectic conservation laws,*

$$D_t w_i^j + \sum_{k=0}^{N-1} (D_1)_{i,k} A_t k_{i,k}^j = 0, \quad i = 1, 2, \dots, N-1, \quad (4.12)$$

and satisfies the total symplecticity in time $\sum_{i=0}^{N-1} w_i^{j+1} = \sum_{i=0}^{N-1} w_i^j$, where

$$\begin{cases} w_i^j = \frac{1}{2} du_i^j \wedge d\varphi_i^j, \\ k_{i,k}^j = \frac{1}{2} du_i^k \wedge d\psi_j^k. \end{cases}$$

To evaluate the local conservation laws of energy, we use the discretizations of the form

$$(R_E)_j^n = D_t E_j^n + (D_1 A_t F^n)_j, \quad (4.13)$$

where

$$\begin{cases} E_j^n = S(Z_j^n) - \frac{1}{2} (Z_j^n)^T K (D_1 Z^n)_j, \\ A_t F_j^n = \frac{1}{2} (A_t Z_j^n)^T K (D_t Z_j^n), \end{cases}$$

$(R_E)_j^n$ is called the residual of LECL.

Under periodic boundary condition, we can define the discrete energy as follows

$$\varepsilon^n = \Delta x \sum_{j=0}^{n-1} E_j^n. \quad (4.14)$$

Theorem 4.6 *For the multi-symplectic Hamiltonian system (4.11), the residual of LECL of equation (3.6) satisfies the following expression*

$$\begin{cases} (R_E)_j^{n+\frac{1}{2}} = -\frac{\alpha}{m\Delta t} (e^{m u_j^{n+1}} - e^{m u_j^n}) + \frac{\alpha}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{m u_j^{n+1}} + e^{m u_j^n}) + \\ \frac{\beta}{m\Delta t} (e^{-m u_j^{n+1}} - e^{-m u_j^n}) + \frac{\beta}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{-m u_j^{n+1}} + e^{-m u_j^n}) + \\ \frac{\gamma}{2m\Delta t} (e^{-2m u_j^{n+1}} - e^{-2m u_j^n}) + \frac{\gamma}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{-2m u_j^{n+1}} + e^{-2m u_j^n}), \end{cases}$$

and global discrete energy of equation (4.11) satisfies the following expression

$$\left\{ \begin{aligned} \varepsilon^n &= \varepsilon^0 + \Delta x \sum_{n=0}^{n-1} \sum_{j=0}^{n-1} \left[-\frac{\alpha}{m} (e^{mu_j^{n+1}} - e^{mu_j^n}) + \frac{\alpha}{2} (u_j^{n+1} - u_j^n) (e^{mu_j^{n+1}} + e^{mu_j^n}) + \right. \\ &\frac{\beta}{m\gamma} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\beta}{2} (u_j^{n+1} - u_j^n) (e^{-mu_j^{n+1}} + e^{-mu_j^n}) + \\ &\left. \frac{\gamma}{2m} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\gamma}{2} (u_j^{n+1} - u_j^n) (e^{-2mu_j^{n+1}} + e^{-2mu_j^n}) \right]. \end{aligned} \right.$$

Proof We rewrite equation (4.11) in a compact form as

$$MD_t z_j^n + K \sum_{k=0}^{N-1} (D_1)_{j,k} A_t z_j^n = \nabla_z S(A_t z_j^n). \tag{4.15}$$

It follows from Wang [10] that the residual of LECL of equation (4.15) satisfies

$$(R_E)_j^{n+\frac{1}{2}} = D_t S(Z_j^n) - (D_t Z_j^n, \nabla_z S(A_t Z_j^n)). \tag{4.16}$$

Substituting equation (3.7) into equation (4.16), we can derive that the residual of LECL of equation (4.11) satisfies the following expression

$$\left\{ \begin{aligned} (R_E)_j^{n+\frac{1}{2}} &= -\frac{\alpha}{m\Delta t} (e^{mu_j^{n+1}} - e^{mu_j^n}) + \frac{\alpha}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{mu_j^{n+1}} + e^{mu_j^n}) + \\ &\frac{\beta}{m\Delta t} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\beta}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{-mu_j^{n+1}} + e^{-mu_j^n}) + \\ &\frac{\gamma}{2m\Delta t} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\gamma}{2\Delta t} (u_j^{n+1} - u_j^n) (e^{-2mu_j^{n+1}} + e^{-2mu_j^n}). \end{aligned} \right.$$

By Wang [10], the discrete energy of equation (4.11) satisfies

$$\varepsilon^n = \varepsilon^0 + \Delta x \sum_{n=0}^{n-1} \sum_{j=0}^{n-1} [S(Z_j^{n+1}) - S(Z_j^n) - (Z_j^{n+1} - Z_j^n)^T S(Z_j^{A_t^n})]. \tag{4.17}$$

Substituting equation (3.7) into equation (4.17), we can derive that the discrete energy of equation (4.11) satisfies the following expression

$$\left\{ \begin{aligned} \varepsilon^n &= \varepsilon^0 + \Delta x \sum_{n=0}^{n-1} \sum_{j=0}^{n-1} \left[-\frac{\alpha}{m} (e^{mu_j^{n+1}} - e^{mu_j^n}) + \frac{\alpha}{2} (u_j^{n+1} - u_j^n) (e^{mu_j^{n+1}} + e^{mu_j^n}) + \right. \\ &\frac{\beta}{m\gamma} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\beta}{2} (u_j^{n+1} - u_j^n) (e^{-mu_j^{n+1}} + e^{-mu_j^n}) + \\ &\left. \frac{\gamma}{2m} (e^{-2mu_j^{n+1}} - e^{-2mu_j^n}) + \frac{\gamma}{2} (u_j^{n+1} - u_j^n) (e^{-2mu_j^{n+1}} + e^{-2mu_j^n}) \right]. \end{aligned} \right.$$

Assume Δt is sufficiently small, then there exist two constants C_1, C_2 independent of Δt and Δx , such that $|(R_E)_j^{n+\frac{1}{2}}| \leq C_1 \Delta t^2, |\varepsilon^{n+1} - \varepsilon^n| \leq C_2 \Delta t^3$.

5. Numerical experiments

In this section, the numerical experiments are presented to illustrate the theoretical results in the previous sections by using the multi-symplectic Fourier pseudospectral method.

Example 5.1 When $m = 1$, the Zhiber-Shabat equation (1.1) has an analytic soliton solution

$$u = \ln \left[\frac{3\gamma(1 - \sqrt{3})}{2\beta} - \frac{9\gamma}{2\beta} \operatorname{sech}^2 \left(\frac{9\alpha\gamma}{4\beta\omega} x - \omega t \right) \right]. \tag{5.1}$$

We consider the problem

$$\begin{cases} u_{xt} + \alpha e^u + \beta e^{-u} + \gamma e^{-2u} = 0, \\ u_0 = \ln\left[\frac{3\gamma(1-\sqrt{3})}{2\beta} - \frac{9\gamma}{2\beta} \sec h^2\left(\frac{9\alpha\gamma}{4\beta\omega}x\right)\right], \\ u(L_1, t) = u(L_2, t), \end{cases} \tag{5.2}$$

where $\alpha = 1, \beta = 1, \gamma = -1, \omega = 1$.

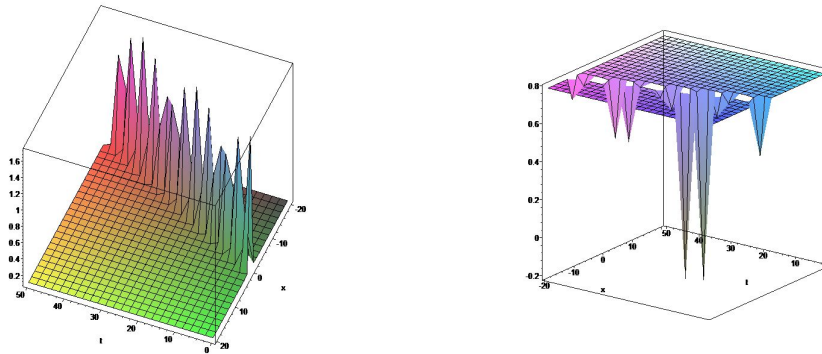


Figure 1 The wave form of the numerical solution Figure 2 The wave form of the numerical solution with $x \in [-20, 20], t \in [0, 50]$

We simulate the solution (5.2) with the initial condition and boundary condition using the multi-symplectic Fourier pseudospectral method (4.11) with $x \in [-20, 20], t \in [0, 50], \Delta t = 0.01$, and $\Delta x = 0.01$. Figure 1 shows the wave form of the numerical solution, and shows that the waves emerge without any changes in their shapes, which indicates that the proposed method simulated the solitary wave well.

Example 5.2 When $m = 1$, the Zhiber-Shabat equation (1.1) has an analytic soliton solution

$$u = \ln\left[\frac{3\gamma(1-\sqrt{3})}{2\beta} + \frac{9\gamma}{2\beta} \csc h^2\left(\frac{9\alpha\gamma}{4\beta\omega}x - \omega t\right)\right]. \tag{5.3}$$

We consider the problem

$$\begin{cases} u_{xt} + \alpha e^u + \beta e^{-u} + \gamma e^{-2u} = 0, \\ u_0 = \ln\left[\frac{3\gamma(1-\sqrt{3})}{2\beta} - \frac{9\gamma}{2\beta} \csc h^2\left(\frac{9\alpha\gamma}{4\beta\omega}x\right)\right], \\ u(L_1, t) = u(L_2, t), \end{cases} \tag{5.4}$$

where $\alpha = 1, \beta = 1, \gamma = -2, \omega = 1$. We simulate the solution (5.4) with the initial conditions and boundary conditions using the multi-symplectic Fourier pseudospectral method (4.11) with $x \in [-20, 20], t \in [0, 50], \Delta t = 0.01$, and $\Delta x = 0.01$. Figure 2 shows the wave form of the numerical solution.

From the results above, we find that the wave-forms keep their amplitudes and velocities invariable throughout the processes of the simulations, which implies that the multi-symplectic Fourier pseudospectral method (4.11) can preserve the local properties of the periodic wave

solution perfectly. The numerical error of the multi-symplectic Fourier pseudospectral method is more regular.

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