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Spherically Symmetric Finsler Metrics with Isotropic E-Curvature

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Abstract In this paper, we obtain a differential equation which characterizes a spherically symmetric Finsler metric with isotropic E-curvature and discuss a particular case using the differential equation we get.

Keywords Finsler metric; spherically symmetric; E-curvature; isotropic E-curvature; projectively flat

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1. Introduction

Let F be a Finsler metric on manifold M. The geodesics of F are characterized locally by the equation $\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+2G^i(x,\frac{\mathrm{d}x}{\mathrm{d}t})=0$, where G^i are coefficients of a spray defined on M denoted by $G(x,y)=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$. A Finsler metric F is called Berwald metric if $G^i=\frac{1}{2}\Gamma^i_{jk}y^jy^k$ are quadratic in $y\in T_xM$ for any $x\in M$. Taking a trace of Berwald curvature yields mean Berwald curvature or E-curvature. The E-curvature is an important quantity defined using the spray of F. It is a kind of non-Riemann quantities [1,2]. Chen and Shen obtained an equivalent condition for a Randers metric to be E-curvature and S-curvature [3]. Then, they studied the relationship between isotropic E-curvature and relatively isotropic Landsberg curvature on a Douglas manifold [4]. Lungu got a condition for Randers spaces to be simultaneously with scalar flag curvature and with constant E-curvature [5]. For (α,β)-metrics in the form $F=\alpha+\varepsilon\beta+k\frac{\alpha^2}{\beta}$, D. Tang obtained an equivalent condition about E-curvature and S-curvature [6]. Tayebi, Nankali and Peyghan proved that every m-root Cartan space of E-curvature reduces to weakly Berwald spaces [7].

On the other hand, a Finsler metric F is said to be spherically symmetric (orthogonally invariant in an alternative terminology in [8]) if F satisfies

$$F(Ax,Ay)=F(x,y) \tag{1.1}$$

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for all $A \in O(n)$, equivalently, if the orthogonal group $O(n)$ acts as isometrics of F. Such metrics were first introduced by Rutz [9]. Spherically symmetric Finsler metrics are the simplest and most important general (α,β)-metrics [10]. It was pointed out in [8] that a Finsler metric F on $\mathbb{B}^n(\mu)$ is a spherically symmetric if and only if there is a function $\phi : [0,\mu) \times \mathbb{R} \to \mathbb{R}$ such that

$$F(x,y) = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big) \tag{1.2}$$

where $(x,y) \in T\mathbb{R}^n(\mu)\backslash\{0\}$, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the standard Euclidean norm and inner product, respectively. The geodesic coefficients of a spherically symmetric Finsler metric $F(x,y) = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ are given by

$$G^i = uPy^i + u^2Qx^i \tag{1.3}$$

where

$$Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \quad u := |y|, \quad r := |x|, \quad s := \frac{\langle x,y\rangle}{|y|} \tag{1.4}$$

and

$$P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}[s\phi + (r^2 - s^2)\phi_s]. \tag{1.5}$$

Many known Finsler metrics are spherically symmetric [8,11]. Such metrics have many good properties [12,16]. Mo, Zhou and Zhu classified the projective spherically symmetric Finsler metrics with constant flag curvature in [13–15]. In [8], Huang and Mo found a PDE to describe a spherically symmetric Finsler metric to be projectively flat. Projectively flat is an important definition in Finsler geometry. Hilbert's 4th Problem is to characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are geodesics [17,18]. Regular distance functions with straight geodesics are projectively flat Finsler metrics. If a Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat, the geodesics of F are straight lines if and only if the spray coefficients of F satisfy $G^i = Py^i$, namely Q in (1.4) is zero.

In this paper, we will study E-curvature of spherically symmetric Finsler metrics. Firstly, we obtain a differential equation which characterizes a spherically symmetric Finsler metric with isotropic E-curvature. Then, we discuss a projectively flat spherically symmetric Finsler metric with isotropic E-curvature using the differential equation we get. We introduce the notation

$$u := |y|, \quad r := |x|, \quad s := \frac{\langle x,y\rangle}{|y|}, \quad \mathbb{B}^n(\mu) := \{x \in \mathbb{R}^n(\mu); |x| < \mu\}.$$

In Section 4, we prove the following

Theorem 1.1 *On $\mathbb{B}^n(\mu)$, a spherically symmetric Finsler metric $F(x,y) = u\phi(r,s)$ has isotropic E-curvature if and only if*

$$(P - sP_s)(n+1) + (r^2 - s^2)(Q_s - sQ_{ss}) = (n+1)(\phi - s\phi_s)c(x)$$

where $c = c(x)$ is a scalar function on M.

In Section 5, we prove

Corollary 1.2 *Let $F(x,y) = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a projectively flat spherically symmetric Finsler metric with isotropic E-curvature on $\mathbb{B}^n(\mu)$. Then $F = F(x,y)$ is a Randers metric.*

2. Preliminaries

In this section, we will give some definitions that will be used in the proof of our main results. Let M be a manifold and let $TM = \cup_{x\in M}T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. A Finsler metric is a Riemannian metric without quadratic restriction. Precisely, a function $F(x,y)$ on TM is called a Finsler metric on a manifold M if it has the following properties:

(i) Regularity: $F(x,y)$ is C^∞ on $TM\backslash\{0\}$;

(ii) Positive homogeneity: $F(x,\lambda y) = \lambda F(x,y),\ \forall\lambda > 0, y \in T_xM$;

(iii) Strong convexity: The $n\times n$ matrix $g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}(y \neq 0)$ is positive definite.

For a Finsler metric $F = F(x,y)$ on a manifold M, the the spray coefficients $G^i = G^i(x,y)$ are defined by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\}$$

where $g_{ij} = \frac{1}{2}[F^2]_{y^iy^i}$ and $(g^{ij}) = (g_{ij})^{-1}$. The Riemann curvature is a family of linear maps $\mathbf{R}_y = R^i{}_k\frac{\partial}{\partial x^i}\bigotimes \mathrm{d}x^k : T_xM \to T_xM$ defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, let $\sigma = \sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\sigma'(0) = y$. Set

$$\mathbf{S}(x,y) := \frac{\mathrm{d}}{\mathrm{d}t}[\tau(\sigma(t),\sigma'(t))]|_{t=0}.$$

$\mathbf{S}$ is called the S-curvature. The S-curvature is introduced by Shen when he studied volume comparison in Riemann-Finsler geometry [1,5]. It is a scalar function $\mathbf{S} : TM \to R$ defined by

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial(\ln\sigma_F)}{\partial y^m}$$

where

$$\sigma_F = \frac{\mathrm{Vol}(\mathbb{B}^m)}{\mathrm{Vol}\{(y^i)\in\mathbb{R}^m|F(x,y^i\frac{\partial}{\partial x^i}) < 1\}}.$$

A Finsler metric F on an n-dimensional manifold M is said to have almost isotropic S-curvature if there is a scalar function $c = c(x)$ on M such that

$$\mathbf{S} = (n+1)\{cF + \eta\}$$

where $\eta = \eta_i(x)y^i$ is a closed 1-form. F is said to have isotropic S-curvature if $\eta = 0$. F is said to have constant S-curvature if $\eta = 0$ and c=constant. There is another quantity associated with S-curvature. The E-tensor $\xi := E_{ij}\mathrm{d}x^i\bigotimes\mathrm{d}x^j$ is defined by

$$E_{ij} := \frac{1}{2}\mathbf{S}_{y^iy^j}(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}(\frac{\partial G^m}{\partial y^m}) \tag{2.1}$$

in local coordinates $x^1,\ldots,x^n$ and $y = \sum_i y^i\frac{\partial}{\partial x^i}$. F is said to have isotropic E-curvature if there is a scalar function $c = c(x)$ on M such that

$$\mathbf{E} = \frac{1}{2}(n+1)cF^{-1}\mathbf{h}. \tag{2.2}$$

Here $\mathbf{h}$ is a family of bilinear forms $\mathbf{h}_y = h_{ij}(x,y)\mathrm{d}x^i \bigotimes \mathrm{d}x^j$ on T_xM, which are defined by $h_{ij} := FF_{y^iy^j}$.

3. *E*-curvature of spherically symmetric Finsler metrics

For a spherically symmetric Finsler metric $F(x,y) = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$, we have already known its geodesic spray coefficients can be written as (1.3). Plugging it into the definition of E-tensor (2.1), we obtain the following proposition.

Proposition 3.1 *Let $F(x,y) = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\mu) \subseteq \mathbb{R}^n$. Let $u := |y|$, $r := |x|$, $s := \frac{\langle x,y\rangle}{|y|}$. Then the E-curvature of F is given by*

$$\begin{aligned}E_{ij} =&[\frac{P_{ss}}{u}(n+1) + 2(\frac{Q_s}{u} - \frac{s}{u}Q_{ss}) - \frac{Q_{sss}}{u}(s^2-r^2)]x^ix^j+\\ &[-\frac{s}{u^2}(n+1)P_{ss} - 2s\frac{Q_s}{u^2} + 2\frac{s^2}{u^2}Q_{ss} + \frac{s}{u^2}Q_{sss}(s^2-r^2)]x^iy^j+\\ &[-\frac{s}{u^2}(n+1)P_{ss} - 2s\frac{Q_s}{u^2} + 2\frac{s^2}{u^2}Q_{ss} + \frac{s}{u^2}Q_{sss}(s^2-r^2)]x^jy^i+\\ &[-(n+1)\frac{P}{u^3} + (n+1)\frac{s}{u^3}P_s + (n+1)\frac{s^2}{u^3}P_{ss} + (3s^2-r^2)\frac{Q_s}{u^3}+\\ &\frac{Q_{ss}}{u^3}(r^2-3s^2)s + \frac{s^2Q_{sss}}{u^3}(r^2-s^2)]y^iy^j+\\ &(\frac{P}{u} - \frac{s}{u}P_s)(n+1) + (\frac{Q_s}{u} - \frac{s}{u}Q_{ss})r^2 + (\frac{s^3}{u}Q_{ss} - \frac{s^2}{u}Q_s).\end{aligned} \tag{3.1}$$

Proof Let F be a spherically symmetric Finsler metric. By a direct computation, we know

$$u_{y^i} = \frac{y^i}{u}, \tag{3.2}$$

$$u_{y^iy^j} = \frac{u^2\delta_{ij} - y^iy^j}{u^3}, \tag{3.3}$$

$$s_{y^i} = \frac{ux^i - sy^i}{u^2}, \tag{3.4}$$

$$s_{y^iy^j} = \frac{3sy^iy^j - ux^iy^j - uy^ix^j - su^2\delta_{ij}}{u^4}. \tag{3.5}$$

From (1.3), (3.2) and (3.4) we have

$$\begin{aligned}\sum_m \frac{\partial G^m}{\partial y^m} &= \sum_m u_{y^m}Py^m + uP_s\sum_m s_{y^m}y^m + nuP + 2Q\langle x,y\rangle + u^2Q_s\sum_m s_{y^m}x^m\\ &= u[(n+1)P + 2sQ + (r^2-s^2)Q_s].\end{aligned} \tag{3.6}$$

By (3.6), we get

$$\begin{aligned}\frac{\partial}{\partial y^i}(\sum_m \frac{\partial G^m}{\partial y^m}) =&(n+1)u_{y^i}P + (n+1)uP_ss_{y^i} + 2Qu + 2sQ_ss_{y^i}u + 2sQu_{y^i} - 2ss_{y^i}uQ_s+\\ &(r^2-s^2)u_{y^i}Q_s + (r^2-s^2)us_{y^i}Q_{ss}\end{aligned} \tag{3.7}$$

where we have used $\frac{\partial r}{\partial y^i}=0$. Differentiating (3.7), we have

$$\begin{aligned}&\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}(\sum_m \frac{\partial G^m}{\partial y^m})\\ &=[(n+1)uP_{ss}+(r^2-s^2)uQ_{sss}-2suQ_{ss}]s_{y^i}s_{y^j}+\\ &\quad[(r^2-s^2)uQ_{ss}+(n+1)uP_s]s_{y^iy^j}+[(n+1)P+2sQ+(r^2-s^2)Q_s]u_{y^iy^j}+\\ &\quad[(n+1)P_s+2Q+(r^2-s^2)Q_{ss}]u_{y^i}s_{y^j}+[(n+1)P_s+(r^2-s^2)Q_{ss}]u_{y^j}s_{y^i}+\\ &\quad 2Q_s s_{y^j}u+2Qu_{y^j}. \end{aligned} \tag{3.8}$$

Substituting (3.2)–(3.5) into (3.8), we conclude the proof. □

4. Isotropic *E*-curvature of spherically symmetric Finsler metrics

In this section, we are going to discuss necessary and sufficient condition for a spherically symmetric Finsler metric with isotropic E-curvature. Let $F(x,y)=u\phi(r,s)$ be a spherically symmetric Finsler metric on $\mathbb{B}^n(\mu)\subseteq\mathbb{R}^n$, where $u=|y|$, $r=|x|$, $s=\frac{\langle x,y\rangle}{|y|}$. By a direct computation, we know

$$F_{y^i}=u_{y^i}\phi+u\phi_s s_{y^i}. \tag{4.1}$$

Differentiating (4.1), we get

$$F_{y^iy^j}=u_{y^iy^j}\phi+(u_{y^j}s_{y^i}+u_{y^i}s_{y^j})\phi_s+us_{y^i}s_{y^j}\phi_{ss}+us_{y^iy^j}\phi_s. \tag{4.2}$$

Substituting (3.2)–(3.5) into (4.2), we obtain

$$F_{y^iy^j}=\frac{1}{u}(\phi-\phi_s)\delta_{ij}+\frac{1}{u}\phi_{ss}x^ix^j-\frac{1}{u^2}\phi_{ss}(x^iy^j+x^jy^i)s-\frac{1}{u^3}(\phi-s\phi_s-s^2\phi_{ss})y^iy^j. \tag{4.3}$$

Suppose that $F(x,y)=u\phi(r,s)$ has isotropic E-curvature, (2.2) holds. Thus

$$\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}(\sum_m\frac{\partial G^m}{\partial y^m})=(n+1)cF_{y^iy^j} \tag{4.4}$$

where $c=c(x)$ is a scalar function on M. Putting together (3.8), (4.3) and (4.4), the following equations can be obtained,

$$P_{ss}(n+1)+2(Q_s-sQ_{ss})+(r^2-s^2)Q_{sss}=\phi_{ss}c(n+1), \tag{4.5}$$

$$(P-sP_s)(n+1)+(r^2-s^2)(Q_s-sQ_{ss})=(n+1)(\phi-s\phi_s)c, \tag{4.6}$$

$$\begin{aligned}&P(n+1)-P_s(n+1)s-P_{ss}(n+1)s^2+Q_s(r^2-3s^2)-Q_{ss}(r^2-3s^2)s-s^2Q_{sss}(r^2-s^2)\\ &\quad=\phi(n+1)c-\phi_s(n+1)sc-\phi_{ss}(n+1)s^2c.\end{aligned} \tag{4.7}$$

Observing (4.5)–(4.7), we can easily deduce that differentiating (4.6), (4.5) is obtained and (4.7) = (4.6) − (4.5) × s^2. The equations above can attribute to (4.6). Moreover, $c=c(x)$ is a function of $r:=|x|$. Conversely, if F satisfies (4.6), then (4.4) holds, namely $F(x,y)=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$ has isotropic E-curvature. This proves Theorem 1.1. □

5. Projectively flat spherically symmetric Finsler metrics with isotropic *E*-curvature

Suppose that a projectively flat spherically symmetric Finsler metric $F(x,y)=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$ has isotropic E-curvature, then $Q=0$, (4.6) can be written as:

$$P-sP_s=(\phi-s\phi_s)c(x). \tag{5.1}$$

By solving (5.1), we get

$$P=\sigma(r)s+c(x)\phi \tag{5.2}$$

where $\sigma=\sigma(r)$ is a scalar function on M. From (1.5), we know

$$P=\frac{1}{2r\phi}(s\phi_r+r\phi_s). \tag{5.3}$$

Thus if the projectively flat spherically symmetric Finsler metric $F(x,y)=u\phi(r,s)$ has isotropic E-curvature, ϕ satisfies

$$\frac{1}{2r\phi}(s\phi_r+r\phi_s)=\sigma(r)s+c(x)\phi, \tag{5.4}$$

$$-\phi_r+s\phi_{rs}+r\phi_{ss}=0. \tag{5.5}$$

Differentiating (5.4) with s, we get

$$\phi_r+s\phi_{rs}+r\phi_{ss}=2r\phi\sigma(r)+2r\phi_s\sigma(r)s+4r\phi\phi_s c(x). \tag{5.6}$$

Plugging (5.5) into (5.6), we know

$$\phi_r=r\phi\sigma(r)+r\phi_s\sigma(r)s+2r\phi\phi_s c(x). \tag{5.7}$$

By (5.7) $\times\, s-$ (5.4), we have

$$(\sigma(r)s^2+2\phi c(x)s+1)\phi_s=\phi\sigma(r)s+2\phi^2c(x). \tag{5.8}$$

For a fixed r, (5.8) is equivalent to the following equation

$$M\mathrm{d}\phi+N\mathrm{d}s=0 \tag{5.9}$$

where $M=\sigma(r)s^2+2\phi c(x)s+1$ and $N=-\phi\sigma(r)s-2\phi^2c(x)$. By a direct computation,

$$\frac{\partial M}{\partial s}=2\sigma(r)s+2\phi c(x),\quad \frac{\partial N}{\partial \phi}=-\sigma(r)s-4\phi c(x).$$

Thus

$$\frac{1}{N}(\frac{\partial M}{\partial s}-\frac{\partial N}{\partial \phi})=-\frac{3}{\phi}. \tag{5.10}$$

By (5.10), the integrating factor $u(\phi)$ of (5.9) can be easily obtained,

$$u(\phi)=\frac{1}{\phi^3}. \tag{5.11}$$

By (5.9) $\times\, u(\phi)$, we have

$$\frac{1}{\phi^3}(\sigma(r)s^2+2\phi c(x)s+1)\mathrm{d}\phi-\frac{1}{\phi^3}(\phi\sigma(r)s+2\phi^2c(x))\mathrm{d}s=0.$$

Then $\mathrm{d}(\frac{1}{\phi^2}\sigma(r)s^2+\frac{4}{\phi}c(x)s+\frac{1}{2\phi^2})=0$. Suppose that $X(r)=\frac{1}{\phi^2}\sigma(r)s^2+\frac{4}{\phi}c(x)s+\frac{1}{2\phi^2}$, we obtain

$$\phi^2X(r)-4\phi c(x)s-(\sigma(r)s^2+\frac{1}{2})=0.$$

Thus

$$\phi(t,s)=\frac{2c(x)s\pm\sqrt{(4c(x)^2+X(r)\sigma(r))s^2+\frac{1}{2}X(r)}}{X(r)}.$$

Due to $F(x,y)\geq 0$, we have

$$\phi(t,s)=\frac{2c(x)s+\sqrt{(4c(x)^2+X(r)\sigma(r))s^2+\frac{1}{2}X(r)}}{X(r)}.$$

It follows that

$$F=\frac{2c(x)\langle x,y\rangle+\sqrt{(4c(x)^2+X(r)\sigma(r))\langle x,y\rangle^2+\frac{1}{2}X(r)|y|^2}}{X(r)}. \tag{5.12}$$

This means F is a Randers metric. Conversely, if F satisfies (5.12), then (4.4) holds, namely $F(x,y)=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$ has isotropic E-curvature. The proof of Corollary is completed. □

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