Journal of Mathematical Research with Applications
Sept., 2015, Vol. 35, No. 5, pp. 581–590
DOI:10.3770/j.issn:2095-2651.2015.05.011
Http://jmre.dlut.edu.cn

A Method for Estimating Dispersion Effects in Unreplicated Two-level Factorial Experiments

Liu YANG
Department of Applied Mathematics, Shanxi University of Finance and Economics, Shanxi 030006*, P. R. China*

Abstract Since the paper of Box and Meyer who first considered the identification and estimation of dispersion effects from unreplicated factorial experiments, various different methods (both iterative and non-iterative) have been proposed for estimating dispersion effects. An overview of various methods was given by Brenneman and Nair and they showed that the modified Harvey (MH) method is better than other methods. For a log-linear or multiplicative model, a non-iterative estimation method of dispersion effects based on residuals averaging from multiple location effect models is proposed in model selection stage, which has been shown smaller Mean Square Errors (MSE) than the MH method in majority of simulated models. And it can apply to the situations with zero or small absolute residuals, but the MH method will be failure. The properties of this estimator are also considered. A real example is used to illustrate the results.

Keywords dispersion effects; unreplicated factorial experiments; mean square errors

MR(2010) Subject Classification 62K15; 62K25

1. Introduction

As emphasized by [1], variation reduction is an important means for quality-improvement of product or production process. Thus, analysis of dispersion effects has led to a renewed attention. A common and popular approach is to conduct two-level (fractional) factorial experiments to estimate dispersion effects and identify active factors by appropriate test methods, such as half-normal plot, lenth method and so on. Then firstly, the levels of active dispersion factors are adjusted to obtain the minimum variance; secondly, the factors with active location effects are used to adjust product or process on target [2].

Traditional methods for identifying both location and dispersion effects typically require replication. Then, sample mean and sample variance are modeled to active location and dispersion factors [3,4]. Excessive number of experimental runs and expensive cost may be impossible in practical manufacturing process. Therefore, authors have proposed some methods for estimating and identifying location and dispersion effects from unreplicated factorial experiments. The problem of identifying location effects from unreplicated experiments has been given a review of various methods by [5]. However, sample variance cannot be obtained from unreplicated factorial experiments and cannot directly model to dispersion factors. A straightforward idea is

Received May 28, 2014; Accepted July 8, 2015
E-mail address: yang_liu@sxu.edu.cn

to model based on residuals. Thus, several non-iterative methods have been proposed to study dispersion effects in unreplicated factorial experiments under the assumption of effect sparsity [6–17]. In particular, [8] systematically studied the properties of several commonly used methods, and showed that the modified Harvey (MH) method appears to perform better than other methods under Mean Square Errors (MSE) criterion via simulated experiments. In an identified model, two iterative methods (maximum likelihood estimation (MLE) and restricted maximum likelihood (REML)) are also considered to estimate dispersion effects [6,8].

In this article, for a log-linear or multiplicative model, a non-iterative estimation method of dispersion effects based on residuals averaging from multiple location effect models is proposed, which has been shown smaller MSE than the MH method proposed by [8] in majority of simulated experiments. We refer to this method as the residuals averaging method, denoted the RA (residuals averaging) method, which can be suitable in situations with zero or small absolute residuals, but the MH method will be failure. We also consider the properties of this estimator in theory. A real example illustrates the results.

The article is organized as follows. Section 2 provides models and corresponding notations. In Section 3, we introduce the RA method in detail and some properties of the RA estimator. In Section 4, we reanalyze the welding experiment previously reported in [18]. The performances of the RA and MH methods are studied through simulated experiments in Section 5.

2. Models and notations

Let $y = (y_1, y_2, \ldots, y_n)'$ be a response vector, X be an $n \times m$ orthogonal design matrix with the first column X_1 of +1s and the remaining columns $X_2, \ldots, X_m$ corresponding to a two-level factorial or fractional factorial experiment. X_k $(k > 1)$ is a vector of +1s and −1s corresponding to the high and low levels of factor k. $S(k+)$ $(S(k-))$ is a set of rows of the + (−) level of factor k. The closure of a set $\mathcal{S}$ is denoted by $\overline{\mathcal{S}}$. Moreover, given a set $\mathcal{S}$, define the set $\mathcal{S}_{-k}$ by $\mathcal{S}_{-k} = \mathcal{S} - \{k\}$.

For $k_1, k_2, k_3 \in \{1, 2, \ldots, n\}$, we write $k_1 = k_2 \circ k_3$ if X_{k_1} can be obtained by elementwise multiplication of X_{k_2} and X_{k_3}, for example, the interaction term $AB = A \circ B$.

Let the location model $L = \{I, l_1, \ldots, l_{p-1}\}$ and for $k \in \{2, \ldots, n\}$, let $L_k = \{k, k \circ l_1, \ldots, k \circ l_{p-1}\}$. For effect k, let L_k^E be the expanded location model, which is the union of the elements of L, the factor k, and all the interactions between the factor k and the elements in L, i.e., $L_k^E = L \bigcup L_k$. I denotes the intercept.

In unreplicated 2^{k-p} designs, there are only $n = 2^{k-p}$ runs, while the total number of possible location parameters is n. If all location effects are active, there are no degrees of freedom to estimate dispersion effects. Thus, effect sparsity principle suggests that only a few location effects are non-negligible. Therefore, we can use normal probability technique, half-normal probability technique or lenth method to identify active location effects. Then, the remaining contrasts can be pooled to estimate dispersion effects. Throughout, we assume that the true location model is correctly fitted via suitable method. Otherwise, misidentifying location model has effect to the estimation of dispersion effects. This problem was considered by [19].

In detail, suppose that the data after a suitable transformation follows the following location-dispersion models:

$$y_i = x_i'\beta + \sigma_i^2\varepsilon_i, \tag{2.1}$$

$$\sigma_i^2 = e^{z_i'\phi}, \quad i = 1, \dots, n, \tag{2.2}$$

where x_i' is the i-th row of $n \times p$ design matrix X in location model. $\beta = (\beta_1, \beta_2, \dots, \beta_p)'$ is the $p \times 1$ unknown parameter vector. β_i is a measure of location effect, called the location effect of factor i. z_i' is the i-th row of $n \times n$ saturated design matrix $Z = (X_1, X_2, \dots, X_n)$ in dispersion model. $\phi = (\phi_1, \phi_2, \dots, \phi_n)'$ is the $n \times 1$ unknown parameter vector. ϕ_i is a measure of dispersion effect, called the dispersion effect of factor i. The identification of dispersion effects is to detect the significant ones of all $\phi_i, i = 2, 3, \dots, n$ based on model (2.2). σ_i^2 is the variance of y_i. ε_i is i.i.d. random variable with standard normal distribution.

Note: In view of the nonnegativity of variance, the log-linear dispersion model (2.2) is used most commonly for modeling variance. It does not require additional constraints on the parameters to ensure that the estimate of variance is positive.

To identify active dispersion effects, effect sparsity principle is also assumed. Next, we primarily consider the estimation of dispersion effects.

3. A residuals averaging method

For factor k, ϕ_k is the dispersion parameter, then by the properties of design matrix X and model (2.2), we know that

$$\phi_k = \frac{1}{n}\Big(\sum_{S(k+)} \log(\sigma_i^2) - \sum_{S(k-)} \log(\sigma_i^2)\Big) \tag{3.1}$$

which shows that it only needs to estimate σ_i^2 when considering the estimation of dispersion parameter ϕ_k.

[8] presented a non-iterative method for estimating dispersion effects in the above-mentioned model, which is based on a set of residuals from the expanded location model. Specifically, if $L = \{I, l_1, \dots, l_{p-1}\}$ denotes the location model, the expanded location model $L_k^E = \{I, l_1, \dots, l_{p-1}, k, k \circ l_1, \dots, k \circ l_{p-1}\}$ for effect $k \in \{2, \dots, n\}$. For example, if the location model $L = \{I, A, B, C\}$, the expanded location models $L_A^E = \{I, A, B, C, AB, AC\}$ for factor A and $L_D^E = \{I, A, B, C, D, AD, BD, CD\}$ for factor D. $\widetilde{X}$ denotes the design matrix of the expanded location model L_k^E, $\widetilde{r}_i$ denotes the corresponding residual from fitted expanded location model, namely $\widetilde{r}_i = y_i - \widetilde{x}_i'\hat{\beta}$, $\widetilde{x}_i'$ is the i-th row of $\widetilde{X}$. This leads to the MH estimator

$$D_k^{MH} = \frac{1}{n}\Big(\sum_{S(k+)} \log(\widetilde{r}_i^2) - \sum_{S(k-)} \log(\widetilde{r}_i^2)\Big). \tag{3.2}$$

By comparing (3.1) and (3.2), we find that $\widetilde{r}_i^2$ is factly regarded as the estimation of σ_i^2. But, it is known that $\widetilde{r}_i^2$ is not unbiased estimation of σ_i^2 from the paper of [20]. Therefore, we need a better estimation of σ_i^2.

Under the assumption of effect sparsity, even though the model is the expanded location model, there are at least half of effects, which is not included in the location model. Then, we consider these columns and all the interactions between these columns and the elements in the expanded location model to be appended into the model. The details are as follows:

Let L be the set of active location effects. For each dispersion factor k, L_k^E is the expanded location model of factor k. Suppose the dimension of L_k^E is p, then there are $n-p$ nonactive location effects, denoting $T=\{l_{p+1},\dots,l_n\}$. $L_k^{(j)}$ denotes the union of L_k^E, l_j, and the interaction between l_j and the elements in L_k^E for $j=p+1,\dots,n$. Let q be the number of different $L_k^{(j)}$, $\widetilde{r}^{(j)}$ be the residual vector obtained from fitted $L_k^{(j)}$. Thus, the mean of $\widetilde{r}^{2(j)}s$ is considered as the estimation of σ_i^2, denoting

$$\overline{r}^2=\frac{\sum_{j=p+1}^{p+q}\widetilde{r}^{2(j)}}{q}$$

where $\widetilde{r}^{2(j)}$ is the square of $\widetilde{r}^{(j)}$. This leads to the statistic

$$D_k^{RA}=\frac{1}{n}\Big(\sum_{S(k+)}\log(\overline{r}_i^2)-\sum_{S(k-)}\log(\overline{r}_i^2)\Big) \tag{3.3}$$

where $\overline{r}_i$ is the i-th element of $\overline{r}$.

From (3.3), we know that $\widetilde{r}^{2(j)}$ can be obtained by fitting different $L_k^{(j)}$, and it is clear that $\sum_{j=p+1}^{p+q}\widetilde{r}^{2(j)}\neq 0$. Thus, the RA method can overcome the disadvantage of other methods such as the MH method (not applicable to the situations of zero or small absolute residuals). Noting that D_k^{RA} has similar structure with the MH method, the following Proposition 3.1 characterizes its unbiasedness under same conditions with the MH method. Proposition 3.2 provides a lower bound of variance of the RA estimator. Proofs are given in the Appendix.

Proposition 3.1 *Let $\mathcal{D}$ be the set of active dispersion effects. If $k\notin\overline{\mathcal{D}}_{-k}$, then D_k^{RA} is unbiased.*

Proposition 3.2 *The RA estimator has a lower bound of variance.*

4. Illustrative example

Consider the welding experiment from National Railway Corporation of Japan first reported by [18] and reanalyzed by several authors. The response is observed tensile strength of a weld. The design matrix and data are given in Table 1 in the same manner as [6]. The effects of nine factors (A to I) were studied in a 16-run 2^{9-5} fractional factorial design.

[6] first reanalyzed the data and showed that active location effects are B and C and active dispersion effects is only C via a normal probability plot. [7,8,19] also reanalyzed this data. Their analysis all agreed that active location effects are B and C. Regarding active dispersion effects, [19] also identified C, while [7] showed the presence of I and H in addition to C. [8] pointed that the MH method identified C,I, and H.

Next, we reanalyze this data using the RA method based on active location model $L=\{I,B,C\}$. Half-normal probability plot of the 15 effects in Figure 1 clearly indicates active dispersion effects in column $2, 13$, and 15 (C, I, and $H = CI$).

	Factor																
		D	H	G	A		$-F$	$-E$						I	B	$-C$	
Run	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	y
1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	43.7
2	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	40.2
3	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	42.4
4	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	44.7
5	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	42.4
6	1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	45.9
7	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	42.2
8	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	40.6
9	1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	42.4
10	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	45.5
11	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	43.6
12	1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	40.6
13	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	44.0
14	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	40.2
15	1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	42.5
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	46.5

Table 1 Design matrix and response

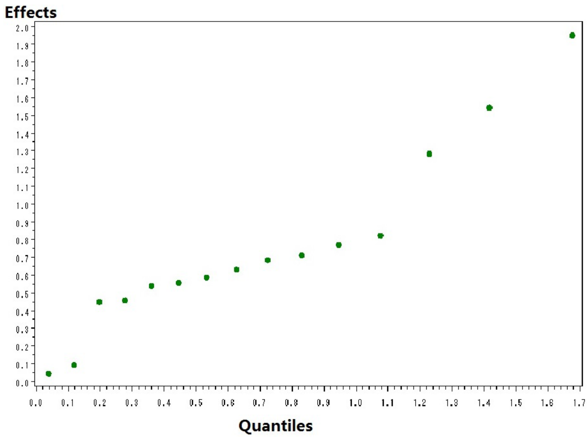

Figure 1 Half-normal plot of dispersion effects

5. Simulated comparison

Effect	True	MH method			RA method			Effect	True	MH method			RA method		
	value	Mean	Std	MSE	Mean	Std	MSE		value	Mean	Std	MSE	Mean	Std	MSE
		D={I,A,B,AB}								D={I,B,C,D}					
A	2.0	3.33	0.73	2.30	3.49	0.49	2.47	A	0	0.01	1.07	1.14	0.00	0.94	0.89
B	2.0	2.01	0.58	0.34	2.01	0.48	0.23	B	2.0	2.01	0.88	0.78	2.01	0.78	0.61
AB	2.0	2.01	0.59	0.35	2.01	0.48	0.23	AB	0	0.01	0.89	0.79	0.01	0.79	0.62
C	0	0.00	0.67	0.45	0.00	0.58	0.34	C	2.0	2.00	0.88	0.78	2.00	0.79	0.62
AC	0	0.00	0.68	0.46	0.00	0.58	0.34	AC	0	0.00	0.89	0.79	0.00	0.79	0.62
BC	0	0.00	0.67	0.45	0.00	0.58	0.33	**BC**	0	1.52	0.82	2.99	1.40	0.74	2.52
ABC	0	0.01	0.67	0.45	0.00	0.58	0.34	ABC	0	0.00	0.83	0.69	0.00	0.75	0.56
D	0	-0.02	0.66	0.43	-0.01	0.56	0.32	D	2.0	1.99	0.88	0.77	2.00	0.78	0.61
AD	0	-0.02	0.66	0.44	-0.02	0.56	0.32	AD	0	0.00	0.88	0.77	0.00	0.78	0.61
BD	0	-0.02	0.66	0.43	-0.01	0.56	0.32	**BD**	0	1.52	0.82	2.99	1.40	0.74	2.52
ABD	0	-0.02	0.67	0.44	-0.01	0.56	0.31	ABD	0	0.00	0.83	0.69	0.00	0.74	0.55
CD	0	0.00	0.66	0.44	0.00	0.57	0.33	**CD**	0	1.52	0.84	3.03	1.40	0.75	2.53
ACD	0	0.00	0.66	0.44	0.00	0.58	0.33	ACD	0	0.01	0.84	0.70	0.00	0.75	0.57
BCD	0	0.00	0.66	0.44	0.00	0.57	0.33	**BCD**	0	1.24	0.87	2.29	1.04	0.80	1.73
$ABCD$	0	0.00	0.67	0.44	0.00	0.57	0.33	$ABCD$	0	0.00	0.86	0.75	0.00	0.80	0.64

Table 2 Simulation of the estimation of dispersion effects under the location model $L = \{\mathrm{I}, A\}$ for various dispersion models

In this section, we compare the RA and MH methods by simulated experiments in the initial dispersion effects estimation step. The simulation is designed to cover a few different location and dispersion model combinations. The true value of each active dispersion effect is 2 and the size is 5000.

Tables 2–5 present the results of simulation, including Mean, Standard deviation (Std) and MSE, which all suggest that the RA method reduces Std well in majority of simulated models. An example is the situation in which $L = \{I, A\}$, $D = \{I, A, B, AB\}$. In this case, both the RA and MH methods are approximately unbiased for factor B (bias $= 0.01$), however, the Std reduces from 0.58 to 0.48 for the RA method. But in situations with $L = \{I, A, B, C\}$, $D = \{I, A, B, C\}$ and $L = \{I, A, B, C, D\}$, $D = \{I, A, B, C\}$, the Stds of the RA method are larger than the MH method for factors $D, AD, BD, ABD, CD, ACD, BCD$ and $ABCD$. This implies that the RA method may be short of robustness for these factors to some extent. The work is underway to explore the reasons.

Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE	Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE
		D={I,A,B,C}								D={I,C,D}					
A	2.0	1.98	0.67	0.45	1.99	0.60	0.37	A	0	0.00	0.89	0.79	-0.01	0.79	0.63
B	2.0	2.02	0.67	0.44	2.01	0.60	0.36	B	0	0.01	0.88	0.78	0.01	0.78	0.61
AB	0	0.02	0.68	0.46	0.01	0.60	0.36	AB	0	0.01	0.89	0.79	0.01	0.79	0.62
C	2.0	2.01	0.94	0.89	2.00	0.82	0.66	C	2.0	2.01	0.67	0.45	2.00	0.60	0.36
AC	0	1.46	0.92	2.98	1.53	0.76	2.91	AC	0	0.00	0.91	0.82	0.00	0.81	0.66
BC	0	1.48	0.94	3.06	1.54	0.78	2.98	BC	0	0.00	0.90	0.80	0.00	0.80	0.63
ABC	0	1.12	0.97	2.20	1.25	0.77	2.16	ABC	0	0.01	0.91	0.82	0.00	0.80	0.64
D	0	0.00	1.12	1.25	-0.01	1.01	1.02	D	2.0	2.00	0.65	0.42	2.00	0.58	0.33
AD	0	-0.01	1.13	1.27	-0.01	1.01	1.02	AD	0	0.00	0.90	0.81	0.00	0.80	0.64
BD	0	0.00	1.13	1.27	-0.01	1.01	1.02	BD	0	0.00	0.89	0.79	0.00	0.81	0.65
ABD	0	-0.01	1.12	1.25	0.00	1.01	1.01	ABD	0	0.00	0.89	0.79	0.00	0.80	0.63
CD	0	-0.01	1.13	1.27	0.00	1.01	1.03	**CD**	0	1.60	0.64	2.97	1.61	0.55	2.88
ACD	0	0.00	1.13	1.27	0.00	1.02	1.04	ACD	0	0.01	0.86	0.74	0.00	0.79	0.63
BCD	0	0.00	1.14	1.30	0.00	1.02	1.04	BCD	0	-0.01	0.84	0.70	0.00	0.78	0.61
$ABCD$	0	0.01	1.13	1.29	0.00	1.02	1.04	$ABCD$	0	0.00	0.84	0.71	0.00	0.78	0.61

Table 3 Simulation of the estimation of dispersion effects under the location model $L = \{\mathrm{I}, A, B\}$ for various dispersion models

Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE	Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE
		D={I,A,B}								D={I,A,B,C}					
A	2.0	2.00	0.76	0.58	2.01	0.64	0.41	A	2.0	2.00	0.92	0.85	2.00	0.80	0.64
B	2.0	2.02	0.77	0.59	2.02	0.64	0.41	B	2.0	2.04	0.91	0.83	2.02	0.80	0.65
AB	0	1.41	0.77	2.59	1.56	0.59	2.78	**AB**	0	1.47	0.92	3.02	1.54	0.76	2.94
C	0	0.01	0.94	0.89	0.00	0.82	0.66	C	2.0	2.01	0.94	0.89	2.00	0.82	0.66
AC	0	0.01	0.93	0.87	0.00	0.81	0.66	**AC**	0	1.16	0.92	2.98	1.53	0.76	2.91
BC	0	0.00	0.94	0.88	0.00	0.82	0.67	**BC**	0	1.48	0.94	3.06	1.54	0.78	2.98
ABC	0	0.00	0.78	0.61	0.00	0.78	0.61	**ABC**	0	0.00	0.78	0.61	0.00	0.78	0.61
D	0	-0.02	0.87	0.77	-0.02	0.81	0.65	D	0	-0.01	0.91	0.83	-0.01	1.03	1.06
AD	0	-0.01	0.88	0.78	-0.01	0.81	0.65	AD	0	0.00	0.91	0.84	-0.01	1.03	1.06
BD	0	-0.01	0.87	0.76	-0.02	0.80	0.64	BD	0	0.00	0.91	0.83	-0.01	1.03	1.06
ABD	0	0.00	0.87	0.76	-0.02	0.80	0.63	ABD	0	0.00	0.91	0.82	0.00	1.02	1.05
CD	0	0.00	0.93	0.86	0.00	0.84	0.70	CD	0	0.00	0.92	0.86	0.00	1.03	1.07
ACD	0	0.00	0.93	0.86	0.00	0.84	0.71	ACD	0	0.00	0.93	0.86	0.00	1.04	1.08
BCD	0	0.00	0.92	0.85	0.00	0.84	0.71	BCD	0	0.00	0.93	0.86	0.00	1.04	1.07
$ABCD$	0	-0.01	0.93	0.87	-0.01	0.85	0.72	$ABCD$	0	0.00	0.92	0.85	0.00	1.04	1.08

Table 4 Simulation of the estimation of dispersion effects under the location model $L = \{\mathrm{I}, A, B, C\}$ for various dispersion models

MSE is a master criterion measuring estimation. If we only consider MSE, the RA method is almost all better than the MH method for active factors. For example, in Table 4 for the case in which $L = \{\text{I}, A, B, C\}$, $D = \{\text{I}, A, B, C\}$, the MSEs of the MH method for factor A, B, and C are 0.85, 0.83, and 0.89, but for the RA method are 0.64, 0.65, and 0.66, respectively. It is not as good as the MH method for the case with $L = \{\text{I}, A\}, D = \{\text{I}, A, B, AB\}$. This also can be seen from Proposition 1 of this paper and Proposition 3 in [8]. Because A, B, and AB are included in the closure of D, the unbiasedness cannot be guaranteed. Thus, we first fit a location model including A, B, and AB, then estimate dispersion effects using H method [21]. Table 6 gives the results. It is obvious that A, B, and AB are all approximately unbiased, i.e., it corrects the biasedness of the RA and MH methods.

D={I,A,B}

Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE
A	2.0	2.01	0.74	0.55	2.01	0.66	0.44
B	2.0	2.01	0.75	0.56	2.02	0.67	0.45
AB	0	1.60	0.72	3.07	1.63	0.63	3.04
C	0	0.00	0.94	0.88	0.00	0.85	0.72
AC	0	0.00	0.94	0.87	0.00	0.84	0.71
BC	0	0.00	0.94	0.89	0.01	0.86	0.73
ABC	0	0.00	1.25	1.57	0.00	1.25	1.57
D	0	-0.02	0.87	0.77	-0.02	0.81	0.65
AD	0	-0.01	0.88	0.78	-0.01	0.81	0.65
BD	0	-0.01	0.87	0.76	-0.02	0.80	0.64
ABD	0	-0.03	1.22	1.49	-0.03	1.22	1.49
CD	0	0.00	0.93	0.86	0.00	0.84	0.70
ACD	0	-0.01	0.97	0.95	-0.01	0.97	0.95
BCD	0	0.01	0.97	0.95	0.01	0.97	0.95
$ABCD$	0	0.00	0.96	0.91	-0.01	0.85	0.73

D={I,A,B,C}

Effect	True value	MH method Mean	MH method Std	MH method MSE	RA method Mean	RA method Std	RA method MSE
A	2.0	2.01	0.90	0.81	2.01	0.82	0.67
B	2.0	2.01	0.92	0.85	2.02	0.82	0.68
AB	0	1.59	0.86	3.27	1.56	0.78	3.03
C	2.0	2.00	0.94	0.88	2.00	0.85	0.72
AC	0	1.58	0.87	3.27	1.55	0.78	3.01
BC	0	1.59	0.89	3.33	1.56	0.80	3.08
ABC	0	1.26	1.23	3.09	1.26	1.23	3.09
D	0	-0.01	0.91	0.83	-0.01	1.03	1.06
AD	0	0.00	0.91	0.84	-0.01	1.03	1.06
BD	0	0.00	0.91	0.83	-0.01	1.03	1.06
ABD	0	0.00	1.24	1.55	0.00	1.24	1.55
CD	0	0.00	0.92	0.86	0.00	1.03	1.07
ACD	0	0.00	1.25	1.56	0.00	1.25	1.56
BCD	0	0.00	1.23	1.51	0.00	1.23	1.51
$ABCD$	0	0.00	1.18	1.40	0.00	1.05	1.09

Table 5 Simulation of the estimation of dispersion effects under the location model $L = \{\text{I}, A, B, C, D\}$ for various dispersion models

Dispersion Effects	True value	H method Mean	H method Std	H method MSE
A	2.0	2.00	0.59	0.35
B	2.0	2.01	0.58	0.34
AB	2.0	2.01	0.59	0.35

Table 6 Simulation under same location and dispersion model $L = \{\text{I}, A, B, AB\}$

Note: In some situations, the RA method has the same results as the MH method. For example, in Table 4 for the case in which $L = \{I, A, B, C\}$ and $D = \{I, A, B, C\}$, both the RA and MH methods are unbiased for factor ABC, and the same Std is 0.78. This result comes from the fact that the expanded location model L^E_{ABC} is a closed set with $\dim L^E_{ABC} = N/2$, and the model based on the RA method is a full model.

6. Appendix

Lemma 6.1 *Let $L_k^{(j)}$ be the union of L_k^E, l_j, and the interaction between l_j and the elements in L_k^E for $j = p+1, \ldots, n$. Then the residuals resulting from $L_k^{(j)}$ with design matrix $\widetilde{X}^{(j)}$ are of the form*

$$\widetilde{r}_i^{(j)} = \begin{cases} \sum_{m \in S(k+)} \widetilde{h}_{im}^{(j)} y_m & i \in S(k+) \\ \sum_{m \in S(k-)} \widetilde{h}_{im}^{(j)} y_m & i \in S(k-) \end{cases}$$

where $\widetilde{H}^{(j)} = I - H = I - \frac{1}{n}\widetilde{X}^{(j)}\widetilde{X}'^{(j)} = (\widetilde{h}^{(j)}_{im})_{i,m=1,\dots,n}$, $\widetilde{r}^{(j)}_i$ *is the* i*-th element of* $\widetilde{r}^{(j)}$.

The proof is the same as the Lemma 2 in [8] and is omitted.

Lemma 6.2 *The means of the residual squares resulting from the RA method are of the form*

$$\overline{r}_i^2 = \widetilde{e}'_{k+}(\Sigma_{k+}^{1/2}\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)}\Sigma_{k+}^{1/2})\widetilde{e}_{k+}, \quad i\in S(k+)$$
$$\overline{r}_i^2 = \widetilde{e}'_{k-}(\Sigma_{k-}^{1/2}\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)}\Sigma_{k-}^{1/2})\widetilde{e}_{k-}, \quad i\in S(k-)$$

where $\widetilde{e}_{k+} = \Sigma_{k+}^{-1/2}e_{k+}$, $\Sigma_{k+} = \mathrm{Var}(e_{k+}) = \mathrm{diag}(\sigma_1^2,\dots,\sigma_{n/2}^2)$, $\widetilde{e}_{k-} = \Sigma_{k-}^{-1/2}e_{k-}$, $\Sigma_{k-} = \mathrm{Var}(e_{k-}) = \mathrm{diag}(\sigma_{n/2+1}^2,\dots,\sigma_n^2)$, $e = (e'_{k+}, e'_{k-})' = y - \widetilde{X}\beta$, $A_i = \frac{\delta_i\delta_i'}{q}$, $\widetilde{H}^{(j)} = \begin{bmatrix} H_1^{(j)} & 0 \\ 0 & H_1^{(j)}\end{bmatrix}$, δ_i *is an* $n/2\times 1$ *vector with the* i*-th element of 1 and the others of 0.*

Proof First, the design matrix $\widetilde{X}^{(j)}$ can be rearranged to satisfy the following condition: $S(k+) = \{1,\dots,n/2\}$ and $S(k-) = \{n/2+1,\dots,n\}$.

By Lemma 6.1, we know that $\widetilde{h}^{(j)}_{ml} = 0$ for $m\in S(k+), l\in S(k-)$ or $m\in S(k-), l\in S(k+)$. And the rearranged $\widetilde{H}^{(j)}$ can be decomposed as

$$\widetilde{H}^{(j)} = \begin{bmatrix} H_1^{(j)} & H_2^{(j)} \\ H_2^{(j)} & H_1^{(j)}\end{bmatrix}$$

where $H_1^{(j)}$ and $H_2^{(j)}$ are $n/2\times n/2$ matrices. This leads to

$$\widetilde{H}^{(j)} = \begin{bmatrix} H_1^{(j)} & 0 \\ 0 & H_1^{(j)}\end{bmatrix}$$

$$\widetilde{r}^{(j)} = \begin{bmatrix} H_1^{(j)} & 0 \\ 0 & H_1^{(j)}\end{bmatrix} y = \begin{bmatrix} H_1^{(j)} & 0 \\ 0 & H_1^{(j)}\end{bmatrix}\begin{pmatrix} e_{k+} \\ e_{k-}\end{pmatrix} = \begin{pmatrix} H_1^{(j)}e_{k+} \\ H_1^{(j)}e_{k-}\end{pmatrix}$$

where $e = (e'_{k+}, e'_{k-})' = y - \widetilde{X}\beta$. For $i\in S(k+)$, $\widetilde{r}_i^{(j)} = \delta_i'H_1^{(j)}e_{k+}, i = 1,\dots,n/2$; $\widetilde{r}_i^{(j)} = \delta_i'H_1^{(j)}e_{k-}, i = n/2+1,\dots,n$, for $i\in S(k-)$, δ_i is an $n/2\times 1$ vector with the i-th element of 1 and the others of 0.

Thus, for $i\in S(k+)$,

$$\begin{aligned}\overline{r}_i^2 =& \frac{\sum_{j=p+1}^{p+q}\widetilde{r}_i^{2(j)}}{q} = \frac{\sum_{j=p+1}^{p+q}(e'_{k+}H_1^{(j)}\delta_i\delta_i'H_1^{(j)}e_{k+})}{q} = \frac{e'_{k+}(\sum_{j=p+1}^{p+q}H_1^{(j)}\delta_i\delta_i'H_1^{(j)})e_{k+}}{q}\\ =& e'_{k+}(\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)})e_{k+} = (\Sigma_{k+}^{1/2}\Sigma_{k+}^{-1/2}e_{k+})'(\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)})(\Sigma_{k+}^{1/2}\Sigma_{k+}^{-1/2}e_{k+})\\ =& \widetilde{e}'_{k+}(\Sigma_{k+}^{1/2}\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)}\Sigma_{k+}^{1/2})\widetilde{e}_{k+}\end{aligned}$$

where $\widetilde{e}_{k+} = \Sigma_{k+}^{-1/2}e_{k+}\sim N(0,I)$, $\Sigma_{k+} = \mathrm{Var}(e_{k+}) = \mathrm{diag}(\sigma_1^2,\dots,\sigma_{n/2}^2)$, $A_i = \frac{\delta_i\delta_i'}{q}$.

A similar argument shows that $\overline{r}_i^2 = \widetilde{e}'_{k-}(\Sigma_{k-}^{1/2}\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)}\Sigma_{k-}^{1/2})\widetilde{e}_{k-}$ for $i\in S(k-)$, where $\widetilde{e}_{k-} = \Sigma_{k-}^{-1/2}e_{k-}\sim N(0,I)$, $\Sigma_{k-} = \mathrm{Var}(e_{k-}) = \mathrm{diag}(\sigma_{n/2+1}^2,\dots,\sigma_n^2)$.

Proof of Proposition 3.1 By Lemmas 6.1 and 6.2,

$$E[D_k^{RA}] = \frac{1}{n}E\Big(\sum_{S(k+)}\log(\overline{r}_i^2) - \sum_{S(k-)}\log(\overline{r}_i^2)\Big)$$

$$=\frac{1}{n}E\Big(\sum_{S(k+)}\log(\widetilde{e}'_{k+}(\Sigma_{k+}^{1/2}B_i\Sigma_{k+}^{1/2})\widetilde{e}_{k+})-\sum_{S(k-)}\log(\widetilde{e}'_{k-}(\Sigma_{k-}^{1/2}B_i\Sigma_{k-}^{1/2})\widetilde{e}_{k-})\Big)$$

where $B_i=\Sigma_{j=p+1}^{p+q}H_1^{(j)}A_iH_1^{(j)}$.

Let c_i be the i-th row of $C=[\widetilde{X}_{\bar{d}_1}|\cdots|\widetilde{X}_{\bar{d}_r}]$, where $\bar{d}_j\in\overline{\mathcal{D}}_{-k}$, let $\phi_C=(\phi_{\bar{d}_1},\ldots,\phi_{\bar{d}_r})'$. Since $k\notin\overline{\mathcal{D}}_{-k}$, the preceding expectation can be rewritten as

$$\begin{aligned}E[D_k^{RA}]=&\frac{1}{n}E\Big(\sum_{S(k+)}\log(\exp(\phi_1+\phi_k)\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})-\\&\sum_{S(k-)}\log(\exp(\phi_1-\phi_k)\widetilde{e}'_{k-}(\widetilde{\Sigma}_{k-}^{1/2}B_i\widetilde{\Sigma}_{k-}^{1/2})\widetilde{e}_{k-})\Big)\\=&\phi_k+\frac{1}{n}E\Big(\sum_{S(k+)}\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})-\sum_{S(k-)}\log(\widetilde{e}'_{k-}(\widetilde{\Sigma}_{k-}^{1/2}B_i\widetilde{\Sigma}_{k-}^{1/2})\widetilde{e}_{k-})\Big)\end{aligned}$$

where

$$\widetilde{\Sigma}_{k+}=\begin{pmatrix}e^{c_1'\phi_C}&0&\cdots&0\\0&e^{c_2'\phi_C}&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&e^{c_{n/2}'\phi_C}\end{pmatrix}^{1/2},\widetilde{\Sigma}_{k-}=\begin{pmatrix}e^{c_{n/2+1}'\phi_C}&0&\cdots&0\\0&e^{c_{n/2+2}'\phi_C}&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&e^{c_n'\phi_C}\end{pmatrix}^{1/2}.$$

According to the results of P_{404} in [8], $c_i=c_{i+n/2}$ for $i=1,\ldots,n/2$, then $\widetilde{\Sigma}_{k+}=\widetilde{\Sigma}_{k-}$. It is clear that for each $i\in S(k+)$, there exists an $i'\in S(k-)$ such that $\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+}$ and $\widetilde{e}'_{k-}(\widetilde{\Sigma}_{k-}^{1/2}B_{i'}\widetilde{\Sigma}_{k-}^{1/2})\widetilde{e}_{k-}$ have the same distribution from the standard normality of $\widetilde{e}_{k+}$ and $\widetilde{e}_{k-}$. Namely, $E[D_k^{RA}]=\phi_k$. This completes the proof. □

Lemma 6.3 *The means of the residual squares resulting from the RA method are such that* $\bar{r}_m^2$ *is independent of* $\bar{r}_{m'}^2$ *whenever* $m\in S(k+)$ *and* $m'\in S(k-)$.

The proof of the Lemma 6.3 is easily obtained from the Corollary 1 in [8] and is omitted.

Proof of Proposition 3.2 By Lemmas 6.2, 6.3 and Proposition 3.1, the variance of the RA estimator can be given by

$$\begin{aligned}\mathrm{Var}(D_k^{RA})=&\frac{1}{n^2}\Big[\mathrm{Var}\Big(\sum_{S(k+)}\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})\Big)+\mathrm{Var}\Big(\sum_{S(k-)}\log(\widetilde{e}'_{k-}(\widetilde{\Sigma}_{k-}^{1/2}B_i\widetilde{\Sigma}_{k-}^{1/2})\widetilde{e}_{k-})\Big]\\=&\frac{2}{n^2}\mathrm{Var}\Big(\sum_{S(k+)}\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})\Big)\\=&\frac{2}{n^2}\sum_{S(k+)}\mathrm{Var}\Big(\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})\Big)+\\&\frac{4}{n^2}\sum_{m,n\in S(k+)}\mathrm{Cov}\Big(\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_m\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+}),\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_n\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})\Big)\\\geq&\frac{2}{n^2}\sum_{S(k+)}\mathrm{Var}\Big(\log(\widetilde{e}'_{k+}(\widetilde{\Sigma}_{k+}^{1/2}B_i\widetilde{\Sigma}_{k+}^{1/2})\widetilde{e}_{k+})\Big).\end{aligned}$$

Acknowledgements We thank the referees for their time and comments.

References

[1] G. TAGUCHI. *Introduction to Quality Engineering: Designing Quality Into Products and Processes.* White Plains, Kraus, 1986.
[2] C. F. Jeff WU, M. HAMADA. *Experiments: Planning, Analysis, and Parameter Design Optimization*. John Wiley & Sons, New York, 2000.
[3] M. S. BARTLETT, D. G. KENDALL. *The statistical analysis of variance-heterogeneity and the logarithmic transformation*. Suppl. J. Roy. Statist. Soc., 1946, **8**: 128–138.
[4] V. N. NAIR, D. PREGIBON. *Analyzing dispersion effects from replicated factorial experiments*. Technometrics, 1988, **30**(3): 247–257.
[5] M. HAMADA, N. BALAKRISHNAN. *Analyzing unreplicated factorial experiments: a review with some new proposals*. Statist. Sinica, 1998, **8**(1): 1–41.
[6] G. E. P. BOX, R. D. MEYER. *Dispersion effects from fractional designs*. Technometrics, 1986, **28**(1): 19–27.
[7] B. BERGMAN, A. HYNEN. *Dispersion effects form unreplicated designs in the 2^{k-p} series*. Technometrics, 1997, **39**: 191–198.
[8] W. A. BRENNEMAN, V. N. NAIR. *Methods for identifying dispersion effects in unreplicated factorial experiments: a critical analysis and proposed strategies*. Technometrics, 2001, **43**: 388–404.
[9] K. WIKLANDER, S. HOLM. *Dispersion effects in unreplicated factorial designs*. Appl. Stoch. Models Bus. Ind., 2003, **19**(1): 13–30.
[10] Chentuo LIAO. *Two-level factorial designs for searching dispersion factors and estimating location main effects*. J. Statist. Plann. Inference, 2006, **136**(11): 4071–4087.
[11] P. M. van de VEN. *On the equivalence of three estimators for dispersion effects in unreplicated two-level factorial designs*. J. Statist. Plann. Inference, 2008, **138**(1): 18–29.
[12] Jihong LI, Gaixian REN, Yu WANG. *Properties of BH estimator of dispersion effect in unreplicated two-level factorial experiments*. Appl. Prob. Stat., 2010, **26**: 179–189. (in Chinese)
[13] Jihong LI, Yu WANG, Xing ZHANG. *Estimation of dispersion effect in two-Level unreplicated factorial experiments*. Acta. Math. Appl. Sin., 2010, **33**: 710–721. (in Chinese)
[14] Yu WANG, Jihong LI, Xia FENG. *The ML estimator of dispersion effects in unreplicated factorial experiments*. J. Systems Sci. Math. Sci., 2011, **31**(7): 804–816. (in Chinese)
[15] Ying CHEN. *A significance test in multi-level orthogonal designs with only one replicate*. Chinese J. Appl. Probab. Statist., 2011, **27**(5): 497–510.
[16] B. BERGQUIST, E. VANHATALO, M. L. NORDENVAAD. *A Bayesian analysis of unreplicated two-Level factorials using effects sparsity, hierarchy, and heredity*. Quality Engineering, 2011, **23**(2): 152–166.
[17] Liu YANG, Yu WANG. *An iterative estimator dispersion effects in unreplicated factorial experiments*. Math. Prac. Theo., 2012, **42**: 144–148. (in Chinese)
[18] G. TAGUCHI, Y. WU. *Introduction to Off-Line Quality Control*. Central Japan Quality Control Association, Tokyo, 1980.
[19] R. N. MCGRATH, K. J. LIN. *Confounding of location and dispersion effects in unreplicated fractional factorials*. J. Qual. Technol., 2001, **33**: 129–139.
[20] R. D. COOK, S. WEISBERG. *Diagnostics for heteroscedasticity in regression*. Biometrika, 1983, **70**: 1–10.
[21] A. C. HARVEY. *Estimating regression models with multiplicative heteroscedasticity*. Econometrica, 1976, **44**: 461–465.