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The Coalgebra Structure over Hom-Hopf Algebra

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Abstract Let (C, α) and (H, β) be Hom-bialgebras and $\omega : C \otimes H \to H \otimes C$ a linear map. We introduce the concept of a Hom-ω-crossed coproduct $(C_\omega \bowtie_\sigma H, \gamma)$ and we give necessary and sufficient conditions for the new object to be a Hom-Hopf algebra.

Keywords Hom-Hopf algebra; Hom-ω-crossed coproduct; ω-crossed coproduct

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1. Introduction

Generalizations of Hopf algebras over fields have quite a long history. The weakening of the (co)associativity leads to Hom-Hopf algebra which is twisted by a linear endomorphism. That is, the associativity of the algebra structure is replaced by the Hom-associativity, namely

$$\alpha(a)(bc) = (ab)\alpha(c),$$

where α is an endomorphism of the algebra. Hom-algebras have been introduced in [1,2]. Hom-coassociativity for a Hom-coalgebra can be considered in a similar way [3]. Also definitions of Hom-bialgebras and Hom-Hopf algebras have been proposed by some scholars in [4–6].

Twisted (or ω-) smash coproduct was introduced by Caenepeel and his collaborators in [7]. The crossed coproduct is well known in the context of Hopf algebras [8,9]. Following the above discussed works, it is very natural to investigate how to construct a Hom-type generalization of ω-smash coproduct coalgebra structure and crossed coproduct coalgebra structure.

In this paper, our aim is to introduce the concept of a Hom-ω-crossed coproduct $(C_\omega \bowtie_\sigma H, \gamma)$ and give necessary and sufficient conditions for the new object to be a Hom-Hopf algebra. This generalizes Caenepeel and Dascalescu's result.

Throughout this paper, we work over a fixed commutative ring k of characteristic 0. Modules, tensor products, and linear maps are all taken over k. We use Sweedler's notation for comultiplication, writing $\triangle(h) = \sum h_1 \otimes h_2$. The antipode of a Hom-Hopf algebra H is denoted by S (or S_H).

2. Hom-ω-crossed product Hopf algebras

In this section, we first recall some concepts and results on Hom-Hopf algebras that we shall need later. We refer the reader to [1,2,4,6] for a detailed exposition on the subject. Then we

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define the Hom-ω-crossed coproduct and derive necessary and sufficient conditions for Hom-ω-crossed coproduct to be a Hom-Hopf algebra.

Definition 2.1 *A Hom-algebra in [4] is a quadruple (A,μ,η,α) consisting of a k-module A, a bilinear map $\mu: A\otimes A\to A$ (the multiplication), a linear self-map $\alpha: A\to A$ (the twisting map) and a homomorphism $\eta: k\to A$ such that*

$$\alpha(ab)=\alpha(a)\alpha(b),\quad \alpha(\eta(1))=\eta(1), \tag{2.1}$$

$$\alpha(a)(bc)=(ab)\alpha(c),\quad a\eta(1)=\alpha(a)=\eta(1)a, \tag{2.2}$$

for all $a,b,c\in A$, where we use the notation $\mu(a\otimes b)=ab$. We call $\eta(1)$ a weak unit of A and write $\eta(1)=1_A$.

Definition 2.2 *A Hom-coalgebra in [4] is a quadruple $(C,\triangle,\varepsilon,\alpha)$ consisting of a k-module C, a linear map $\triangle: C\to C\otimes C$ (the comultiplication), a linear self-map $\alpha: A\to A$ (the twisting map) and a homomorphism $\varepsilon: C\to k$ such that*

$$\triangle(\alpha(c))=\sum\alpha(c_1)\otimes\alpha(c_2),\quad \varepsilon(\alpha(c))=\varepsilon(c), \tag{2.3}$$

$$\sum\alpha(c_1)\otimes\triangle(c_2)=\sum\triangle(c_1)\otimes\alpha(c_2),\quad \sum c_1\varepsilon(c_2)=\alpha(c)=\sum\varepsilon(c_1)c_2, \tag{2.4}$$

for all $c\in C$.

Definition 2.3 *A Hom-bialgebra in [4] is a quintuple $(H,\mu,\eta,\triangle,\varepsilon,\alpha)$ in which (H,μ,η,α) is a Hom-algebra, $(H,\triangle,\varepsilon,\alpha)$ is a Hom-coalgebra such that $\triangle$ and ε are algebra maps, that is, for any $h,g\in H$,*

$$\triangle(hg)=\triangle(h)\triangle(g),\quad \triangle(1_H)=1_H\otimes 1_H, \tag{2.5}$$

$$\varepsilon(hg)=\varepsilon(h)\varepsilon(g). \tag{2.6}$$

Observe that a Hom-bialgebra is neither associative nor coassociative, unless of course $\alpha=\mathrm{Id}$, in which case we have a bialgebra. Instead of (co)associativity, in a Hom-bialgebra we have Hom-(co)associativity, $\mu\circ(\alpha\otimes\mu)=\mu\circ(\mu\otimes\alpha)$ and $(\alpha\otimes\triangle)\circ\triangle=(\triangle\otimes\alpha)\circ\triangle$. So, roughly speaking, the degree of non-(co)associativity in a Hom-bialgebra is measured by how far the twisting map α deviates from the identity map.

Definition 2.4 *A Hom-bialgebra (H,α) is a Hom-Hopf algebra in [4] if there exists a morphism (called antipode) $S: H\to H$ such that*

$$S\circ\alpha=\alpha\circ S,\quad S*\mathrm{Id}=\eta\circ\varepsilon=\mathrm{Id}*S. \tag{2.7}$$

In the following lemma, we give some elementary properties of the antipode.

Lemma 2.5 *Let (H,α) be a Hom-Hopf algebra. Then, by [4], for any $h,g\in H$, we have*

$$S(hg)=S(g)S(h),\quad S(1_H)=1_H, \tag{2.8}$$

$$\triangle(S(h))=S(h_2)\otimes S(h_1),\quad \varepsilon\circ S=\varepsilon. \tag{2.9}$$

We know that S is a (co)algebra anti-homomorphism.

From now on, we assume that the twisting map α is a bijective linear map. For the Hom-associativity, we obtain the formula

$$a(bc) = (\alpha^{-1}(a)b)\alpha(c) \tag{2.2'}$$

which is clearly equivalent to the first formula of (2.2).

Let $(C, \triangle, \varepsilon, \alpha)$ and $(H, \triangle, \varepsilon, \beta)$ be Hom-coalgebras over commutative ring k and consider a linear map $\omega : C \otimes H \to H \otimes C$. We write

$$\omega(c \otimes h) = \sum {}^{\omega}h \otimes {}^{\omega}c$$

for all $c \in C$ and $h \in H$. Then ω is called Hom-conormal if,

$$(I_H \otimes \varepsilon_C)\omega(c \otimes h) = \varepsilon_C(c)\beta(h), \quad (\varepsilon_H \otimes I_C)\omega(c \otimes h) = \varepsilon_H(h)\alpha(c). \tag{2.10}$$

Let $\sigma \in \mathrm{Hom}_k(C, H \times H)$ and write $\sigma(c) = \sum \sigma_1(c) \otimes \sigma_2(c)$. Then (C, H) is called an $(\alpha, \beta, \sigma, \omega)$-compatible pair if,

$$\sigma(\alpha(c)) = (\beta \otimes \beta)\sigma(c), \tag{2.11}$$

or

$$\sum \sigma_1(\alpha(c)) \otimes \sigma_2(\alpha(c)) = \sum \beta(\sigma_1(c)) \otimes \beta(\sigma_2(c)), \tag{2.11'}$$

and

$$\sum \alpha({}^{\omega}c) \otimes \beta({}^{\omega}h) = \sum {}^{\omega}\alpha(c) \otimes {}^{\omega}\beta(h). \tag{2.12}$$

Definition 2.6 *Let (C, H) be an $(\alpha, \beta, \sigma, \omega)$-compatible pair and ω be a Hom-conormal linear map. Let $\gamma = \alpha \otimes \beta : C \otimes H \to C \otimes H$ be a linear self-map. Let $C_\omega \bowtie_\sigma H$ be the coalgebra whose underlying vector space is $C \otimes_k H$, with comultiplication and counit given by*

$$\triangle_{C_\omega \bowtie_\sigma H}(c \bowtie h) = \sum c_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))) \otimes {}^{\omega}\alpha^{-2}(c_{21}) \bowtie \sigma_2(c_{22})\beta^{-1}(h_2),$$

$$\varepsilon_{C_\omega \bowtie_\sigma H}(c \bowtie h) = \varepsilon_C(c)\varepsilon_H(h),$$

for all $c \in C$ and $h \in H$, where we have written $c \bowtie h$ for the tensor $c \otimes h$. If the quadruple $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ forms a Hom-coalgebra, then it is called a Hom-ω-crossed coproduct of C and H and we denote it by $(C_\omega \bowtie_\sigma H, \gamma)$.

In the following theorem, we give necessary and sufficient conditions for $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ to be a Hom-ω-crossed coproduct.

Theorem 2.7 *Let (C, H) be an $(\alpha, \beta, \sigma, \omega)$-compatible pair and ω be a Hom-conormal linear map. Then $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ is a Hom-ω-crossed coproduct, if and only if the following conditions hold ($\overline{\omega} = \omega$):*

$$\sum \varepsilon_H(\sigma_1(c))\sigma_2(c) = \sum \sigma_1(c)\varepsilon_H(\sigma_2(c)) = \varepsilon_C(c)1_H, \tag{2.13}$$

$$\sum \beta^{-1}({}^{\overline{\omega}\omega}\beta(h)) \otimes {}^{\overline{\omega}}c_1 \otimes \alpha({}^{\omega}\alpha^{-1}(c_2)) = \sum {}^{\omega}\beta(h) \otimes ({}^{\omega}c)_1 \otimes ({}^{\omega}c)_2, \tag{2.14}$$

$$\sum \beta(\sigma_1(\alpha^2(c_1))\beta^{-3}(({}^{\omega}\beta(h))_1)) \otimes \sigma_2(\alpha^2(c_1))\beta^{-2}(({}^{\omega}\beta(h))_2) \otimes \alpha({}^{\omega}\alpha^{-1}(c_2))$$

$$= \sum {}^{\omega}(\sigma_1(\alpha^2(c_2))\beta^{-1}(h_1)) \otimes \beta^{-1}({}^{\overline{\omega}}(\sigma_2(\alpha^2(c_2))h_2)) \otimes {}^{\overline{\omega}\omega}\alpha^{-1}(c_1), \tag{2.15}$$

$$\sum \beta^3(\sigma_1(c_1))(\sigma_1(\alpha(c_2))\beta^{-2}(h_1))_1 \otimes \beta^2(\sigma_2(c_1))(\sigma_1(\alpha(c_2))\beta^{-2}(h_1))_2 \otimes \beta(\sigma_2(\alpha(c_2)))h_2$$
$$= \sum {}^{\omega}\beta^2(\sigma_1(c_2)\beta^{-3}(h_1)) \otimes \beta^3(\sigma_1({}^{\omega}\alpha^{-1}(c_1)))(\sigma_2(c_2)\beta^{-2}(h_2))_1 \otimes$$
$$\beta^2(\sigma_2({}^{\omega}\alpha^{-1}(c_1)))(\sigma_2(c_2)\beta^{-2}(h_2))_2. \tag{2.16}$$

Proof For any $c \in C$ and $h \in H$, we have

$$(I \otimes \varepsilon_{C_\omega \bowtie_\sigma H}) \triangle_{C_\omega \bowtie_\sigma H} (c \bowtie h)$$
$$= \sum c_1 \varepsilon_C({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))\varepsilon_H(\sigma_2(c_{22})\beta^{-1}(h_2))$$
$$\overset{(2.10)}{=} \sum c_1 \varepsilon_C(\alpha^{-2}(c_{21})) \bowtie \beta^{-1}(\beta^2(\sigma_1(c_{22})\beta^{-1}(h)))\varepsilon_H(\sigma_2(c_{22}))$$
$$\overset{(2.13)}{=} \sum c_1 \varepsilon_C(\alpha^{-2}(c_{21}))\varepsilon_C(c_{22}) \bowtie \beta^{-1}(\beta^2(h))$$
$$= \gamma(c \bowtie h),$$

$$(\varepsilon_{C_\omega \bowtie_\sigma H} \otimes I) \triangle_{C_\omega \bowtie_\sigma H} (c \bowtie h)$$
$$= \sum \varepsilon_C(c_1)\, {}^{\omega}\alpha^{-2}(c_{21}) \bowtie \varepsilon_H(\beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\sigma_2(c_{22})\beta^{-1}(h_2)$$
$$\overset{(2.10)}{=} \sum \varepsilon_C(c_1)\alpha^{-1}(c_{21}) \bowtie \varepsilon_H(\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))\sigma_2(c_{22})\beta^{-1}(h_2)$$
$$= \sum \varepsilon_C(c_1)\alpha^{-1}(c_{21}) \bowtie \varepsilon_H(\sigma_1(c_{22}))\sigma_2(c_{22})h$$
$$\overset{(2.13)}{=} \sum \varepsilon_C(c_1)\varepsilon_C(c_{22})\alpha^{-1}(c_{21}) \bowtie \beta(h)$$
$$= \gamma(c \bowtie h),$$

$$\triangle_{C_\omega \bowtie_\sigma H} \gamma(c \bowtie h)$$
$$= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(\alpha(c_{22}))\beta^{-1}(h_1))) \otimes {}^{\omega}\alpha^{-1}(c_{21}) \bowtie \sigma_2(\alpha(c_{22}))h_2$$
$$\overset{(2.11)}{=} \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\omega}\beta^2(\sigma_1(c_{22})\beta^{-2}(h_1))) \otimes {}^{\omega}\alpha^{-1}(c_{21}) \bowtie \beta(\sigma_2(c_{22})\beta^{-1}(h_2))$$
$$\overset{(2.12)}{=} \sum \alpha(c_1) \bowtie {}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)) \otimes \alpha({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta(\sigma_2(c_{22})\beta^{-1}(h_2))$$
$$= (\gamma \otimes \gamma) \triangle_{C_\omega \bowtie_\sigma H} (c \bowtie h).$$

$$(\triangle_{C_\omega \bowtie_\sigma H} \otimes \gamma) \triangle_{C_\omega \bowtie_\sigma H} (c \bowtie h)$$
$$= \sum \triangle_{C_\omega \bowtie_\sigma H}(c_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))) \otimes \alpha({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta(\sigma_2(c_{22}))h_2$$
$$= \sum c_{11} \bowtie \beta^{-1}({}^{\overline{\omega}}\beta(\sigma_1(c_{122})\beta^{-3}(({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))_1))) \otimes$$
$${}^{\overline{\omega}}\alpha^{-2}(c_{121}) \bowtie \sigma_2(c_{122})\beta^{-2}(({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))_2) \otimes \alpha({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta(\sigma_2(c_{22}))h_2$$
$$= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta(\sigma_1(\alpha^{-1}(c_{2211}))\beta^{-3}(({}^{\omega}\beta(\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1)))_1))) \otimes$$
$${}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \sigma_2(\alpha^{-1}(c_{2211}))\beta^{-2}(({}^{\omega}\beta(\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1)))_2) \otimes$$
$$\alpha({}^{\omega}\alpha^{-4}(c_{2212})) \bowtie \beta(\sigma_2(\alpha^{-1}(c_{222})))h_2$$
$$= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}\omega}(\sigma_1(\alpha^{-1}(c_{2212}))\beta^{-1}((\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1))_1))) \otimes$$
$${}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\sigma_2(\alpha^{-1}(c_{2212}))(\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1))_2)) \otimes$$
$${}^{\overline{\overline{\omega}}\omega}\alpha^{-4}(c_{2211}) \bowtie \beta(\sigma_2(\alpha^{-1}(c_{222})))h_2$$

$$
\begin{aligned}
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta^{-1}({}^{\omega}(\beta^3(\sigma_1(\alpha^{-3}(c_{2212})))(\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1))_1)))\otimes \\
&{}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta^2(\sigma_2(\alpha^{-3}(c_{2212})))(\sigma_1(\alpha^{-1}(c_{222}))\beta^{-2}(h_1))_2))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}({}^{\omega}\alpha^{-3}(c_{2211})) \bowtie \beta(\sigma_2(\alpha^{-1}(c_{222})))h_2 \\
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta^{-1}({}^{\omega}(\beta^3(\sigma_1(\alpha^{-3}(c_{2221})))(\sigma_1(\alpha^{-2}(c_{2222}))\beta^{-2}(h_1))_1)))\otimes \\
&{}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta^2(\sigma_2(\alpha^{-3}(c_{2221})))(\sigma_1(\alpha^{-2}(c_{2222}))\beta^{-2}(h_1))_2))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}({}^{\omega}\alpha^{-2}(c_{221})) \bowtie \beta(\sigma_2(\alpha^{-2}(c_{2222})))h_2 \\
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta^{-1}({}^{\omega\overline{\overline{\overline{\omega}}}}\beta^2(\sigma_1(\alpha^{-3}(c_{2222}))\beta^{-3}(h_1))))\otimes \\
&{}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta^3(\sigma_1({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2221})))(\sigma_2(\alpha^{-3}(c_{2222}))\beta^{-2}(h_2))_1))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}({}^{\omega}\alpha^{-2}(c_{221})) \bowtie \beta^2(\sigma_2({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2221})))(\sigma_2(\alpha^{-3}(c_{2222}))\beta^{-2}(h_2))_2 \\
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta^{-1}({}^{\omega\overline{\overline{\overline{\omega}}}}\beta(\sigma_1(\alpha^{-2}(c_{2222}))\beta^{-2}(h_1))))\otimes \\
&{}^{\overline{\omega}}\alpha^{-1}(c_{21}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta(\sigma_1(\alpha^2({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2221}))))\beta^{-1}((\sigma_2(\alpha^{-2}(c_{2222}))\beta^{-1}(h_2))_1)))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}({}^{\omega}\alpha^{-2}(c_{221})) \bowtie \sigma_2(\alpha^2({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2221})))\beta^{-1}((\sigma_2(\alpha^{-2}(c_{2222}))\beta^{-1}(h_2))_2) \\
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}}\beta^{-1}({}^{\omega\overline{\overline{\overline{\omega}}}}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\otimes \\
&{}^{\overline{\omega}}\alpha^{-2}(c_{211}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta(\sigma_1(\alpha^2({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2122}))))\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_1)))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}({}^{\omega}\alpha^{-3}(c_{2121})) \bowtie \sigma_2(\alpha^2({}^{\overline{\overline{\overline{\omega}}}}\alpha^{-4}(c_{2122})))\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_2) \\
&= \sum \alpha(c_1) \bowtie \beta^{-1}({}^{\overline{\omega}\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))\otimes \\
&{}^{\overline{\omega}}\alpha^{-2}(c_{211}) \bowtie \beta^{-1}({}^{\overline{\overline{\omega}}}(\beta(\sigma_1(\alpha(({}^{\omega}\alpha^{-3}(c_{212}))_2)))\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_1)))\otimes \\
&{}^{\overline{\overline{\omega}}}\alpha^{-1}(({}^{\omega}\alpha^{-3}(c_{212}))_1) \bowtie \sigma_2(\alpha(({}^{\omega}\alpha^{-3}(c_{212}))_2))\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_2) \\
&= \sum \alpha(c_1) \bowtie {}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))\otimes \\
&({}^{\omega}\alpha^{-2}(c_{21}))_1 \bowtie \beta^{-1}({}^{\overline{\omega}}(\beta(\sigma_1(({}^{\omega}\alpha^{-2}(c_{21}))_{22}))\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_1)))\otimes \\
&{}^{\overline{\omega}}\alpha^{-2}(({}^{\omega}\alpha^{-2}(c_{21}))_{21}) \bowtie \sigma_2(({}^{\omega}\alpha^{-2}(c_{21}))_{22})\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_2) \\
&= \sum \alpha(c_1) \bowtie {}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)) \otimes \triangle_{C_\omega \bowtie_\sigma H}({}^{\omega}\alpha^{-2}(c_{21}) \bowtie \sigma_2(c_{22})\beta^{-1}(h_2)) \\
&= (\gamma \otimes \triangle_{C_\omega \bowtie_\sigma H})\triangle_{C_\omega \bowtie_\sigma H}(c \bowtie h).
\end{aligned}
$$

This shows that $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega\bowtie_\sigma H}, \varepsilon_{C_\omega\bowtie_\sigma H}, \gamma)$ is a Hom-ω-crossed coproduct.

Conversely, by $(I\otimes\varepsilon_{C_\omega\bowtie_\sigma H})\triangle_{C_\omega\bowtie_\sigma H}(c \bowtie 1_H) = \alpha(c) \bowtie 1_H$ and $(\varepsilon_{C_\omega\bowtie_\sigma H}\otimes I)\triangle_{C_\omega\bowtie_\sigma H}(c \bowtie 1_H) = \alpha(c) \bowtie 1_H$, we have

$$\sum c_1\varepsilon_C({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta^{-1}({}^{\omega}\beta^2(\sigma_1(c_{22})))\varepsilon_H(\sigma_2(c_{22})) = \alpha(c) \bowtie 1_H,$$

and

$$\sum \varepsilon_C(c_1)\,{}^{\omega}\alpha^{-2}(c_{21}) \bowtie \varepsilon_H({}^{\omega}\beta^2(\sigma_1(c_{22})))\beta(\sigma_2(c_{22})) = \alpha(c) \bowtie 1_H.$$

By applying $\varepsilon_C\otimes I_H$ to both sides of the equation above and then by using (2.10), we get (2.13). By $(\triangle_{C_\omega\bowtie_\sigma H}\otimes\gamma)\triangle_{C_\omega\bowtie_\sigma H}(c \bowtie h) = (\gamma\otimes\triangle_{C_\omega\bowtie_\sigma H})\triangle_{C_\omega\bowtie_\sigma H}(c \bowtie h)$, we have

$$\sum c_{11} \bowtie \beta^{-1}({}^{\overline{\omega}}\beta(\sigma_1(c_{122})\beta^{-3}(({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))_1)))\otimes$$

$$\begin{aligned}
&{}^{\overline{\omega}}\alpha^{-2}(c_{121}) \bowtie \sigma_2(c_{122})\beta^{-2}(({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))_2) \otimes \alpha({}^{\omega}\alpha^{-2}(c_{21})) \bowtie \beta(\sigma_2(c_{22}))h_2\\
&= \sum \alpha(c_1) \bowtie {}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)) \otimes ({}^{\omega}\alpha^{-2}(c_{21}))_1 \bowtie\\
&\beta^{-1}({}^{\overline{\omega}}(\beta(\sigma_1(({}^{\omega}\alpha^{-2}(c_{21}))_{22})\beta^{-2}((\sigma_2(c_{22})\beta^{-1}(h_2))_1)))\otimes\\
&{}^{\overline{\omega}}\alpha^{-2}(({}^{\omega}\alpha^{-2}(c_{21}))_{21}) \bowtie \sigma_2(({}^{\omega}\alpha^{-2}(c_{21}))_{22})\beta^{-1}((\sigma_2(c_{22})\beta^{-1}(h_2))_2).
\end{aligned}$$

Now we apply $\varepsilon_C \otimes I_H \otimes I_C \otimes \varepsilon_H \otimes I_C \otimes \varepsilon_H$ to both sides of the equation above and then by using (2.13), we get (2.14). We apply $\varepsilon_C \otimes I_H \otimes \varepsilon_C \otimes I_H \otimes I_C \otimes \varepsilon_H$ to both sides of the equation above and then by using (2.10) and (2.13), we get (2.15). Applying $\varepsilon_C \otimes I_H \otimes \varepsilon_C \otimes I_H \otimes \varepsilon_C \otimes I_H$ to both sides of the equation above and then using (2.10), we get (2.16). □

Let (C,α) and (H,β) be Hom-bialgebras and $\omega : C\otimes H \to H\otimes C$ a linear map. If the Hom-ω-crossed coproduct coalgebra structure with the tensor product Hom-algebra structure makes $C\otimes H$ into a Hom-bialgebra, we then call this Hom-bialgebra a Hom-ω-crossed coproduct bialgebra. The next theorem gives necessary and sufficient conditions for Hom-ω-crossed coproduct to be a Hom-Hopf algebra.

Theorem 2.8 *Let (C,α) and (H,β) be Hom-bialgebras and (C,H) be a $(\alpha,\beta,\sigma,\omega)$-compatible pair and ω be a Hom-conormal linear map. Then $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega\bowtie_\sigma H}, \varepsilon_{C_\omega\bowtie_\sigma H}, \gamma)$ is a Hom-ω-crossed coproduct bialgebra if and only if the conditions (2.13)–(2.16) and the following conditions hold ($\overline{\omega} = \omega$):*

$$\sum \beta(\sigma_1(d))h_1 \otimes \sigma_2(d)h_2 = \sum h_1\beta(\sigma_1(d)) \otimes h_2\sigma_2(d), \tag{2.17}$$

$$\sum \beta^{-1}({}^{\omega}\beta(hg)) \otimes {}^{\omega}(cd) = \sum \beta^{-1}({}^{\omega}\beta(h)\,{}^{\overline{\omega}}\beta(g)) \otimes {}^{\omega}c\,{}^{\overline{\omega}}d, \tag{2.18}$$

$$\sum \beta^4(\sigma_1(cd)) \otimes \beta^3(\sigma_2(cd)) = \sum \beta^4(\sigma_1(c)\sigma_1(d)) \otimes \beta^3(\sigma_2(c)\sigma_2(d)). \tag{2.19}$$

Furthermore, if (C,α) and (H,β) are Hom-Hopf algebras with antipodes S_C and S_H, then $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega\bowtie_\sigma H}, \varepsilon_{C_\omega\bowtie_\sigma H}, \gamma)$ is a Hom-Hopf algebra with an antipode given by

$$S_{C_\omega\bowtie_\sigma H}(c\bowtie h) = \sum S_C({}^{\omega}\alpha^{-2}(c_2)) \bowtie S_H(\sigma_1^{-1}(c_1)\beta^{-3}({}^{\omega}h))\sigma_2^{-1}(c_1).$$

Proof From Theorem 2.7, it follows that $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega\bowtie_\sigma H}, \varepsilon_{C_\omega\bowtie_\sigma H}, \gamma)$ is a Hom-coalgebra if and only if the conditions (2.13)–(2.16) hold. It is clear that $\varepsilon_{C_\omega\bowtie_\sigma H}$ is an algebra map follows that ε_C and ε_H are algebra map. To show that $\triangle_{C_\omega\bowtie_\sigma H}$ is an algebra map of $C_\omega \bowtie_\sigma H$, we compute, for all $c,d \in C$ and $h,g\in H$,

$$\begin{aligned}
&\triangle_{C_\omega\bowtie_\sigma H}((c\bowtie h)(d\bowtie g)) = \triangle_{C_\omega\bowtie_\sigma H}(cd\bowtie hg)\\
&= \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22}d_{22})\beta^{-2}(h_1g_1))) \otimes {}^{\omega}\alpha^{-2}(c_{21}d_{21}) \bowtie \sigma_2(c_{22}d_{22})\beta^{-1}(h_2g_2)\\
&\overset{(2.19)}{=} \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta((\sigma_1(c_{22})\sigma_1(d_{22}))(\beta^{-2}(h_1)\beta^{-2}(g_1))))\otimes\\
&\quad {}^{\omega}\alpha^{-2}(c_{21}d_{21}) \bowtie (\sigma_2(c_{22})\sigma_2(d_{22}))(\beta^{-1}(h_2)\beta^{-1}(g_2))\\
&\overset{(2.2)}{=} \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta((\sigma_1(c_{22})\beta^{-2}(\beta(\sigma_1(d_{22}))\beta^{-1}(h_1)))\beta^{-1}(g_1)))\otimes\\
&\quad {}^{\omega}\alpha^{-2}(c_{21}d_{21}) \bowtie (\sigma_2(c_{22})\beta^{-1}(\sigma_2(d_{22})\beta^{-1}(h_2)))g_2\\
&\overset{(2.17)}{=} \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta((\sigma_1(c_{22})\beta^{-2}(\beta^{-1}(h_1)\beta(\sigma_1(d_{22}))))\beta^{-1}(g_1)))\otimes
\end{aligned}$$

$$
\begin{aligned}
&{}^{\omega}\alpha^{-2}(c_{21}d_{21}) \bowtie (\sigma_2(c_{22})\beta^{-1}(\beta^{-1}(h_2)\sigma_2(d_{22})))g_2 \\
&\overset{(2.2)}{=} \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta((\sigma_1(c_{22})\beta^{-2}(h_1))(\sigma_1(d_{22})\beta^{-2}(g_1))))\otimes \\
&\quad {}^{\omega}(\alpha^{-2}(c_{21})\alpha^{-2}(d_{21})) \bowtie (\sigma_2(c_{22})\beta^{-1}(h_2))(\sigma_2(d_{22})\beta^{-1}(g_2)) \\
&\overset{(2.18)}{=} \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))\,{}^{\overline{\omega}}\beta(\sigma_1(d_{22})\beta^{-2}(g_1)))\otimes \\
&\quad {}^{\omega}\alpha^{-2}(c_{21})\,{}^{\overline{\omega}}\alpha^{-2}(d_{21}) \bowtie (\sigma_2(c_{22})\beta^{-1}(h_2))(\sigma_2(d_{22})\beta^{-1}(g_2)) \\
&= \triangle_{C_\omega \bowtie_\sigma H}(c \bowtie h)\, \triangle_{C_\omega \bowtie_\sigma H}(d \bowtie g).
\end{aligned}
$$

Conversely, if $\triangle_{C_\omega \bowtie_\sigma H}$ is an algebra map, then we have

$$
\begin{aligned}
&\sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22}d_{22})\beta^{-2}(h_1g_1))) \otimes {}^{\omega}\alpha^{-2}(c_{21}d_{21}) \bowtie \sigma_2(c_{22}d_{22})\beta^{-1}(h_2g_2) \\
&\quad = \sum c_1d_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))\,{}^{\overline{\omega}}\beta(\sigma_1(d_{22})\beta^{-2}(g_1))) \otimes {}^{\omega}\alpha^{-2}(c_{21})\,{}^{\overline{\omega}}\alpha^{-2}(d_{21}) \bowtie \\
&\qquad (\sigma_2(c_{22})\beta^{-1}(h_2))(\sigma_2(d_{22})\beta^{-1}(g_2)).
\end{aligned}
$$

Now applying $\varepsilon_C \otimes I_H \otimes I_C \otimes \varepsilon_H$ to both sides of the equation above, we get (2.18). In the above equality, if let $h = g = 1_H$, we apply $\varepsilon_C \otimes I_H \otimes \varepsilon_C \otimes I_H$ to both sides of the equation above, and get (2.19); if let $c = 1_C, g = 1_H$, we apply $\varepsilon_C \otimes I_H \otimes \varepsilon_C \otimes I_H$ to both sides of the equation above, and get (2.17).

Thus, we complete the proof of the first part. To prove that $S_{C_\omega \bowtie_\sigma H}$ is an antipode of $C_\omega \bowtie_\sigma H$, we compute

$$
\begin{aligned}
S_{C_\omega \bowtie_\sigma H}(\gamma(c \bowtie h)) &= \sum S_C({}^{\omega}\alpha^{-1}(c_2)) \bowtie S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-3}({}^{\omega}\beta(h)))\sigma_2^{-1}(\alpha(c_1)) \\
&\overset{(2.12)}{=} \sum S_C(\alpha({}^{\omega}\alpha^{-2}(c_2))) \bowtie S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-2}({}^{\omega}h))\sigma_2^{-1}(\alpha(c_1)) \\
&\overset{(2.11)}{=} \sum S_C(\alpha({}^{\omega}\alpha^{-2}(c_2))) \bowtie S_H(\beta(\sigma_1^{-1}(c_1)\beta^{-3}({}^{\omega}h)))\beta(\sigma_2^{-1}(c_1)) \\
&= \gamma(S_{C_\omega \bowtie_\sigma H}(c \bowtie h)),
\end{aligned}
$$

$$
\begin{aligned}
&S_{C_\omega \bowtie_\sigma H} * I(c \bowtie h) \\
&\quad = \sum S_{C_\omega \bowtie_\sigma H}(c_1 \bowtie \beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))({}^{\omega}\alpha^{-2}(c_{21}) \bowtie \sigma_2(c_{22})\beta^{-1}(h_2)) \\
&\quad = \sum (S_C({}^{\overline{\omega}}\alpha^{-2}(c_{12})) \bowtie \\
&\qquad S_H(\sigma_1^{-1}(c_{11})\beta^{-3}({}^{\overline{\omega}}\beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))))\sigma_2^{-1}(c_{11}))({}^{\omega}\alpha^{-2}(c_{21}) \bowtie \sigma_2(c_{22})\beta^{-1}(h_2)) \\
&\quad = \sum S_C({}^{\overline{\omega}}\alpha^{-2}(c_{12}))\,{}^{\omega}\alpha^{-2}(c_{21}) \bowtie \\
&\qquad (S_H(\sigma_1^{-1}(c_{11})\beta^{-3}({}^{\overline{\omega}}\beta^{-1}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1)))))\sigma_2^{-1}(c_{11}))(\sigma_2(c_{22})\beta^{-1}(h_2)) \\
&\quad \overset{(2.12)}{=} \sum S_C(\alpha^{-1}({}^{\overline{\omega}}\alpha^{-1}(c_{12})))\,{}^{\omega}\alpha^{-2}(c_{21}) \bowtie \\
&\qquad (S_H(\sigma_1^{-1}(c_{11})\beta^{-4}({}^{\overline{\omega}\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\sigma_2^{-1}(c_{11}))(\sigma_2(c_{22})\beta^{-1}(h_2)) \\
&\quad \overset{(2.4)}{=} \sum S_C(\alpha^{-1}({}^{\overline{\omega}}\alpha^{-2}(c_{211})))\,{}^{\omega}\alpha^{-3}(c_{212}) \bowtie \\
&\qquad (S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-4}({}^{\overline{\omega}\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\sigma_2^{-1}(\alpha(c_1)))(\sigma_2(c_{22})\beta^{-1}(h_2)) \\
&\quad \overset{(2.14)}{=} \sum S_C(\alpha^{-1}(({}^{\omega}\alpha^{-2}(c_{21}))_1))\alpha^{-1}(({}^{\omega}\alpha^{-2}(c_{21}))_2) \bowtie \\
&\qquad (S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-3}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\sigma_2^{-1}(\alpha(c_1)))(\sigma_2(c_{22})\beta^{-1}(h_2))
\end{aligned}
$$

$$= \sum \varepsilon_C(\alpha^{-1}({}^{\omega}\alpha^{-2}(c_{21})))1_C \bowtie$$
$$(S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-3}({}^{\omega}\beta(\sigma_1(c_{22})\beta^{-2}(h_1))))\sigma_2^{-1}(\alpha(c_1)))(\sigma_2(c_{22})\beta^{-1}(h_2))$$
$$\overset{(2.10)}{=} \sum \varepsilon_C(c_{21})1_C \bowtie (S_H(\sigma_1^{-1}(\alpha(c_1))\beta^{-1}(\sigma_1(c_{22})\beta^{-2}(h_1)))\sigma_2^{-1}(\alpha(c_1)))(\sigma_2(c_{22})\beta^{-1}(h_2))$$
$$\overset{(2.2)}{=} \sum 1_C \bowtie \beta(S_H(\beta^{-1}(\sigma_1^{-1}(\alpha(c_1))\sigma_1(\alpha(c_2)))\beta^{-2}(h_1)))(\beta^{-1}(\sigma_2^{-1}(\alpha(c_1))\sigma_2(\alpha(c_2)))\beta^{-1}(h_2))$$
$$= \sum \varepsilon_C(c)1_C \bowtie \beta(S_H(\beta^{-1}(h_1)))h_2$$
$$= \varepsilon_C(c)\varepsilon_H(h)1_C \bowtie 1_H.$$

Similarly, we can show that $S_{C_\omega \bowtie_\sigma H}$ satisfies $I * S_{C_\omega \bowtie_\sigma H}(c \bowtie h) = \varepsilon_C(c)\varepsilon_H(h)1_C \bowtie 1_H$ by using (2.10) and (2.14). This completes the proof. □

Remark 2.9 (1) If linear map σ is ordinary, that is, $\sigma(c) = \varepsilon_C(c)1_H \otimes 1_H$, for all $c \in C$, then Hom-ω-crossed coproduct $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ is a Hom-ω-smash coproduct in [10] and Theorems 2.7 and 2.8 are exactly Theorems 2.7 and 2.8 in [10], respectively. Furthermore, if α and β are identity maps, that is, $\gamma(c \otimes h) = c \otimes h$, for all $c \in C$ and $h \in H$, then Hom-ω-crossed coproduct $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ is a ω-smash coproduct in [7] and Theorems 2.7 and 2.8 are exactly Theorem 3.4 and Corollary 4.8 in [7], respectively.

(2) If (C, α) is a left H-comodule Hom-bialgebra with left comodule structure map $\rho : c \to H \otimes C$, $\rho(c) = \sum c_{(-1)} \otimes c_{(0)}$, and set $\omega(c \otimes h) = \sum c_{(0)}h \otimes c_{(0)}$, then Hom-$\omega$-crossed coproduct $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ is a Hom-crossed coproduct in [11] and Theorems 2.7 and 2.8 are exactly Theorems 2.1 and 2.2 in [11], respectively. Furthermore, if α and β are identity maps, that is, $\gamma(c \otimes h) = c \otimes h$, for all $c \in C$ and $h \in H$, then Hom-ω-crossed coproduct $(C_\omega \bowtie_\sigma H, \triangle_{C_\omega \bowtie_\sigma H}, \varepsilon_{C_\omega \bowtie_\sigma H}, \gamma)$ is a crossed coproduct in [8,9] and Theorem 2.7 is exactly Corollary 2.5 in [8] and Corollary 2.1 in [9], respectively, and Theorem 2.8 is exactly Theorem 2.2 in [9].

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