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H-Module Algebra Structures On $M_2(\mathbb{F})$

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Abstract Let $\mathbb{F}$ be an algebraically closed field of characteristic 0, H be an eight-dimensional non-semisimple Hopf algebra which is neither pointed nor unimodular and $M_2(\mathbb{F})$ be the full matrix algebra of 2×2 over $\mathbb{F}$. In this paper, we discuss and classify all H-module algebra structures on $M_2(\mathbb{F})$.

Keywords H-module algebras; square root of matrix; Sweedler algebra

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1. Introduction

The notion of Hopf algebra actions on algebras was introduced by Beattie [1,2] in 1976. A duality theorem for Hopf module algebras was studied by Blattner and Montgomery [3] in 1985. It generalized the corresponding theorem of group actions. Later, many mathematicians were engaged in the theory of Hopf algebra actions [4–6]. In recent years, the classification of finite-dimensional Hopf algebra actions on algebras has drawn people's attention extensively. In [7], Chen and Zhang classified the Yetter-Drinfeld $\mathbb{H}_4$-module algebra structures on $M_2(k)$ over a field k of characteristic $\neq 2$. Gordieko [8] described and classified $\mathbb{H}_4$-module algebra structures on full matrix algebra $M_n(\mathbb{F})$ over an algebraically closed field of characteristic $\neq 2$.

The classification of all Hopf algebras with dimension ≤ 11 over algebraically closed field $\mathbb{F}$ was done by Williams [9], and conformed by Stefan [10]. The result of eight-dimensional Hopf algebras tells us that there are six types of eight-dimensional nonsemisimple Hopf algebras up to isomorphism. Among these, the type $(H''_{C_4})^*$ is neither pointed nor unimodular [11], which makes it much more difficult to describe completely the $(H''_{C_4})^*$-actions on $M_n(\mathbb{F})$ for $n \geq 3$. The aim of this paper is to discuss and classify all $(H''_{C_4})^*$-actions on $M_2(\mathbb{F})$ by the actions of a Hopf subalgebra and theory on square root of matrix.

2. Preliminaries

Firstly, we recall some basic concepts and results on Hopf algebra actions on algebras from [12].

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Let H be a Hopf algebra. Denote the comultiplication and the counit Δ and ε, respectively. We also fix some notations as follows:

- $\mathbb{N}$: the set of natural numbers,
- $[\underline{n}] = \{1, 2, \ldots, n-1, n\}$ for any $n \in \mathbb{N}$,
- $M_n(k)$: the full matrix algebra of $n \times n$-matrices over k,
- E_n: $n \times n$-identity matrix,
- $GL_n(k)$: the multiplicative group of the invertible matrices in $M_n(k)$,
- A standard basis of $M_2(k)$: $E_{11}, E_{22}, E_{12}, E_{21}$,
- $S_1 \times S_2$: the Cartesian product of two sets S_1 and S_2.

Let A be an algebra over a field k. We call A a left H-module algebra if A is a left H-module, and

$$h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_A = \varepsilon(h) 1_A$$

for all $h \in H$ and $a, b \in A$, where $\Delta(h) = \sum_{(h)} (h_1 \otimes h_2)$. The reader can also refer to [13, 14] for more notions about Hopf algebras and Hopf algebra actions.

Let $P_1, P_2 \in M_n(k)$. We say P_1 and P_2 are weakly similar if $P_2 = \alpha O P_1 O^{-1}$ for some $O \in GL_n(k)$ and $\alpha \in k^* = k \backslash \{0\}$. Obviously, two similar matrices are weakly similar, and weak similarity is an equivalence relation on $M_n(k)$. We have the following fact [15].

Lemma 2.1 *Let φ_1 and φ_2 be two inner automorphisms of $M_n(k)$. If $\varphi_1(a) = P_1 a P_1^{-1}, \varphi_2(a) = P_2 a P_2^{-1}$ for all $a \in M_n(k)$, where $P_1, P_2 \in GL_n(k)$, then $\varphi_1 = \varphi_2$ if and only if P_1 is weakly similar to P_2.*

Let $(H''_{C_4})^* = \mathbb{F}\langle g, x \rangle / (g^4 - 1, x^2, gx - \omega xg)$ be the eight-dimensional non-semisimple Hopf algebra which is neither pointed nor unimodular. Its Hopf algebra structure is given by

$$\Delta(g) = g \otimes g - 2g^3 x \otimes gx, \quad \Delta(x) = g^2 \otimes x + x \otimes 1,$$

where $\omega \in \mathbb{F}$ is a primitive 4-th root of unity.

From now on we denote $H''''_{\mathbb{C}_4}$ by H. Let $K = \mathbb{F}1 \oplus \mathbb{F}g^2 \oplus \mathbb{F}x \oplus \mathbb{F}g^2 x$ and $\mathbb{H}_4 = \langle c, \nu \mid c^2 = 1, \nu^2 = 0, \nu g + g\nu = 0 \rangle$ be the four-dimensional Sweedler Hopf algebra. It is clear that K is a Hopf subalgebra of H and $K \cong \mathbb{H}_4$ as Hopf algebras. So we have the following lemma from [8].

Lemma 2.2 *All K-module algebra structures on $M_2(\mathbb{F})$ are as follows. For any $C = (c_{ij})_{2\times 2} \in M_2(\mathbb{F})$,*

(i) $g^2 \cdot C = C$, $x \cdot C = 0$;

(ii) $g^2 \cdot C = \mathrm{diag}(1, -1) C \,\mathrm{diag}(1, -1)$, $x \cdot C = 0$;

(iii) $g^2 \cdot C = \mathrm{diag}(1, -1) C \,\mathrm{diag}(1, -1)$, $x \cdot C = \begin{pmatrix} a(c_{12}+c_{21}) & c_{11}-c_{22} \\ -a(c_{11}-c_{22}) & a(c_{12}+c_{21}) \end{pmatrix}$,

and these module algebras are not isomorphic for different $a \in \mathbb{F}$.

3. The main result and proof

In this section we mainly describe and classify all the H-module algebra structures on $M_2(\mathbb{F})$. First we give the main result of this paper as follows.

Theorem 3.1 *Up to isomorphism, there are four H-module algebra structures on $M_2(\mathbb{F})$ such that for all $C \in M_2(\mathbb{F})$,*

(i) $g \cdot C = C$, $x \cdot C = 0$;

(ii) $g \cdot C = \mathrm{diag}(1,-1)C\,\mathrm{diag}(1,-1)$, $x \cdot C = 0$;

(iii) $g \cdot C = \mathrm{diag}(1,\omega)C\,\mathrm{diag}(1,\omega)$, $x \cdot C = 0$;

(iv) The matrices of g and x as linear transformations on the standard basis are $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 2\omega \\ 0 & 0 & 0 & -\omega \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, *respectively.*

Proof By (i) and (ii) of the Lemma 2.2 in Section 2, we know all K-module structures on $M_2(\mathbb{F})$. That is, the action of g^2 and x on $M_2(\mathbb{F})$ is clear. Therefore, to describe all H-module algebra structures on $M_2(\mathbb{F})$, we only need find the action of g satisfying the following conditions

$$gx \cdot E_{ij} = \omega(xg \cdot E_{ij}),$$
$$g \cdot (E_{ij}E_{kl}) = (g \cdot E_{ij})(g \cdot E_{kl}) - 2(g^2 \cdot (gx \cdot E_{ij}))(gx \cdot E_{kl})$$

for all $i,j,k,l \in [2]$. In particular, when the action of x on $M_2(\mathbb{F})$ is zero, we have

$$g \cdot (E_{ij}E_{kl}) = (g \cdot E_{ij})(g \cdot E_{kl}) \text{ for all } i,j,k,l \in [2].$$

Since $M_2(\mathbb{F})$ as an algebra can be generated by the standard basis, the action of g on $M_2(\mathbb{F})$ is an algebraic automorphism. Since all algebraic automorphisms of $M_2(\mathbb{F})$ are inner, we may assume $g \cdot C = PCP^{-1}$ for some $P \in GL_2(\mathbb{F})$. Thus $g^2 \cdot C = P^2CP^{-2}$. In fact, by (i) and (ii) of Lemma 2.2, we have $g^2 \cdot C = C$ or $g^2 \cdot C = \mathrm{diag}(1,-1)C\,\mathrm{diag}(1,-1)$.

If $g^2 \cdot C = C$, then $P^2C = CP^2$ for all $C \in M_2(\mathbb{F})$ and P must be weakly similar to $\mathrm{diag}(1,1)$ or $\mathrm{diag}(1,-1)$.

If $g^2 \cdot C = \mathrm{diag}(1,-1)C\,\mathrm{diag}(1,-1)$, then P^2 must be weakly similar to $P = \mathrm{diag}(1,\omega)$. Therefore, we have proven (i), (ii) and (iii). The proof of (iv) is much more complex, we need make some preparation.

When the action of x is not zero, the action of g need not be an algebraic automorphism. Hence, to describe the action of g on $M_2(\mathbb{F})$, we need find the matrix of g as linear transformation. By Lemma 2.2, the matrix of g^2 on the standard basis is $B = \mathrm{diag}(1,1,-1,-1)$. Let

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 0 & -a & -1 \\ 0 & 0 & -a & -1 \\ 1 & -1 & 0 & 0 \\ -a & a & 0 & 0 \end{pmatrix}$$

be the matrices of g and the matrix of x, respectively, where $a \in \mathbb{F}$.

First, we find those matrices D such that

$$D^2 = B, \quad DX = \omega X D, \tag{3.1}$$

$$g \cdot (E_{ij}E_{kl}) = (g \cdot E_{ij})(g \cdot E_{kl}) - 2(g^2 \cdot (gx \cdot E_{ij}))(gx \cdot E_{kl}) \text{ for all } i, j, k, l \in [\underline{2}]. \tag{3.2}$$

We know that a square root of a matrix $Y \in M_n(\mathbb{F})$ is a matrix $Z \in M_n(\mathbb{F})$ such that $Z^2 = Y$. It is obvious that the square root of a diagonal matrix must exist. We also know that a square root of an invertible diagonal matrix Y must be diagonalizable, any eigenvalue of the square root is a square root of some eigenvalue of Y (see [15]).

Since B is invertible, every square root of B is diagonalizable. It is clear that every square root of B must be similar to one of the following nine types:

$$\begin{array}{lll} \mathrm{I}: \mathrm{diag}(1,1,\omega,\omega), & \mathrm{II}: \mathrm{diag}(1,1,-\omega,-\omega), & \mathrm{III}: \mathrm{diag}(-1,-1,\omega,\omega), \\ \mathrm{IV}: \mathrm{diag}(-1,-1,-\omega,-\omega), & \mathrm{V}: \mathrm{diag}(1,1,\omega,-\omega), & \mathrm{VI}: \mathrm{diag}(-1,-1,\omega,-\omega), \\ \mathrm{VII}: \mathrm{diag}(1,-1,\omega,\omega), & \mathrm{VIII}: \mathrm{diag}(1,-1,-\omega,-\omega), & \mathrm{IX}: \mathrm{diag}(-1,1,\omega,-\omega). \end{array}$$

Let W be the set of above nine types. Thus D has the form PQP^{-1} for some $P \in GL_4(\mathbb{F})$ and $Q \in W$, and $D^2 = (PQP^{-1})^2 = PQ^2P^{-1} = \mathrm{diag}(1,1,-1,-1)$. Thus $P\mathrm{diag}(1,1,-1,-1) = \mathrm{diag}(1,1,-1,-1)P$. Direct calculation shows that $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ for some $P_i \in GL_2(\mathbb{F})$, $i \in [\underline{2}]$. Let $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. Then $D = PQP^{-1} = \begin{pmatrix} P_1Q_1P_1^{-1} & 0 \\ 0 & P_2Q_2P_2^{-1} \end{pmatrix}$. Since $|P_iP_i^{-1}| = 1$, we may assume $|P_i| = 1$.

Let $P_1 = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}, P_2 = \begin{pmatrix} t_5 & t_6 \\ t_7 & t_8 \end{pmatrix}$ with $t_1t_4 - t_2t_3 = t_5t_8 - t_6t_7 = 1, t_i \in \mathbb{F}$. Then all square roots of B consist of the following nine classes.

(I) : $\mathrm{diag}(1,1,\omega,\omega)$; (II) : $\mathrm{diag}(1,1,-\omega,-\omega)$;

(III) : $\mathrm{diag}(-1,-1,\omega,\omega)$; (IV) : $\mathrm{diag}(-1,-1,-\omega,-\omega)$;

(V) : $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega(t_5t_8 + t_6t_7) & -2\omega t_5t_6 \\ 0 & 0 & 2\omega t_7t_8 & -\omega(t_5t_8 + t_6t_7) \end{pmatrix}$;

(VI) : $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \omega(t_5t_8 + t_6t_7) & -2\omega t_5t_6 \\ 0 & 0 & 2\omega t_7t_8 & -\omega(t_5t_8 + t_6t_7) \end{pmatrix}$;

(VII) : $\begin{pmatrix} t_1t_4 + t_2t_3 & -2t_1t_2 & 0 & 0 \\ 2t_3t_4 & -(t_1t_4 + t_2t_3) & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$;

$$\text{(VIII)}: \begin{pmatrix} t_1t_4+t_2t_3 & -2t_1t_2 & & \\ 2t_3t_4 & -(t_1t_4+t_2t_3) & & \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix};$$

$$\text{(IX)}: \begin{pmatrix} t_1t_4+t_2t_3 & -2t_1t_2 & 0 & 0 \\ 2t_3t_4 & -(t_1t_4+t_2t_3) & 0 & 0 \\ 0 & 0 & \omega(t_5t_8+t_6t_7) & -2\omega t_5t_6 \\ 0 & 0 & 2\omega t_7t_8 & -\omega(t_5t_8+t_6t_7) \end{pmatrix}.$$

First it is easy to see that the matrices $\{D\}'s$ in (I)–(IV) do not satisfy the condition $DX=\omega XD$.

In (V): Let

$$X_1=\begin{pmatrix} -a & -1 \\ -a & -1 \end{pmatrix},\ X_2=\begin{pmatrix} 1 & -1 \\ -a & a \end{pmatrix} \text{ and } D_1=\begin{pmatrix} t_5t_8+t_6t_7 & -2t_5t_6 \\ 2t_7t_8 & -(t_5t_8+t_6t_7) \end{pmatrix}.$$

Then $DX=\omega XD$ is equivalent to $X_1=-X_1D_1$ and $X_2=D_1X_2$. More precisely,

$$X_1=\begin{pmatrix} at_5t_8+at_6t_7+2t_7t_8 & -2at_5t_6-t_5t_8-t_6t_7 \\ at_5t_8+at_6t_7+2t_7t_8 & -2at_5t_6-t_5t_8-t_6t_7 \end{pmatrix};$$
$$X_2=\begin{pmatrix} t_5t_8+t_6t_7+2at_5t_6 & -t_5t_8-t_6t_7-2at_5t_6 \\ 2t_7t_8+at_5t_8+at_6t_7 & -2t_7t_8-at_5t_8-at_6t_7 \end{pmatrix}.$$

Thus

$$\begin{cases} at_5t_8+at_6t_7+2t_7t_8=-a, \\ t_5t_8+t_6t_7+2at_5t_6=1, \\ t_5t_8-t_6t_7=1. \end{cases}$$

Now, we begin to solve the above equations. Firstly,

(1) If $a=0$, then $t_7t_8=0,\ t_6t_7=0$ and $t_5t_8=1$.

(2) If $a\neq 0$, then $t_7t_8=a^2t_5t_6-a,\ t_5t_8+t_6t_7=1-2at_5t_6$.

It follows that

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega(1-2at_5t_6) & -2\omega t_5t_6 \\ 0 & 0 & \omega(2a^2t_5t_6-2a) & -\omega(1-2at_5t_6) \end{pmatrix}.$$

We denote t_5t_6 by t, then

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega(1-2at) & -2\omega t \\ 0 & 0 & \omega(2a^2t-2a) & -\omega(1-2at) \end{pmatrix} \text{ with } t\in\mathbb{F}.$$

In (VI): $DX = \omega XD$ is equivalent to the following equations:

$$\begin{cases} at_5t_8 + at_6t_7 + 2t_7t_8 = a, \\ t_5t_8 + t_6t_7 + 2at_5t_6 = -1, \\ t_5t_8 - t_6t_7 = 1. \end{cases}$$

By solving the above equations we obtain that

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \omega(-1-2at) & -2\omega t \\ 0 & 0 & \omega(2a^2t+2a) & \omega(1+2at) \end{pmatrix} \text{ with } t \in \mathbb{F}.$$

In (VII): $DX = \omega XD$ is equivalent to the following equations:

$$\begin{cases} t_1t_4 + t_2t_3 - 2t_1t_2 = -1; \\ t_1t_4 + t_2t_3 - 2t_3t_4 = 1; \\ t_1t_4 - t_2t_3 = 1. \end{cases}$$

By solving the above equations we get

$$D = \begin{pmatrix} 1+2t & -2-2t & 0 & 0 \\ 2t & -1-2t & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \text{ with } t \in \mathbb{F}.$$

In (VIII): $DX = \omega XD$ is equivalent to the following equations:

$$\begin{cases} t_1t_4 + t_2t_3 - 2t_1t_2 = 1; \\ 2t_3t_4 - t_1t_4 - t_2t_3 = 1; \\ t_1t_4 - t_2t_3 = 1. \end{cases}$$

By solving the above equations we obtain

$$D = \begin{pmatrix} 1+2t & -2t & 0 & 0 \\ 2+2t & -1-2t & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix} \text{ with } t \in \mathbb{F}.$$

In (IX): $DX=\omega XD$ is equivalent to the following equations:

$$\begin{cases} t_1t_4+t_2t_3-2t_1t_2-2at_5t_6-t_5t_8-t_6t_7=0, \\ -at_1t_4-at_2t_3+2at_1t_2-at_5t_8-at_6t_7-2t_7t_8=0, \\ at_1t_4+at_2t_3-2t_3t_4-at_5t_8-at_6t_7-2t_7t_8=0, \\ 2t_3t_4-t_1t_4-t_2t_3-2at_5t_6-t_5t_8-t_6t_7=0, \\ t_1t_4+t_2t_3-2t_3t_4-t_5t_8-t_6t_7-2at_5t_6=0, \\ t_1t_4+t_2t_3-2t_1t_2+t_5t_8+t_6t_7+2at_5t_6=0, \\ -at_1t_4-at_2t_3+2at_3t_4-2t_7t_8-at_5t_8-at_6t_7=0, \\ at_1t_4+at_2t_3-2at_1t_2-2t_7t_8-at_5t_8-at_6t_7=0, \\ t_1t_4-t_2t_3=1, \\ t_5t_8-t_6t_7=1. \end{cases}$$

Solving the above equations reveals that they are incompatible. That is, there is no such D which can satisfy simultaneously the conditions $D^2=B$ and $DX=\omega XD$. In conclusion, we have the following

Lemma 3.2 *$D^2=B$ and $DX=\omega XD$ if and only if D is one of the following forms ($t\in\mathbb{F}$):*

(i) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega(1-2at) & -2\omega t \\ 0 & 0 & \omega(2a^2t-2a) & -\omega(1-2at) \end{pmatrix};$$

(ii) $$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\omega(1+2at) & -2\omega t \\ 0 & 0 & \omega(2a+2a^2t) & \omega(1+2at) \end{pmatrix};$$

(iii) $$\begin{pmatrix} 1+2t & -2-2t & 0 & 0 \\ 2t & -1-2t & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix};$$

(iv) $$\begin{pmatrix} 1+2t & -2t & 0 & 0 \\ 2+2t & -1-2t & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}.$$

For convenience, for all $i,j,k,l\in[\underline{2}]$, we denote by $L(i,j,k,l)$ and $R(i,j,k,l)$ the left side and the right side of (3.2), respectively.

Lemma 3.3 *Let D be any matrix from Lemma 3.2. Then the action of g attached to D satisfies $L(i,j,k,l)=R(i,j,k,l)$ for all $i,j,k,l\in[\underline{2}]$ if and only if*

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 2\omega \\ 0 & 0 & 0 & -\omega \end{pmatrix}.$$

Moreover, the matrices of x and gx on the standard basis are

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{respectively.}$$

Proof

Case 1 The matrices of g and gx are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega - 2\omega at & -2\omega t \\ 0 & 0 & \omega(2a^2t - 2a) & -\omega(1-2at) \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & -\alpha & -1 \\ 0 & 0 & -\alpha & -1 \\ \omega & -\omega & 0 & 0 \\ -\omega\alpha & \omega\alpha & 0 & 0 \end{pmatrix}, \text{ respectively,}$$

where $\alpha, \beta \in \mathbb{F}$. On the one hand, we have

$$L(1,1,1,1) = g \cdot (E_{11}E_{11}) = g \cdot E_{11} = E_{11},$$

$$L(2,2,2,2) = g \cdot (E_{22}E_{22}) = g \cdot E_{22} = E_{22}.$$

On the other hand,

$$R(1,1,1,1) = (g \cdot E_{11})(g \cdot E_{11}) - 2(g^2 \cdot (gx \cdot E_{11}))(gx \cdot E_{11}) = (1+2a)E_{11} + 2aE_{22},$$

$$R(2,2,2,2) = (g \cdot E_{22})(g \cdot E_{22}) - 2(g^2 \cdot (gx \cdot E_{22}))(gx \cdot E_{22}) = 2aE_{11} + (1+2a)E_{22}.$$

Therefore, the equations

$$L(1,1,1,1) = R(1,1,1,1) \text{ and } L(2,2,2,2) = R(2,2,2,2)$$

hold if and only if $1 + 2a = 1,\ 2a = 0$. It is clear that $a = 0$. In addition,

$$L(1,1,2,1) = g(E_{11}E_{21}) = 0,$$

$$R(1,1,2,1) = (g \cdot E_{11})(g \cdot E_{21}) - 2(g^2 \cdot (gx \cdot E_{11}))(gx \cdot E_{21}) = -2\omega(1+t)E_{12}.$$

By $L(1,1,2,1) = R(1,1,2,1)$ we have $t = -1$. At the same time, the matrices of g and x are

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 2\omega \\ 0 & 0 & 0 & -\omega \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ respectively.}$$

Let $S = \{(1,1,1,1), (2,2,2,2), (1,1,2,1)\}, U = \underline{[2]} \times \underline{[2]} \times \underline{[2]} \times \underline{[2]}$ and $U \diagdown S$ be the complement of S in U. Direct calculation shows that $L(i,j,k,l) = R(i,j,k,l)$ for all $(i,j,k,l) \in U \diagdown S$.

Case 2 The matrices of g and gx are

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \omega(-1-2at) & -\omega(2t) \\ 0 & 0 & \omega(2a^2t+2a) & \omega(1+2at) \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & 0 & a & 1 \\ 0 & 0 & a & 1 \\ -\omega & \omega & 0 & 0 \\ \omega a & -\omega a & 0 & 0 \end{pmatrix}, \text{ respectively.}$$

It is easy to see that

$$L(1,1,1,1) = -E_{11}, \quad R(1,1,1,1) = (1+2\alpha)E_{11} + (2\alpha)E_{22},$$

$$L(2,2,2,2) = -E_{22}, \quad R(2,2,2,2) = (2\alpha)E_{11} + (1+2\alpha)E_{22}.$$

Therefore, the equations

$$L(1,1,1,1) = R(1,1,1,1) \text{ and } L(2,2,2,2) = R(2,2,2,2)$$

hold if and only if $-1 = 1 + 2a$, $0 = 2a$. Obviously, such a does not exist.

Similarly, we can also prove that any D in (iii) and (iv) of Lemma 3.2 does not satisfy (3.2). The proof of Lemma is completed. □

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