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Higher Order Teodorescu Operators in Superspace

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Abstract We investigate some fundamental properties of the higher order Teodorescu operators which are defined by the high order Cauchy-Pompeiu formulas in superspace. Moreover, we get an expansion of Almansi type for k-supermonogenic functions in sense of the Teodorescu operators. By the expansion, a Morera type theorem, a Painleve theorem and a uniqueness theorem for k-supermonogenic functions are obtained.

Keywords superspace; Teodorescu operator; k-supermonogenic functions; Morera type theorem; Painleve theorem; uniqueness theorem

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1. Introduction

This operator is nothing else than a 2-dimensional weak singular integral operator over a domain in the complex plane, which is right inverse to the Cauchy-Riemann operator. It reads as follows:

$$Tg(z) = -\frac{1}{\pi}\int\int_{\Omega}\frac{g(\zeta)}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta, \quad \zeta = \xi + i\eta,$$

where Ω is a bounded domain in $\mathbb{C}$. It is worth noticing that the Teodoreseu integral kernel is the Cauchy kernel, which itself is a convolution kernel obtained by translating the fundamental solution $\frac{1}{\pi z}$ to the Cauchy-Riemann operator $\overline{\partial}$ (see [1]). Recently, great progress has been made in Teodorescu operators. Without claiming completeness, we mention some of them. One is the Teodorescu operator in several complex variables. In this case, the kernel of the operator is not holomorphic but still harmonic, which formally mimics the Martinelli-Bochner kernel [2]. Another direction is the Teodorescu operator in the Euclidean Clifford analysis which is a hypercomplex function theory with functions defined in the Euclidean space $\mathbf{R}^m$ and taking values in an orthogonal Clifford algebra. In this case, the kernel of the operator is monogenic, obtained by the fundamental solution for the Dirac operator ∂_x (i.e., the vector derivative $\sum_{i=1}^{m} e_i\partial_{x_i}$) (see [3]). However, we are interested in the Teodorescu operator in superspace. The kernel of

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the operator is supermonogenic, obtained by the fundamental solution for the Dirac operator in superspace.

Superspaces, developed during the second half of the previous century, are spaces equipped with both a set of commuting variables and a set of anti-commuting variables (generating the so-called Grassmann algebra) in order to describe the properties of bosons and fermions in Quantum Mechanics. In recent years, Sommen, DeBie and others have studied a superspace of dimension $(m, 2n)$ with a novel approach. Their approach is not based on algebraic geometry as in [4], nor on differential geometry as in [5], but Clifford analysis [6]. They constructed a Laplace and a Dirac operator, acting on functions depending on both commuting and anti-commuting variables [7]. Furthermore the fundamental solutions of these differential operators were obtained in [8]. Besides, they defined integration in superspace by Berezin integration [9,10]. Moreover, using a distributional approach to integration in superspace, they obtained a Stokes formula, a Morera theorem, etc [11]. Based on their work, we want to investigate the Almansi type expansion in superspace.

In 1899, the Almansi expansion for polyharmonic functions was given, which was equivalent to the Fischer decomposition for polynomials [12]. The results in the case of complex analysis and Clifford analysis have been well developed in [13,14]. But as we know, up to now there is no hint on the Almansi expansion in superspace. We try to fill part of this gap. In [15], we studied the Almansi type expansion in superspace by constructing special integral operators. Then, we discussed the Almansi type expansion in superspace from normalized systems [16,17]. In this paper, we investigate the Almansi type expansion in superspace by the high order Teodorescu operators, inspired by an expansion of polyanalytic functions applying the iterate operator of Pompeiu due to Pascai [18].

The paper is organized as follows. We start with a short introduction to Clifford analysis in $\mathbf{R}^{m|2n}$. We introduce the higher order Cauchy-Pompeiu formula in superspace [19]. By the formula, the operators T_i are defined, which are the so-called higher order Teodorescu operators. These operators are singular integral operators over a domain in $\mathbf{R}^{m|2n}$, with Berezin integration. The kernel functions of the operators are described with the help of the fundamental solutions for all natural powers of the super Dirac operator (i.e., the Dirac operator in $\mathbf{R}^{m|2n}$), where the kernel functions containing anti-commuting variables are more complicated than the kernel functions in $\mathbf{R}^m$. Next, we investigate some basic properties of the operators, such as the relations between the operators T_i and Cauchy type integral, the relations of the operators T_i and T_{i-1}. The most important is to obtain an expansion for k-supermonogenic functions applying the integral operators. Besides, by the expansion, we obtain a Morera type theorem, a Painleve theorem and a uniqueness theorem for k-supermonogenic functions.

2. Preliminaries

2.1. Superspace

We study the superspace

$$\mathbf{R}^{m|2n} = \{(x, \grave{x}) \mid x = (x_1, \ldots, x_m) \in \mathbf{R}^m, \grave{x} = (\grave{x}_1, \ldots, \grave{x}_{2n}) \in \Lambda_{2n}\},$$

by introducing

$$\mathrm{Alg}(x_i, \grave{x}_j) \otimes \mathrm{Alg}(e_i, \grave{e}_j) = \mathrm{Alg}(x_i, e_i; \grave{x}_j, \grave{e}_j),\ i = 1, \ldots, m;\ j = 1, \ldots, 2n,$$

where the scalar algebra $\mathcal{P} = \mathrm{Alg}(x_i, \grave{x}_j)$ and the Clifford algebra $\mathcal{C} = \mathrm{Alg}(e_i, \grave{e}_j)$, respectively satisfy the following relations:

$$\begin{cases} x_i x_j = x_j x_i, & i, j \in \{1, \ldots, m\}, \\ \grave{x}_i \grave{x}_j = -\grave{x}_j \grave{x}_i, & i, j \in \{1, \ldots, 2n\}, \\ x_i \grave{x}_j = \grave{x}_j x_i, & i \in \{1, \ldots, m\},\ j \in \{1, \ldots, 2n\}, \end{cases}$$

and

$$\begin{cases} e_j e_k + e_k e_j = -2\delta_{jk}, & j, k \in \{1, \ldots, m\}, \\ \grave{e}_{2j} \grave{e}_{2k} - \grave{e}_{2k} \grave{e}_{2j} = 0, & j, k \in \{1, \ldots, n\}, \\ \grave{e}_{2j-1} \grave{e}_{2k-1} - \grave{e}_{2k-1} \grave{e}_{2j-1} = 0, & j, k \in \{1, \ldots, n\}, \\ \grave{e}_{2j-1} \grave{e}_{2k} - \grave{e}_{2k} \grave{e}_{2j-1} = \delta_{jk}, & j, k \in \{1, \ldots, n\}, \\ e_j \grave{e}_k + \grave{e}_k e_j = 0, & j \in \{1, \ldots, m\},\ k \in \{1, \ldots, 2n\}. \end{cases}$$

Moreover, the elements of the two algebras can commute with each other. When $n = 0$, we have that $\mathcal{C} \cong \mathbf{R}_{0,m}$, with $\mathbf{R}_{0,m}$ being the standard orthogonal Clifford algebra.

When $m = 0$, $\mathcal{P} \otimes \mathcal{C} = \Lambda_{2n} \otimes \mathcal{W}_{2n}$, where the Grassmann algebra $\Lambda_{2n} = \mathrm{Alg}(\grave{x}_1, \ldots, \grave{x}_{2n})$, and the Weyl algebra $\mathcal{W}_{2n} = \mathrm{Alg}(\grave{e}_1, \ldots, \grave{e}_{2n})$.

The most important element of the algebra $\mathcal{P} \otimes \mathcal{C}$ is the super vector variable:

$$x = \underline{x} + \underline{\grave{x}},\ \text{with } \underline{x} = \sum_{i=1}^{m} x_i e_i,\ \text{and } \underline{\grave{x}} = \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j.$$

One can calculate that

$$x^2 = \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{x}_{2j} - \sum_{i=1}^{m} x_i^2 = \underline{\grave{x}}^2 + \underline{x}^2,$$

where $\underline{x}^2 = -\sum_{i=1}^{m} x_i^2$.

Finally, we define a more general function space as:

$$C^k(\Omega) \otimes \Lambda_{2n} \otimes \mathcal{C},$$

where $C^k(\Omega)$ denotes the space of the k-times continuously differentiable real-valued functions defined in some domain $\Omega \subset \mathbf{R}^m$. We use the notation $C^k(\Omega)_{m|2n} = C^k(\Omega) \otimes \Lambda_{2n}$.

2.2. Differential operators in superspace

The left super Dirac operator is defined by

$$\partial_x \cdot = \partial_{\underline{\grave{x}}} \cdot - \partial_{\underline{x}} \cdot = 2\sum_{j=1}^{n} (\grave{e}_{2j} \partial_{\grave{x}_{2j-1}} \cdot - \grave{e}_{2j-1} \partial_{\grave{x}_{2j}} \cdot) - \sum_{i=1}^{m} e_i \partial_{x_i} \cdot .$$

Similarly, we define the right super Dirac operator by

$$\cdot\partial_x = -\cdot\partial_{\grave{\underline{x}}} - \cdot\partial_{\underline{x}}.$$

A direct calculation shows that $\partial_x(x) = (x)\partial_x = m - 2n = M$, where M is the so-called super-dimension. The physical meaning of this super-dimension was discussed in [9].

The square of the super Dirac operator is the super Laplace operator

$$\Delta = \partial_x^2\cdot = 4\sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}}\partial_{\grave{x}_{2j}}\cdot - \sum_{i=1}^{m} \partial_{x_i}^2\cdot .$$

Definition 2.1 *A function $f(x) \in C^k(\Omega)_{m|2n}\otimes C$ is called left k-supermonogenic (k-supermonogenic in short) in an open set $\Omega \subset \mathbf{R}^m$ if $\partial_x^k f(x) = 0$.*

2.3. Integration in superspace

The integration in superspace is defined by

$$\int_{\mathbf{R}^{m|2n}} \cdot = \int_{\mathbf{R}^m} \mathrm{d}V(\underline{x}) \int_B \cdot = \int_B \int_{\mathbf{R}^m} \cdot \mathrm{d}V(\underline{x}),$$

where $\mathrm{d}V(\underline{x}) = \mathrm{d}x_1 \cdots \mathrm{d}x_m$ is the usual Lebesgue measure in $\mathbf{R}^m$, and

$$\int_B \cdot = \pi^{-n}\partial_{\grave{x}_{2n}} \cdots \partial_{\grave{x}_1}\cdot$$

used on Λ^{2n} is the so-called Berezin integration.

2.4. Fundamental solutions

The fundamental solutions for the natural powers of the classical Laplace operator Δ_b are well-known [13].

We denote by $\nu_{2l}^{m|0}(\underline{x}), l = 1, 2, \ldots$, a sequence of such fundamental solutions, satisfying

$$\begin{cases} \Delta_b^j \nu_{2l}^{m|0}(\underline{x}) = \nu_{2l-2j}^{m|0}(\underline{x}), & j < l, \\ \Delta_b^l \nu_{2l}^{m|0}(\underline{x}) = \delta(\underline{x}), & j = l, \end{cases}$$

where $\delta(\underline{x})$ is the classical Dirac distribution in $\mathbf{R}^m$. Their explicit form depends both on the dimension m and on l. More specifically, in the case where m is odd we have that

$$\nu_{2l}^{m|0}(\underline{x}) = \frac{r^{2l-m}}{\gamma_{l-1}},\ \gamma_l = (-1)^{l+1}(2-m)4^l l! \frac{\Gamma(l+2-\frac{m}{2})}{\Gamma(2-\frac{m}{2})} \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})},\ r = \sqrt{-\underline{x}^2}.$$

The formulae for m even are more complicated and can be found in [13].

In the sequel, we will show the fundamental solutions for the natural powers of the super Laplace and Dirac operators.

Lemma 2.2 ([8]) *The function $\nu_{2k}^{m|2n}(x)$ defined by*

$$\nu_{2k}^{m|2n}(x) = \pi^n \sum_{l=0}^{n} 4^l \frac{(l+k-1)!}{(n-l)!(k-1)!} \nu_{2l+2k}^{m|0}(\underline{x})\grave{\underline{x}}^{2n-2l}$$

is a fundamental solution for the operator Δ^k.

The lemma means that $\Delta^k \nu_{2k}^{m|2n}(x) = \delta(x)$, where $\delta(x) = \delta(\underline{x})\frac{\pi^n}{n!}(\underline{\grave{x}})^{2n}$ is the super Dirac distribution in $\mathbf{R}^{m|2n}$.

In a similar vein one can obtain the fundamental solution $\nu_{2k+1}^{m|2n}(x)$ for the operator $\Delta^k\partial_x$, by calculating $\partial_x \nu_{2k+2}^{m|2n}(x)$. This leads to

Lemma 2.3 ([8]) *The function $\nu_{2k}^{m|2n}(x)$ defined by*

$$\begin{aligned}\nu_{2k+1}^{m|2n}(x) =&\pi^n \sum_{l=0}^{n-1} 2\frac{4^l(l+k)!}{(n-l-1)!k!}\nu_{2l+2k+2}^{m|0}(\underline{x})\underline{\grave{x}}^{2n-2l-1} - \\ &\pi^n \sum_{l=0}^{n} \frac{4^l(l+k)!}{(n-l)!k!}\nu_{2l+2k+1}^{m|0}(\underline{x})\underline{\grave{x}}^{2n-2l}\end{aligned}$$

is a fundamental solution for the operator $\Delta^k\partial_x$.

The lemma means that $\partial_x^{2k+1}\nu_{2k+1}^{m|2n}(x) = \nu_{2k+1}^{m|2n}(x)\partial_x^{2k+1} = \delta(x)$, where $\delta(x)$ is the super Dirac distribution in $\mathbf{R}^{m|2n}$.

2.5. Higher order Cauchy-Pompeiu formula in superspace

In complex analysis, a special case of the Cauchy-Pompeiu formula is the Cauchy formula of holomorphic functions which is deduced from the Grass theorem. Analogously to this, the Cauchy-Pompeiu formula in superspace is also a consequence of the Stokes formula in superspace [11]. Inspired by the above-mentioned results, we developed further these ideas to construct the higher order Cauchy-Pompeiu formulae in superspace [19].

The Clifford analysis version of Stokes formula in $\mathbf{R}^{m|2n}$ are given in the following lemma.

Lemma 2.4 ([11]) *For $\beta \in \Lambda_{2n}$, and $f, g \in C^1(\Omega)_{m|2n} \otimes \mathcal{C}$, with $\overline{\Omega} \subset \mathbf{R}^m$ a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Omega$ the following holds:*

$$\int_\Omega \int_B [(f\widehat{\beta}\partial_x)g + f\beta(\partial_x g)]\mathrm{d}V(\underline{x}) = -\int_{\partial\Omega}\int_B f\beta \mathrm{d}\sigma_{\underline{x}} g + \int_\Omega \int_B f(\beta\partial_{\underline{\grave{x}}})g\mathrm{d}V(\underline{x}).$$

Note that with $f\widehat{\beta}\partial_{\underline{\grave{x}}}$ we mean the fermionic Dirac operator acting from the left on $f\beta$ but β is not derived. We cannot switch β and $\partial_{\underline{\grave{x}}}$ from place because of the anticommuting variables.

In the sequel, we will need the following lemma.

Lemma 2.5 ([6]) *Let Ω be as stated before, and let $f \in C^1(\Omega) \otimes \mathbf{R}_{0,m}$. Let $B(\underline{y}, R)$ be a ball of radius R and center $\underline{y}$ contained in Ω. Further let the functions $\nu_k^{m|0}(\underline{x}-\underline{y})$ be defined as in Section 2.4. Then the following holds:*

$$\lim_{R\to 0}\int_{B(\underline{y},R)} \nu_k^{m|0}(\underline{x}-\underline{y})f(\underline{x})\mathrm{d}V(\underline{x}) = 0,\ \ k \in \mathbf{N},$$

$$\lim_{R\to 0}\int_{\partial B(\underline{y},R)} \nu_k^{m|0}(\underline{x}-\underline{y})\mathrm{d}\sigma_{\underline{x}} f(\underline{x}) = \begin{cases} 0, & k > 1, \\ -f(\underline{y}), & k = 1. \end{cases}$$

Now we can show the higher order Cauchy-Pompeiu formulas in superspace which is the most important lemma in this section, as follows:

Lemma 2.6 ([19]) *Let Ω be as stated before and $\overline{\Omega}$ a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Omega$. Let $f(x) \in C^k(\Omega)_{m|2n} \otimes \mathcal{C}$ and let the functions $\nu_j^{m|2n}(x-y)$ be the fundamental solutions for the powers of the super Dirac operator ∂_x. Then*

$$\int_{\partial\Omega}\int_B \sum_{j=1}^{k}(-1)^{j+1}\nu_j^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^{j-1}f(x) + (-1)^{k+1}\int_{\Omega}\int_B \nu_k^{m|2n}(x-y)[\partial_x^k f(x)]\mathrm{d}V(\underline{x})$$

$$= \begin{cases} 0, & \underline{y} \in \mathbf{R}^m \backslash \overline{\Omega}. \\ -f(y), & \underline{y} \in \Omega. \end{cases} \tag{2.1}$$

3. Higher order Teodorescu operators in superspace

As a consequence of this higher order Cauchy-Pompeiu formulae in superspace, we shall be able to define higher order Teodorescu operators in superspace. In other words, the last term in equality (2.1) for the case $\underline{y} \in \Omega$ suggests the following definition of higher order Teodorescu operators in superspace.

Definition 3.1 *Let Ω be as stated before. If $f(x) \in C^i(\Omega)_{m|2n} \otimes \mathcal{C}$, then we define the operators T_i by*

$$T_i f(y) = (-1)^i \int_{\Omega}\int_B \nu_i^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x}), \quad i = 1, 2, \ldots, \tag{3.1}$$

where $\nu_i^{m|2n}(x-y)$ are the fundamental solutions for the operators ∂_x^i. It means that in the distributional sense that $\partial_x^i \nu_i^{m|2n}(x-y) = \delta(x-y)$, where $\delta(x-y)$ stands for the super Dirac distribution in $\mathbf{R}^{m|2n}$. The operators T_i are the so-called higher order Teodorescu operators in superspace, and regarded as the Cauchy principal value of singular integral operators. For $i = 1$, the operator T_1 is the Teodorescu operator T, which is the right inverses of the super Dirac operator ∂_x. It is a meaningful generalization of the Teodorescu operator in the Euclidean Clifford analysis which is the right inverse of the Dirac operator $\partial_{\underline{x}}$. Especially we denote f as $T_0 f$.

We now begin with investigating the links between the operators T_i and Cauchy type integral.

Theorem 3.2 *Let Ω be as stated before and let $f(x) \in C^k(\Omega)_{m|2n} \otimes \mathcal{C}$. If $f(x)$ is k-supermonogenic, then for $\underline{y} \in \Omega$,*

$$T_i f(y) = \sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial\Omega}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x), \quad i = 1, 2, \ldots. \tag{3.2}$$

Proof Suppose that $f(x) \in C^k(\Omega)_{m|2n} \otimes \mathcal{C}$ is k-supermonogenic. We consider a ball $\Gamma = B(\underline{y}, R) \subset \Omega$. By Lemma 2.4 for the case $\beta = 1$, we obtain

$$\sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial(\Omega\backslash\Gamma)}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)$$

$$
\begin{aligned}
&= (-1)^{i+1}\int_{\partial(\Omega\backslash\Gamma)}\int_B \nu_{i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}} f(x)+\\
&\quad (-1)^{i+2}\int_{\partial(\Omega\backslash\Gamma)}\int_B \nu_{i+2}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x f(x)+\cdots+\\
&\quad (-1)^{i+k}\int_{\partial(\Omega\backslash\Gamma)}\int_B \nu_{i+k}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^k f(x)\\
&= (-1)^{i}\int_{\Omega\backslash\Gamma}\int_B \Big[(\nu_{i+1}^{m|2n}(x-y)\partial_x)f(x)+\nu_{i+1}^{m|2n}(x-y)(\partial_x f(x))\Big]\mathrm{d}V(\underline{x})+\\
&\quad (-1)^{i+1}\int_{\Omega\backslash\Gamma}\int_B \Big[(\nu_{i+2}^{m|2n}(x-y)\partial_x)f(x)+\nu_{i+2}^{m|2n}(x-y)(\partial_x^2 f(x))\Big]\mathrm{d}V(\underline{x})+\cdots+\\
&\quad (-1)^{i+k-1}\int_{\Omega\backslash\Gamma}\int_B \Big[(\nu_{i+k}^{m|2n}(x-y)\partial_x)f(x)+\nu_{i+k}^{m|2n}(x-y)(\partial_x^k f(x))\Big]\mathrm{d}V(\underline{x})\\
&= (-1)^{i}\int_{\Omega\backslash\Gamma}\int_B \Big[\nu_{i}^{m|2n}(x-y)f(x)+\nu_{i+1}^{m|2n}(x-y)(\partial_x f(x))\Big]\mathrm{d}V(\underline{x})+\\
&\quad (-1)^{i+1}\int_{\Omega\backslash\Gamma}\int_B \Big[\nu_{i+1}^{m|2n}(x-y)f(x)+\nu_{i+2}^{m|2n}(x-y)(\partial_x^2 f(x))\Big]\mathrm{d}V(\underline{x})+\cdots+\\
&\quad (-1)^{i+k-1}\int_{\Omega\backslash\Gamma}\int_B \Big[\nu_{i+k-1}^{m|2n}(x-y)f(x)+\nu_{i+k}^{m|2n}(x-y)(\partial_x^k f(x))\Big]\mathrm{d}V(\underline{x})\\
&= (-1)^{i}\int_{\Omega\backslash\Gamma}\int_B \nu_{i}^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x}).
\end{aligned}
$$

On the one side, we have

$$
\begin{aligned}
&\lim_{R\to 0}\sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial(\Omega\backslash\Gamma)}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)\\
&= \lim_{R\to 0}(-1)^i\int_{\Omega\backslash\Gamma}\int_B \nu_i^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x})\\
&= (-1)^i\int_{\Omega}\int_B \nu_i^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x})-(-1)^i\lim_{R\to 0}\int_{\Gamma}\int_B \nu_i^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x}).
\end{aligned}
$$

Now we calculate the second integral in previous equality.

Case 1 $i=2s$ is even. From Lemma 2.2, we have

$$
\begin{aligned}
&\lim_{R\to 0}\int_{\Gamma}\int_B \nu_{2s}^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x})\\
&\quad = \lim_{R\to 0}\int_{\Gamma}\int_B \pi^n\sum_{l=0}^{n}4^l\frac{(l+s-1)!}{(n-l)!(s-1)!}\nu_{2l+2s}^{m|0}(\underline{x}-\underline{y})(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2l}f(x)\mathrm{d}V(\underline{x}).
\end{aligned}
$$

Due to the linearity it suffices to prove this formula for $f(x)=f_1(\underline{x})f_2(\grave{\underline{x}})$, where $f_1(\underline{x})\in C^k(\Omega)\otimes\mathbf{R}_{0,m}$ and $f_2(\grave{\underline{x}})\in\Lambda_{2n}\otimes\mathcal{W}_{2n}$. By Lemma 2.5, we obtain

$$
\lim_{R\to 0}\int_{\Gamma}\nu_{2l+2s}^{m|0}(\underline{x}-\underline{y})f_1(\underline{x})\mathrm{d}V(\underline{x})\int_B(\grave{\underline{x}}-\grave{\underline{y}})^{2n-2l}f_2(\grave{\underline{x}})=0.
$$

Case 2 $i=2s+1$ is odd. By Lemmas 2.3 and 2.5, it can be similarly proved. Therefore, we have

$$
\lim_{R\to 0}\int_{\Gamma}\int_B \nu_i^{m|2n}(x-y)f(x)\mathrm{d}V(\underline{x})=0. \tag{3.3}
$$

On the other side, we have

$$\begin{aligned}&\lim_{R\to 0}\sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial(\Omega\setminus\Gamma)}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)\\ &=\sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial\Omega}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)-\\ &\quad\sum_{j=0}^{k-1}(-1)^{j+i+1}\lim_{R\to 0}\int_{\partial\Gamma}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x).\end{aligned}$$

The proof of the second integral in previous equality is similar to the equality (3.3). Thus, we have

$$\lim_{R\to 0}\int_{\partial\Gamma}\int_B \nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)=0,\quad j=0,1,2,\ldots,k-1.$$

Combining these two sides, we have the conclusion. □

This gives rise to the following properties of higher order Teodorescu operators in superspace.

Corollary 3.3 *Let Ω be as stated before and let $f(x)\in C^k(\Omega)_{m|2n}\otimes\mathcal{C}$. If $f(x)$ is k-supermonogenic, then for $\underline{y}\in\Omega$, $\partial_y T_i f(y)=T_{i-1}f(y)$, $i=1,2,\ldots,k$.*

Proof From Theorem 3.2, we have

$$\begin{aligned}\partial_y T_i f(y)=&\sum_{j=0}^{k-1}(-1)^{j+i+1}\int_{\partial\Omega}\int_B \partial_y\nu_{j+i+1}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)\\ =&\sum_{j=0}^{k-1}(-1)^{j+i}\int_{\partial\Omega}\int_B \partial_y\nu_{j+i}^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x)\\ =&T_{i-1}f(y).\quad\square\end{aligned}$$

Corollary 3.3 implies that $T_k f$ provides a particular solution to the inhomogeneous equation $\partial_y^k\omega=f(y)$, where $f(y)$ is k-supermonogenic.

These above results pave the way for investigating the Almansi type expansion for k-supermonogenic functions in the sequel.

Theorem 3.4 *Let Ω be as stated before, and $\overline{\Omega}$ be a compact oriented differentiable m-dimensional manifold with smooth boundary $\partial\Omega$. If the function $f(y)\in C^k(\Omega)_{m|2n}\otimes\mathcal{C}$ is k-supermonogenic, then there exist unique supermonogenic functions $f_0,\ldots,f_{k-1}$ in the domain Ω such that*

$$f(y)=f_0(y)+T_1f_1(y)+\cdots+T_{k-1}f_{k-1}(y),\tag{3.4}$$

where

$$f_j(y)=-\int_{\partial\Omega}\int_B \nu_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x),\quad j=0,\ldots,k-1,\tag{3.5}$$

and the function $\nu_1^{m|2n}(x-y)$ is the fundamental solution for the super Dirac operator.

Conversely, if the functions $f_0,\ldots,f_{k-1}$ are supermonogenic, then the sum in (3.4) is a k-supermonogenic function.

Proof First, for any $\underline{y} \in \Omega$,

$$\partial_y f_j(y) = -\int_{\partial\Omega}\int_B \partial_y \nu_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x) = 0.$$

Secondly, from Corollary 3.3, we have

$$\begin{aligned}\partial_y^j f(y) =&\partial_y^j\left[f_0(y) + T_1 f_1(y) + \cdots + T_{k-1}f_{k-1}(y)\right]\\ =&f_j(y) + T_1[f_{j+1}(y) + \cdots + T_{k-1-j}f_{k-1}(y)]\\ =&f_j(y) + T_1(\partial_y^{j+1}f(y)).\end{aligned} \tag{3.6}$$

On the other side, by Lemma 2.6 for the case $k = 1$, we obtain

$$\begin{aligned}\partial_y^j f(y) =& -\int_{\partial\Omega}\int_B \nu_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x) - \int_{\Omega}\int_B \nu_1^{m|2n}(x-y)\partial_x^{j+1}f(x)\mathrm{d}V(\underline{x})\\ =& -\int_{\partial\Omega}\int_B \nu_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x) + T_1(\partial_y^{j+1}f(y)).\end{aligned} \tag{3.7}$$

By comparing (3.6) with (3.7), we have

$$f_j(y) = -\int_{\partial\Omega}\int_B \nu_1^{m|2n}(x-y)\mathrm{d}\sigma_{\underline{x}}\partial_x^j f(x).$$

Conversely, if the functions $f_0, \ldots, f_{k-1}$ are supermonogenic, then by Corollary 3.3, we have

$$\partial_y^k(f_0(y) + T_1 f_1(y) + \cdots + T_{k-1}f_{k-1}(y)) = 0.$$

Therefore, we obtain the conclusion. □

As the reader may have noticed, the new issue of this result is that it is valid for the domain in superspace, without assuming in the star domain. Furthermore, it connects k-supermonogenic functions with supermonogenic functions, and leads to the properties of k-supermonogenic functions in the next section.

4. Fundamental theorems for k-supermonogenic functions

Lemma 4.1 ([11]) *A function $f \in C^0(\Omega)_{m|2n} \otimes \mathcal{C}$ (with Ω being an open subset of $\mathbf{R}^m$) is supermonogenic in Ω if and only if*

$$\int_{\partial I}\int_B \alpha \mathrm{d}\sigma_{\underline{x}} f - \int_I\int_B (\alpha\partial_{\dot{\underline{x}}}) f \mathrm{d}V(\underline{x}) = 0, \tag{4.1}$$

for every interval $I \subset \Omega$ and for every α in Λ_{2n}.

Applying Lemma 4.1 (i.e., Morera theorem for supermonogenic functions), we get the following Morera type theorem for k-supermonogenic functions.

Theorem 4.2 *Let Ω be as stated before. A function $f \in C^k(\Omega)_{m|2n} \otimes \mathcal{C}$ is k-supermonogenic if and only if*

$$\int_{\partial I}\int_B \alpha \mathrm{d}\sigma_{\underline{y}}[\partial^j f - T_1(\partial^{j+1}f)] - \int_I\int_B (\alpha\partial_{\dot{\underline{y}}})[\partial^j f - T_1(\partial^{j+1}f)]\mathrm{d}V(\underline{y}) = 0,$$

where $j = 0, 1, \ldots, k-1$, $I \subset \Omega$, and $\alpha \in \Lambda_{2n}$.

In [15], a Painleve theorem and a uniqueness theorem for supermonogenic functions were given. By Theorem 3.4, we obtain a Painleve theorem and a uniqueness theorem for k-supermonogenic functions as follows:

Theorem 4.3 *Let Ω be as stated before and Ω' be open in $\mathbf{R}^{m-1}$ such that $\Omega' \cap \mathbf{R}^m = \Omega'$. If $f(x) \in C^k(\Omega)_{m|2n} \otimes \mathcal{C}$ is k-supermonogenic in $\Omega \setminus \Omega'$, then $f(x)$ is k-supermonogenic in Ω.*

Theorem 4.4 *If a function f is k-supermonogenic in the open connected set $\Omega \subset \mathbf{R}^m$ and vanishes in the open set $\Sigma \subset \Omega$, then f is identically zero in Ω.*

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