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Higher-Order Attraction of Pullback Attractors for Parabolic Equations Involving Grushin Operators

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Abstract The higher-order attraction of pullback attractors for non-autonomous parabolic equations involving Grushin operators is considered. Firstly, the maximum principle is studied. Next, the higher-order integrability of the difference of weak solutions is established. Finally, the higher-order attraction is proved.

Keywords non-autonomous dynamical system; higher-order attraction; maximum principle; pullback attractor; Grushin operators

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1. Introduction

The long time behaviour of dynamical systems is one of the most important problems of modern mathematical physics. By the study of attractor, we can reduce the original system and capture more information implied in systems. For autonomous system, global attractor is usually used to study the long time behaviour of dynamical systems [1]. As extension of the concept of global attractor, in 1986, Haraux [2] provided uniform attractor apt to the asymptotic behaviour of non-autonomous systems. It is remarkable that the conditions ensuring the existence of the uniform attractor parallel those for autonomous case. However, one drawback of the uniform attractor is that it need not be invariant. Moreover, it is well-known that the trajectories may be unbounded for many non-autonomous systems when time tends to infinity and there does not exist the uniform attractor for these systems. In order to overcome this drawback, pulback attractor has been introduced for non-autonomous case. In the recent years, the existence of pullback attractors has been proved for some partial differential equations [3–5]. Meanwhile, with new problems and different force terms, pullback $\mathscr{D}$-attractor has been introduced [6].

One of the class of degenerate equations ([7–11]) that has been studied widely in recent years is the class of equations involving an operator of Grushin type

$$G_r u = \triangle_{x_1} u + |x_1|^{2r} \triangle_{x_2} u, \quad (x_1, x_2) \in \mathcal{O} \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},\ r \geqslant 0,$$

which was introduced firstly in [12]. As $r = 0$, then $G_0 = \Delta$ and (1) reduces to a semilinear reaction-diffusion equation, and G_r, when $r > 0$, is not elliptic in domains in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

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intersecting with the hyperplane $\{x_1 = 0\}$.

For autonomous system with Grushin operators, that is, the force term is independent of time, [9,11] considered the long time behaviour of solution. For non-autonomous system, Anh [8] considered the existence of pullback $\mathscr{D}$-attractor in $L^2(\mathcal{O})$ for non-autonomous parabolic equations involving Grushin operators. Later, Binh [10] proved the regularity and exponential growth of pullback attractor in the space $\mathcal{S}_0^1(\mathcal{O}) \cap L^{2p-2}(\mathcal{O})$ with force term $f \in W^{1,2}_{\text{loc}}(\mathbb{R}; L^2(\mathcal{O}))$ satisfying

$$\int_{-\infty}^{t} e^{\lambda s}(|f(s)|_2^2 + |f'(s)|_2^2)\mathrm{d}s < \infty.$$

But as $f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\mathcal{O}))$ with

$$\int_{-\infty}^{t} e^{\lambda s}|f(s)|_2^2\mathrm{d}s < \infty,$$

it is impossible that the weak solution belongs to $\mathcal{S}_0^1(\mathcal{O}) \cap L^{2p-2}(\mathcal{O})$, furthermore, we cannot prove the existence of pullback attractor in $\mathcal{S}_0^1(\mathcal{O}) \cap L^{2p-2}(\mathcal{O})$. Then, can we study the higher-order attraction for non-autonomous parabolic equations? Sun and Yuan [13], Xiao and Sun [14] considered the results for semi-linear reaction-diffusion equations in non-cylindrical domains. But for degenerate parabolic equation involving Grushin operators, higher-order attraction remains open.

In this paper, we consider the following initial boundary value problem for a non-autonomous parabolic equation involving Grushin operators

$$\begin{cases} \dfrac{\partial u}{\partial t} - G_r u + g(u) = f(t) \quad \text{in} \quad Q_\tau, \\ u = 0 \ \text{ on } \Sigma_\tau, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}, \end{cases} \tag{1}$$

where $\tau \in \mathbb{R}$, $u_\tau : \mathcal{O}_\tau \to \mathbb{R}$ and $f : Q_\tau \to \mathbb{R}$ are given, and $g \in C^1(\mathbb{R}, \mathbb{R})$ is also a given function, for which there exist nonnegative constants α_1, α_2, β and l, and $p \geqslant 2$, such that

$$-\beta + \alpha_1|s|^p \leqslant g(s)s \leqslant \beta + \alpha_2|s|^p, \quad \forall\, s \in \mathbb{R} \tag{2}$$

and

$$g'(s) \geqslant -l, \quad \forall\, s \in \mathbb{R}. \tag{3}$$

We obtain the following main result:

Theorem 1.1 *Under the assumptions (2), (3). Let $f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\mathcal{O}))$ satisfy*

$$\int_{-\infty}^{t} e^{\lambda s}|f(s)|_2^2\mathrm{d}s < \infty. \tag{4}$$

Let $U(t,\tau)$ be the process generated by the weak solutions of (1) and $\hat{\mathscr{A}} = \{\mathscr{A}(t) : \ t \in \mathbb{R}\}$ be the pullback $\mathscr{D}_\lambda$-attractor of $U(t,\tau)$ in $L^2(\mathcal{O})$. Then for any $\delta \in [0,\infty)$, any $\hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$, the following properties hold:

(i) $\hat{\mathscr{A}}$ *is* $L^{2+\delta}$*-pullback* $\mathscr{D}_\lambda$*-attracting, that is,*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{L^{2+\delta}(\mathcal{O})}(U(t,\tau)D(\tau),\mathscr{A}(t)) = 0 \quad \text{for all } t\in\mathbb{R}; \tag{5}$$

(ii) There exist two sequences $T(t,\delta,\hat{D},\hat{\mathscr{A}})$ *(which depends only on* t,δ, $\hat{D}$ *and* $\hat{\mathscr{A}}$*) and* $M_\delta(t)$ *(which depends only on* $t,\delta,N(r)$ *and* $\int_{-\infty}^t e^{\lambda s}|f(s)|_2^2\mathrm{d}s$*), such that*

$$\int_{\mathcal{O}} |U(t,\tau)u_\tau - v(t)|^{2+\delta}\mathrm{d}x \leqslant M_\delta(t) \quad \text{for any } t-\tau \geqslant T(t,\delta,\hat{D},\hat{\mathscr{A}}), \tag{6}$$

where $v(\tau)\in\mathscr{A}(\tau)$ $(\tau\in\mathbb{R})$ *is a (arbitrary) fixed complete trajectory of* $U(t,\tau)$*.*

The paper is organized as follows. In Section 2, we recall some concepts and results about pullback $\mathscr{D}$-attractor, and introduce the function spaces, weak solution and known results. To make the test function used later meaningful, in Section 3, we establish the maximum principle (Theorem 3.3). Finally, in Section 4, we establish the higher-order integrability of the difference of weak solutions (Theorem 4.2) and obtain higher-order attraction (Theorem 1.1).

2. Preliminaries

In this section, we recall the notations and related results about pullback attractor, and introduce the function spaces used later and weak solution of problem (1).

• Pullback $\mathscr{D}$-attractor

We consider a process (also called a two-parameter semigroup) U on a Banach space X, i.e., a family $\{U(t,\tau); -\infty<\tau\leqslant t<+\infty\}$ of continuous mappings $U(t,\tau): X\to X$, such that

$$U(\tau,\tau)x = x \quad \text{and} \quad U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau\leqslant r\leqslant t.$$

Suppose $\mathscr{D}$ is a nonempty class of parameterized sets $\hat{D} = \{D(t): t\in\mathbb{R}\}\subset\mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 2.1 *The process* $U(\cdot,\cdot)$ *is said to be pullback* $\mathscr{D}$*-asymptotically compact if for any* $t\in\mathbb{R}$*, any* $\hat{D}\in\mathscr{D}$*, any sequence* $\tau_n\to-\infty$ *and any sequence* $x_n\in D(\tau_n)$*, the sequence* $\{U(t,\tau_n)x_n\}_{n=1}^\infty$ *is precompact in* X*.*

Definition 2.2 *It is said that* $\hat{B}\in\mathscr{D}$ *is pullback* $\mathscr{D}$*-absorbing for the process* $U(\cdot,\cdot)$ *if for any* $t\in\mathbb{R}$ *and any* $\hat{D}\in\mathscr{D}$*, there exists a* $\tau_0=\tau_0(t,\hat{D})\leqslant t$ *such that*

$$U(t,\tau)D(\tau)\subset B(t) \quad \text{for all } \tau\leqslant\tau_0(t,\hat{D}).$$

Definition 2.3 *The family* $\hat{\mathscr{A}} = \{\mathscr{A}(t): \mathscr{A}(t)\in\mathcal{P}(X), t\in\mathbb{R}\}$ *is said to be a pullback* $\mathscr{D}$*-attractor for the process* $U(\cdot,\cdot)$*, if:*

(1) $\mathscr{A}(t)$ *is compact in* X *for all* $t\in\mathbb{R}$*;*

(2) $\hat{\mathscr{A}}$ *is pullback* $\mathscr{D}$*-attracting, i.e.,*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),\mathscr{A}(t)) = 0 \quad \text{for all } \hat{D}\in\mathscr{D} \text{ and all } t\in\mathbb{R};$$

(3) $\hat{\mathscr{A}}$ *is invariant, i.e.,* $U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t)$ *for any* $-\infty<\tau\leqslant t<\infty$*.*

The following abstract result is important to deduce our main result:

Theorem 2.4 ([13]) *Let X, Y, Z be three Banach spaces satisfying $Z \hookrightarrow Y \hookrightarrow X$ with continuous embeddings, respectively. Let $U(\cdot,\cdot)$ be a process defined on X and $W(t,\tau)$ $(-\infty < \tau \leqslant t < \infty)$ be a family operators defined on X satisfying*

$$U(t,\tau)\cdot = v(t) + W(t,\tau)\big(\cdot - v(\tau)\big) \quad \text{for all } \tau \leqslant t.$$

Moreover, assume further that

(a) $U(\cdot,\cdot)$ has a pullback $\mathscr{D}$-attractor $\hat{\mathscr{A}} = \{\mathscr{A}(t)|\ t \in \mathbb{R}\}$ in X, and $\hat{\mathscr{A}} \in \mathscr{D}$;

(b) $\hat{v} = \{v(t) : t \in \mathbb{R}\} \in \mathscr{D}$ is a complete trajectory of $U(t,\tau)$;

(c) there exists $\hat{\mathscr{B}}_0 = \{B_0(t)|\ t \in \mathbb{R}\}$ with $B_0(t)$ bounded in Z for each $t \in \mathbb{R}$, satisfying that for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathscr{D}$, there exists a $\tau_0 = \tau_0(t, \hat{D}) \leqslant t$ such that

$$W(t,\tau)\big(D(\tau) - v(\tau)\big) \subset B_0(t) \quad \text{for all } \tau \leqslant \tau_0. \tag{7}$$

Then, the following hold:

(i) $\hat{\mathscr{B}} = \{v(t)\}_{t\in\mathbb{R}} + \hat{\mathscr{B}}_0 := \{B(t) = v(t) + B_0(t)|\ t \in \mathbb{R}\}$ is a $\mathscr{D}$-absorbing set in X for the process $U(\cdot,\cdot)$;

(ii) $\mathrm{dist}_X\big(\hat{\mathscr{A}}, \hat{\mathscr{B}}\big) = 0$, i.e.,

$$\mathrm{dist}_X\big(\mathscr{A}(t),\, v(t) + B_0(t)\big) = \mathrm{dist}_X\big(\mathscr{A}(t) - v(t),\, B_0(t)\big) = 0 \quad \text{for all } t \in \mathbb{R}; \tag{8}$$

(iii) if $B_0(t)$ is closed in X for all $t \in \mathbb{R}$, then

$$\mathscr{A}(t) - v(t) \subset B_0(t) \quad \text{for all } t \in \mathbb{R}; \tag{9}$$

moreover, if assume further that the space Y satisfies $\|\cdot\|_Y \leqslant C\|\cdot\|_X^{\theta}\|\cdot\|_Z^{1-\theta}$ for some $\theta \in (0,1]$ and constant C, then for any $\hat{D} \in \mathscr{D}$ and any $t \in \mathbb{R}$,

$$\mathrm{dist}_Y\big(U(t,\tau)D(\tau),\, \mathscr{A}(t)\big) \to 0 \quad \text{as } \tau \to -\infty. \tag{10}$$

• **Function spaces**

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} (N_1, N_2 \geqslant 1)$ with smooth boundary $\partial\mathcal{O}$,

$$Q_\tau := \bigcup_{t\in(\tau,\infty)} \mathcal{O} \times t, \quad \Sigma_\tau := \bigcup_{t\in(\tau,\infty)} \partial\mathcal{O} \times t,$$

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{O} \times t, \quad \Sigma_{\tau,T} := \bigcup_{t\in(\tau,T)} \partial\mathcal{O} \times t.$$

For a fixed finite time interval $[\tau,T]$, let $(X, \|\cdot\|_X)$ $(t \in [\tau,T])$ be a family of Banach spaces such that $X \subset L^1_{\mathrm{loc}}(\mathcal{O})$ for all $t \in [\tau,T]$. For any $1 \leqslant q \leqslant \infty$, we denote by $L^q(\tau,T;X)$ the vector space of all functions $u \in L^1_{\mathrm{loc}}(Q_{\tau,T})$ such that $u(t) = u(\cdot,t) \in X$ a.e., $t \in (\tau,T)$, and the function $\|u(\cdot)\|_X$ defined by $t \mapsto \|u(t)\|_X$, belongs to $L^q(\tau,T)$.

By definition, we consider on $L^q(\tau,T;X)$ the norm given by

$$\|u\|_{L^q(\tau,T;X)} := \|\|u(\cdot)\|_X\|_{L^q(\tau,T)}.$$

The space $\mathcal{S}_0^1(\mathcal{O})$ is defined as the closure of $C_0^1(\bar{\mathcal{O}})$ with respect to the norm

$$\|u\| = \Big(\int_{\mathcal{O}} (|\nabla_{x_1} u|^2 + |x_1|^{2r} |\nabla_{x_2} u|^2) \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Then $\mathcal{S}_0^1(\mathcal{O})$ is a Hilbert space w.r.t. the scalar product

$$((u,v)) := \int_{\mathcal{O}} (\nabla_{x_1} u \nabla_{x_1} v + |x_1|^{2r} \nabla_{x_2} u \nabla_{x_2} v) \mathrm{d}x.$$

We denote by $|\cdot|_2$ $(\cdot,\cdot)$ the norms and scalar products in $L^2(\mathcal{O})$ and by $\|\cdot\|$, $((\cdot,\cdot))$ the norms and scalar products in $\mathcal{S}_0^1(\mathcal{O})$.

The following lemma is necessary in later work. We can refer to [7] for more details.

Lemma 2.5 *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} (N_1, N_2 \geqslant 1)$. Then the following embeddings hold:*

(i) $\mathcal{S}_0^1(\mathcal{O}) \hookrightarrow L^{2_r^*}(\mathcal{O})$ continuously,

(ii) $\mathcal{S}_0^1(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$ compactly for $p \in [1, 2_r^*)$, where $2_r^* = \frac{2N(r)}{N(r)-2}$, $N(r) = N_1 + (r+1)N_2$.

It is known (see [11]) that there exists a complete orthonormal system of eigenvectors e_j corresponding to the eigenvalues λ_j, such that

$$-G_r e_j = \lambda_j e_j, \quad j = 1, 2, \ldots, \text{ and } 0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots,$$

where $\lambda_1 = \inf\{\frac{\|u\|^2}{|u|_2^2}, u \in \mathcal{S}_0^1(\mathcal{O}), \ u \neq 0\}$.

- **Weak solutions**

For the readers' convenience, we recall the definition of different solutions about equation (1).

For each $T > \tau$, consider the auxiliary problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - G_r u + g(u) = f(t) \quad \text{in } Q_{\tau,T}, \\ u = 0 \ \text{ on } \Sigma_{\tau,T}, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}, \end{cases} \tag{11}$$

where $\tau \in \mathbb{R}$, $u_\tau : \mathcal{O} \to \mathbb{R}$.

Let

$$V := L^2(\tau, T; \mathcal{S}_0^1(\mathcal{O})) \cap L^p(\tau, T; L^p(\mathcal{O})), \quad V^* := L^2(\tau, T; \mathcal{S}^{-1}(\mathcal{O})) + L^{p'}(\tau, T; L^{p'}(\mathcal{O})).$$

Definition 2.6 ([8]) *A function $u = u(x,t)$ defined in $Q_{\tau,T}$ is said to be a weak solution of (11) if $u \in V$, $\frac{\partial u}{\partial t} \in V^*$ and for any $\varphi \in V$,*

$$\int_\tau^T \int_{\mathcal{O}} \Big(\frac{\partial u}{\partial t} \varphi + \nabla_{x_1} u \nabla_{x_1} \varphi + |x_1|^{2r} \nabla_{x_2} u \nabla_{x_2} \varphi + g(u)\varphi \Big) \mathrm{d}x\mathrm{d}t = \int_\tau^T \int_{\mathcal{O}} f(t) \varphi \mathrm{d}x\mathrm{d}t.$$

Definition 2.7 (Weak solution) *A function $u : Q_\tau \to \mathbb{R}$ is called a weak solution of (1) if for any $T > \tau$, the restriction of u on $Q_{\tau,T}$ is a weak solution of (11).*

Theorem 2.8 ([8]) *Assume that (2), (3) and (4) hold, for any $\tau \in \mathbb{R}$, $u_\tau \in L^2(\mathcal{O})$ given. Then* *problem (1) has a unique weak solution u.*

Lemma 2.9 ([11]) *If $u \in V$ and $\frac{\partial u}{\partial t} \in V^*$, then $u \in C([\tau, T]; L^2(\mathcal{O}))$.*

According to Theorem 2.8 and Lemma 2.9, we define the process $U(t,\tau): L^2(\mathcal{O}) \to L^2(\mathcal{O})$ for any $-\infty < \tau \leqslant t < +\infty$. And denote by $\mathcal{R}_\lambda$ the set of $\rho: \mathbb{R} \to [0, +\infty)$ such that

$$e^{\lambda\tau}\rho^2(\tau) \to 0, \quad \tau \to -\infty.$$

Denote by $\mathscr{D}_\lambda$ the family of set class $\hat{\mathcal{D}} := \{\mathcal{D}(t) | \mathcal{D}(t) \subset L^2(\mathcal{O}),\ \forall\, t \in \mathbb{R}, \mathcal{D}(t) \neq \emptyset\}$ such that for each $\rho_{\hat{\mathcal{D}}} \in \mathcal{R}_\lambda$, $\mathcal{D}(t) \subset \{u \in L^2(\mathcal{O}) : |u(t)|_2 \leqslant \rho_{\hat{\mathcal{D}}}\}$.

Theorem 2.10 *Assume that (2), (3) and (4) hold. Then the process corresponding to (1) has a pullback $\mathscr{D}_\lambda$-attractor in $L^2(\mathcal{O})$.*

3. Maximum principle

The main purpose of this section is to apply the Stampacchia's truncation method to establish some L^∞ a priori estimates for the weak solution, which will guarantee the test functions used in next section meaningful.

Throughout this section, let the initial data $(u_\tau, f) \in \big(H_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O}),\ L^\infty(Q_{\tau,T})\big)$. Then, for the regular data $(u_\tau,\ f)$, from Theorem 2.8, we know that there exists a unique weak solution.

Lemma 3.1 *For any $k > 0$ and any $\phi \in \mathcal{S}_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$, the following equality holds:*

$$\begin{aligned}&\int_{\mathcal{O}} \big(\nabla_{x_1}\phi\nabla_{x_1}(|\phi|^k\phi) + |x_1|^{2r}\nabla_{x_2}\phi\nabla_{x_2}(|\phi|^k\phi)\big)\mathrm{d}x \\ &= (k+1)\big(\frac{2}{k+2}\big)^2 \int_{\mathcal{O}} \big(\big|\nabla_{x_1}|\phi|^{\frac{k+2}{2}}\big|^2 + |x_1|^{2r}\big|\nabla_{x_2}|\phi|^{\frac{k+2}{2}}\big|^2\big)\mathrm{d}x. \qquad (12)\end{aligned}$$

Proof Since $\phi \in \mathcal{S}_0^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$, we know that $|\phi|^k\phi$ and $|\phi|^{\frac{k+2}{2}}$ also belongs to $\mathcal{S}_0^1(\mathcal{O})$. Hence, the integrals in (12) make sense. Then, we need only to show

$$\begin{aligned}\frac{1}{k+1}\int_{\mathcal{O}} \nabla_{x_1}\phi\nabla_{x_1}(|\phi|^k\phi)\mathrm{d}x &= \int_{\mathcal{O}} |\phi|^k|\nabla_{x_1}\phi|^2\mathrm{d}x \\ &= \int_{\mathcal{O}(\phi\geqslant 0)} |\phi|^k|\nabla_{x_1}\phi|^2\mathrm{d}x + \int_{\mathcal{O}(\phi\leqslant 0)} |\phi|^k|\nabla_{x_1}\phi|^2\mathrm{d}x \\ &= \int_{\mathcal{O}(\phi\geqslant 0)} |\phi^{\frac{k}{2}}\nabla_{x_1}\phi|^2\mathrm{d}x + \int_{\mathcal{O}(\phi\leqslant 0)} |(-\phi)^{\frac{k}{2}}\nabla_{x_1}(-\phi)|^2\mathrm{d}x \\ &= \frac{4}{(k+2)^2}\Big(\int_{\mathcal{O}(\phi\geqslant 0)} |\nabla_{x_1}\phi^{\frac{k+2}{2}}|^2\mathrm{d}x + \int_{\mathcal{O}(\phi\leqslant 0)} |\nabla_{x_1}(-\phi)^{\frac{k+2}{2}}|^2\mathrm{d}x\Big) \\ &= \frac{4}{(k+2)^2}\int_{\mathcal{O}} \big|\nabla_{x_1}|\phi|^{\frac{k+2}{2}}\big|^2\mathrm{d}x. \quad \Box\end{aligned}$$

Lemma 3.2 ([13]) *Let $f \in L^2_{\rm loc}(\mathbb{R}; L^2(\mathcal{O}))$ and satisfy (4). Then, for each $T \in \mathbb{R}$, there is a family $\{f_m\} \subset L^\infty_{\rm loc}(Q_{-\infty,T})$ such that*

$$\text{for any (fixed) } \tau \in (-\infty, T), \quad f_m \to f \ \text{ in } L^2(\tau, T; L^2(\mathcal{O})) \qquad (13)$$

and for any $t \in (-\infty, T)$,

$$\int_{-\infty}^{t} e^{\lambda s}|f_m(s)|_2^2 \mathrm{d}s \leqslant 2\int_{-\infty}^{t} e^{\lambda s}|f(s)|_2^2 \mathrm{d}s + \frac{1}{4} \quad \text{for all } m = 1, 2, \ldots. \tag{14}$$

Recall $Q_{-\infty,T} = \cup_{t\in(-\infty,T)}\mathcal{O} \times \{t\}$. *The family* $\{f_m\}$ *may depend on* T.

Fix a function $\mathscr{H}(\cdot) \in C^1(\mathbb{R})$ such that

$$\begin{cases} \text{(i)} \ \ |\mathscr{H}'(\mathrm{s})| \leqslant \mathrm{M} < \infty, \quad \forall\, \mathrm{s} \in \mathbb{R}, \\ \text{(ii)} \ \ \mathscr{H} \text{ is strictly increasing on } (0, \infty), \\ \text{(iii)} \ \ \mathscr{H}(\mathrm{s}) = 0, \quad \forall\, \mathrm{s} \leqslant 0; \end{cases} \tag{15}$$

and define

$$H(s) = \int_0^s \mathscr{H}(\sigma)\mathrm{d}\sigma. \tag{16}$$

Theorem 3.3 (L^∞-estimate) *Assume that* g *satisfies (2). Then, for any* $-\infty < \tau \leqslant T < \infty$ *and any initial data* $(u_\tau,\, f) \in \big(H_0^1(\mathcal{O}_\tau) \cap L^\infty(\mathcal{O}_\tau),\, L^\infty(Q_{\tau,T})\big)$, *the unique weak solution* u *of (11) belongs to* $L^\infty(Q_{\tau,T})$.

Proof From the assumption (2), we know that there is a positive constant M_0 such that

$$g(s) > 0 \ \text{ as } s \geqslant M_0 \ \text{ and } \ g(s) < 0 \ \text{ as } s \leqslant -M_0. \tag{17}$$

Denote $K' := \max\{\|u_\tau\|_{L^\infty(\mathcal{O})},\, \|f\|_{L^\infty(Q_{\tau,T})}\}$. From the assumption (2), we know that there is a positive constant M depending on β, α_1 and K' such that

$$g(s) > K' \ \text{ as } s \geqslant M \ \text{ and } \ g(s) < -K' \ \text{ as } s \leqslant -M. \tag{18}$$

Define $K := \max\{K',\, M\} + 1$.

Since $u \in L^2(\tau, T;\, \mathcal{S}_0^1(\mathcal{O})) \cap L^p(\tau, T;\, L^p(\mathcal{O}))$, we have that

$$\mathscr{H}(u(t) - K) \in \mathcal{S}_0^1(\mathcal{O}) \cap L^p(\mathcal{O}) \ \text{ a.e., } t \in (\tau, T) \tag{19}$$

and

$$\mathscr{H}(u(t) - K) \in L^2(\tau, T;\, \mathcal{S}_0^1(\mathcal{O})) \cap L^p(\tau, T;\, L^p(\mathcal{O})), \tag{20}$$

so, $\mathscr{H}(u(t) - K)$ can be selected as a test function.

Hence, from the definition of weak solution, we have

$$\begin{aligned} &\int_\tau^T \int_{\mathcal{O}} u'(x,s)\mathscr{H}(u(s) - K)\mathrm{d}x\mathrm{d}s - \int_\tau^T \int_{\mathcal{O}} G_r u(s)\mathscr{H}(u(s) - K)\mathrm{d}x\mathrm{d}s \\ &\quad = -\int_\tau^T \int_{\mathcal{O}} g(u(x,s))\mathscr{H}(u(s) - K)\mathrm{d}x\mathrm{d}s + \int_\tau^T \int_{\mathcal{O}} f(x,s)\mathscr{H}(u(s) - K)\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{21}$$

where for any $\varphi \in L^2(\tau, T;\, \mathcal{S}_0^1(\mathcal{O})) \cap L^p(\tau, T;\, L^p(\mathcal{O}))$,

$$-\int_\tau^T \int_{\mathcal{O}} G_r u(s)\varphi \mathrm{d}x\mathrm{d}s = \int_\tau^T \int_{\mathcal{O}} \nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2r}\nabla_{x_2} u \nabla_{x_2}\varphi \mathrm{d}x\mathrm{d}s$$

(recall that $G_r u = \triangle_{x_1} u + |x_1|^{2r}\triangle_{x_2} u$), and from (20) we know that all of integrals above make sense.

In the following, we will estimate each term in (21) one by one.

From (19) and the properties of $\mathscr{H}(\cdot)$, we have

$$\begin{aligned}&-\int_\tau^T\int_{\mathcal{O}} G_r u(s)\mathscr{H}(u(s)-K)\mathrm{d}x\mathrm{d}s\\ &=\int_\tau^T\int_{\mathcal{O}} \mathscr{H}'(u(s)-K)(|\nabla_{x_1}u(s)|^2+|x_1|^{2r}|\nabla_{x_2}u(s)|^2)\mathrm{d}x\mathrm{d}s\geqslant 0.\end{aligned}\tag{22}$$

Secondly, from the definition of K', (20) and the fact that $(T-\tau)\times \mathrm{mes}(\mathcal{O})<\infty$, we know that

$$0\leqslant\int_\tau^T\int_{\mathcal{O}} K'\mathscr{H}(u(s)-K)\mathrm{d}x\mathrm{d}s<\infty,$$

which, combined with (17) and the definition of K, implies that

$$-\int_\tau^T\int_{\mathcal{O}} \big(g(u(x,s))-K'\big)\mathscr{H}(u(s)-K)\mathrm{d}x\mathrm{d}s\leqslant 0.$$

Similarly, we can deduce that

$$\int_\tau^T\int_{\mathcal{O}} \big(f(x,s)-K'\big)\mathscr{H}(u(s)-K)\mathrm{d}x\mathrm{d}s\leqslant 0.$$

Therefore, inserting the above estimates into (21), we obtain that

$$\int_\tau^T\int_{\mathcal{O}} u'(x,s)\mathscr{H}(u(s)-K)\mathrm{d}x\mathrm{d}s\leqslant 0,$$

that is,

$$\int_{\mathcal{O}} H(u(x,t)-K)\mathrm{d}x-\int_{\mathcal{O}} H(u(x,\tau)-K)\mathrm{d}x\leqslant 0\ \text{ a.e., } t\in[\tau,T].$$

Consequently, from the definition of K and $H(\cdot)$, we have that

$$\int_{\mathcal{O}} H(u(x,\tau)-K)\mathrm{d}x=0$$

and $H(u(x,t)-K)=0$ a.e., on $\mathcal{O}$, a.e., $t\in[\tau,T]$.

Hence,

$$u(x,t)\leqslant K\ \text{ a.e., on } \mathcal{O},\ \text{ a.e., } t\in[\tau,T].\tag{23}$$

Similarly, defining $\tilde{\mathscr{H}}(s)=\mathscr{H}(-s)$ and replacing $\mathscr{H}(u(s)-K)$ by $\tilde{\mathscr{H}}(u(s)+K)$ in (21), we can deduce that

$$u(x,t)\geqslant -K\ \text{ a.e., on } \mathcal{O},\ \text{ a.e., } t\in[\tau,T].\tag{24}$$

Summarizing (23) and (24), we prove the solution is bounded.

4. Higher-order attraction of pullback $\mathscr{D}_\lambda$-attractors

Throughout this section, let

$$\hat{v}:=\{v(t):\ t\in\mathbb{R}\}\ \text{ with } v(t)\in\mathscr{A}(t),\ \ \forall\, t\in\mathbb{R}\tag{25}$$

denote a fixed complete trajectory of $U(t,\tau)$, that is

$$U(t,\tau)v(\tau)=v(t)\ \text{ for any } -\infty<\tau\leqslant t<\infty.$$

For any $\hat{D} = \{D(t) : \ t \in \mathbb{R}\} \in \mathscr{D}_\lambda$ and $u_\tau \in D(\tau)$, set $u(t) = U(t, \tau)u_\tau$. For any (fixed) $T \in \mathbb{R}$, throughout this subsection, we choose and fix a family $\{f_m\} \subset L^\infty_{\mathrm{loc}}(Q_{-\infty,T})$ such that

$$\text{the family } \{f_m\} \text{ satisfies the conditions (13) and (14) in Lemma 3.2.} \tag{26}$$

Then, for any $\tau < T$, there are two sequences $\{(u_{\tau m}, f_m)\}$ and $\{(v_{\tau m}, f_m)\}$ $(i = 1, 2)$ satisfying

$$u_{\tau m}, v_{\tau m} \in \mathcal{S}^1_0(\mathcal{O}) \cap L^\infty(\mathcal{O}) \ \text{ and } \ f_m \in L^\infty(Q_{\tau,T}), \tag{27}$$

such that

$$u_{\tau m} \to u_\tau, \ v_{\tau m} \to v_\tau \text{ in } L^2(\mathcal{O}) \ \text{ and } \ f_m \to f \text{ in } L^2(\tau, T; L^2(\mathcal{O})) \text{ as } m \to \infty, \tag{28}$$

where u_m and v_m are the unique weak solutions of (11) corresponding to $(u_{\tau m}, f_m)$ and $(v_{\tau m}, f_m)$, respectively.

From (28), we can assume that

$$|u_{\tau m}|_2^2 \leqslant 2|u_\tau|_2^2 + 1 \ \text{ and } \ |v_{\tau m}|_2^2 \leqslant 2|v_\tau|_2^2 + 1 \ \text{ for all } m = 1, 2, \ldots. \tag{29}$$

Denote

$$w_m(t) = u_m(t) - v_m(t) \ \text{ for any } \tau \leqslant t \leqslant T, \tag{30}$$

then $w_m(t)$ $(m = 1, 2, \ldots)$ is the unique solution of the following equation:

$$\begin{cases} \dfrac{\partial w_m}{\partial t} - G_r w_m + g(u_m) - g(v_m) = 0, \ \text{ in } \ Q_{\tau,T}, \\ w_m = 0, \ \text{ on } \Sigma_{\tau,T}, \\ w_m(\tau, x) = u_{\tau m} - v_{\tau m}, \quad x \in \mathcal{O}. \end{cases} \tag{31}$$

Applying Theorem 3.3, we know that $u_m, v_m \in L^\infty(Q_{\tau,T})$ for each $m = 1, 2, \ldots$, and so

$$w_m = u_m - v_m \in L^2(\tau, T; \mathcal{S}^1_0(\mathcal{O})) \cap L^\infty(Q_{\tau,T})$$

and for any $0 \leqslant \theta < \infty$, $|w_m|^\theta w_m \in L^2(\tau, T;\, \mathcal{S}^1_0(\mathcal{O})) \cap L^\infty(Q_{\tau,T})$. Consequently, we can multiply (31) by $|w_m|^\theta w_m$ for any $\theta \in [0, \infty)$, and then applying Lemma 3.1, we obtain that

$$\begin{aligned} &\frac{1}{\theta+2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m\|^{\theta+2}_{L^{\theta+2}(\mathcal{O})} + \frac{4(\theta+1)}{(\theta+2)^2}\int_{\mathcal{O}}\Big(\big|\nabla_{x_1}|w_m(t)|^{\frac{\theta+2}{2}}\big|^2 + |x_1|^{2r}\big|\nabla_{x_2}|w_m(t)|^{\frac{\theta+2}{2}}\big|^2\Big)\mathrm{d}x \\ &\quad = -\int_{\mathcal{O}}\big(g(u_m) - g(v_m)\big)|w_m|^\theta w_m \mathrm{d}x \leqslant l\|w_m(t)\|^{\theta+2}_{L^{\theta+2}(\mathcal{O})} \ \text{ a.e., } t \in (\tau, T). \end{aligned} \tag{32}$$

The main purpose of this subsection is, based on (32), to deduce some pullback L^q-type a priori estimates about w_m. More precisely, we will prove the following main result of this section:

Theorem 4.1 *Let $\hat{D} \in \mathscr{D}_\lambda$, $\hat{v}$ be the fixed complete trajectory given in (25) and T be a fixed time. Assume further that $f_m, u_{\tau m}, v_{\tau m}$ satisfy (26), (28) and (29). Then, for each $t \in (\tau, T)$ and each $k = 0, 1, 2, \ldots$, there exist two positive constant sequences $\tilde{T}_k(t, \hat{D}, \hat{v})$ (which depends only on k, t, λ, $\hat{D}$ and $\hat{v}$) and $\tilde{M}_k(t)$ (which depends only on $t, k, \lambda, N(r)$ and $\int_{-\infty}^t e^{\lambda s}|f_m(s)|_2^2\mathrm{d}s$), such that for any $m = 1, 2, \ldots$, the solution w_m of (31) satisfies*

$$\int_{\mathcal{O}} |w_m(t)|^{2(\frac{N(r)}{N(r)-2})^k}\mathrm{d}x \leqslant \tilde{M}_k(t) \ \text{ for any } t - \tau \geqslant \tilde{T}_k(t, \hat{D}, \hat{v}), \tag{A_k}$$

and

$$\int_s^{s+1}\Big(\int_{\mathcal{O}}|w_m(\sigma)|^{2(\frac{N(r)}{N(r)-2})^{k+1}}\mathrm{d}x\Big)^{\frac{N(r)-2}{N(r)}}\mathrm{d}\sigma\leqslant\tilde{M}_k(t)\ \text{ for any } s-\tau\geqslant\tilde{T}_k(t,\hat{D},\hat{v}).\qquad(B_k)$$

Proof At first, since u_m is a weak solution, by using of (2) we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}|u_m(s)|_2^2+\|u_m(s)\|^2+2\alpha_1\|u_m\|^p_{L^p(\mathcal{O})}\leqslant\frac{1}{\lambda}|f_m(s)|_2^2+2\beta|\mathcal{O}|\ \text{ a.e., } s\in(\tau,T),$$

where λ is the first eigenvalue of $-G_r$ in $\mathcal{S}_0^1(\mathcal{O})$. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}s}|u_m(s)|_2^2+\lambda|u_m(s)|_2^2+2\alpha_1\|u_m\|^p_{L^p(\mathcal{O})}\leqslant\frac{1}{\lambda}|f_m(s)|_2^2+2\beta|\mathcal{O}|\ \text{ a.e., } s\in(\tau,T).\qquad(33)$$

Then applying Gronwall lemma to (33), we obtain that

$$|u_m(t)|_2^2\leqslant e^{-\lambda(t-\tau)}|u_m(\tau)|_2^2+\frac{1}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}|f_m(s)|_2^2\mathrm{d}s+\frac{2\beta}{\lambda}|\mathcal{O}|(T-\tau),\quad\forall\ t\in(\tau,T).$$

Similarly, about v_m we have

$$|v_m(t)|_2^2\leqslant e^{-\lambda(t-\tau)}|v_m(\tau)|_2^2+\frac{1}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}|f_m(s)|_2^2\mathrm{d}s+\frac{2\beta}{\lambda}|\mathcal{O}|(T-\tau),\quad\forall\ t\in(\tau,T).$$

Therefore, for any $t\in(\tau,T)$,

$$|w_m(t)|_2^2\leqslant 2e^{-\lambda(t-\tau)}\big(|u_m(\tau)|_\tau^2+|v_m(\tau)|_\tau^2\big)+\frac{4}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}|f_m(s)|_2^2\mathrm{d}s+\frac{8\beta}{\lambda}|\mathcal{O}|(T-\tau).$$

For each $t\in\mathbb{R}$, we set $\tilde{M}_0'(t)$ the positive number given by

$$\tilde{M}_0'(t)=\frac{8}{\lambda}e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}|f(s)|_2^2\mathrm{d}s+\frac{8\beta}{\lambda}|\mathcal{O}|(T-\tau)+\frac{e^{-\lambda t}}{\lambda}.\qquad(34)$$

Then, from (26) and (29) we have that

$$|w_m(t)|_2^2\leqslant 4e^{-\lambda(t-\tau)}\big(|u_\tau|_2^2+|v(\tau)|_2^2+1\big)+\tilde{M}_0'(t).\qquad(35)$$

Therefore, note that $u_\tau\in D(\tau)$ with $\hat{D}=\{D(t):\ t\in\mathbb{R}\}\in\mathscr{D}_\lambda$ and $\hat{v}\in\mathscr{D}_\lambda$, for each $t\in\mathbb{R}$, from (35) we know that there is a $T'(t,\hat{D},\hat{v})$ such that

$$|w_m(t)|_2^2\leqslant\tilde{M}_0'(t)+1\ \text{ for all } t-\tau\geqslant T'(t,\hat{D},\hat{v}).\qquad(36)$$

Taking $\theta=0$ in (32) and integrating with respect to time t, we obtain that

$$\int_s^{s+1}\int_{\mathcal{O}}\Big(\big|\nabla_{x_1}|w_m(t)|\big|^2+|x_1|^{2r}\big|\nabla_{x_2}|w_m(t)|\big|^2\Big)\mathrm{d}x\mathrm{d}t\leqslant(l+1)(\tilde{M}_0'(t)+1)\qquad(37)$$

for all $s-\tau\geqslant T'(t,\hat{D},\hat{v})$. On the other hand, from the embedding $\mathcal{S}_0^1(\mathcal{O})\hookrightarrow L^{\frac{2N(r)}{N(r)-2}}(\mathcal{O})$, we know that there is a constant $c_{N(r)}$ such that

$$\|\phi\|_{L^{\frac{2N(r)}{N(r)-2}}(\mathcal{O})}\leqslant c_{N(r)}\|\phi\|,\quad\forall\ \phi\in\mathcal{S}_0^1(\mathcal{O}).\qquad(38)$$

Hence, (37) implies that

$$\int_s^{s+1}\|w_m(t)\|^2_{L^{\frac{2N(r)}{N(r)-2}}(\mathcal{O})}\mathrm{d}t\leqslant c^2_{N(r)}(l+1)(\tilde{M}_0'(t)+1)\ \text{ for all } s-\tau\geqslant T'(t,\hat{D},\hat{v}).\qquad(39)$$

Set

$$\tilde{M}_0(t) = \big(1 + c^2_{N(r)}(l+1)\big)\big(\tilde{M}'_0(t) + 1\big) \quad \text{and} \quad \tilde{T}_0(t, \hat{D}, \hat{v}) = T'(t, \hat{D}, \hat{v}), \tag{40}$$

from (36) and (39) we know that (A_0) and (B_0) hold.

By induction, we assume (A_k) and (B_k) hold for $k \geqslant 0$.

In the following, we will show that (A_{k+1}) and (B_{k+1}) hold.

Taking $\theta = 2(\frac{N(r)}{N(r)-2})^{k+1} - 2$ in (32), then we obtain that

$$\begin{aligned}&\frac{1}{2}\Big(\frac{N(r)-2}{N(r)}\Big)^{k+1}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m\|^{2(\frac{N(r)}{N(r)-2})^{k+1}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} + \\ &\quad\frac{2\big(\frac{N(r)}{N(r)-2}\big)^{k+1}-1}{\big(\frac{N(r)}{N(r)-2}\big)^{2(k+1)}}\int_{\mathcal{O}}\Big(\big|\nabla_{x_1}|w_m(t)|^{(\frac{N(r)}{N(r)-2})^{k+1}}\big|^2 + |x_1|^{2r}\big|\nabla_{x_2}|w_m(t)|^{(\frac{N(r)}{N(r)-2})^{k+1}}\big|^2\Big)\mathrm{d}x \\ &\leqslant l\|w_m(t)\|^{2(\frac{N(r)}{N(r)-2})^{k+1}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} \quad \text{a.e., } t \in (\tau, T),\end{aligned}$$

that is, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w_m\|^{2(\frac{N(r)}{N(r)-2})^{k+1}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} + \big(4(\frac{N(r)}{N(r)-2})^{k+1} - 2\big)(\frac{N(r)-2}{N(r)})^{k+1}\cdot \\ &\qquad\int_{\mathcal{O}}\big(|\nabla_{x_1}|w_m(t)|^{(\frac{N(r)}{N(r)-2})^{k+1}}|^2 + |x_1|^{2r}|\nabla_{x_2}|w_m(t)|^{(\frac{N(r)}{N(r)-2})^{k+1}}|^2\big)\mathrm{d}x \\ &\leqslant 2l(\frac{N(r)}{N(r)-2})^{k+1}\|w_m(t)\|^{2(\frac{N(r)}{N(r)-2})^{k+1}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} \quad \text{a.e., } t \in (\tau, T)\end{aligned} \tag{41}$$

and so,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_m\|^{2(\frac{N(r)}{N(r)-2})^{k}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} \leqslant 2l(\frac{N(r)}{N(r)-2})^{k}\|w_m(t)\|^{2(\frac{N(r)}{N(r)-2})^{k}}_{L^{2(\frac{N(r)}{N(r)-2})^{k+1}}(\mathcal{O})} \quad \text{a.e., } t \in (\tau, T). \tag{42}$$

Applying the uniform Gronwall lemma to (42) and (B_k), we obtain that

$$\int_{\mathcal{O}}|w_m(t)|^{2(\frac{N(r)}{N(r)-2})^{k+1}}\mathrm{d}x \leqslant C_{\tilde{M}_k(t),l,N(r),k} \quad \text{for any } t - \tau \geqslant \tilde{T}_k(t, \hat{D}, \hat{v}) + 1. \tag{43}$$

And, for any $s - \tau \geqslant \tilde{T}_k(t, \hat{D}, \hat{v}) + 1$, we integrate (41) over $[s, s+1]$ and obtain that

$$\int_s^{s+1}\||w_m(x,\sigma)|^{(\frac{N(r)}{N(r)-2})^{k+1}}\|^2\mathrm{d}\sigma \leqslant C'_{\tilde{M}_k(t),l,N(r),k}. \tag{44}$$

On the other hand, from Theorem 3.3 and Lemma 3.1, we know that

$$|w_m(\cdot,t)|^{(\frac{N(r)}{N(r)-2})^{k+1}} \in \mathcal{S}^1_0(\mathcal{O}) \quad \text{for a. e., } t \in (\tau, T). \tag{45}$$

Hence, applying (38) to $|w_m(\cdot,t)|^{(\frac{N(r)}{N(r)-2})^{k+1}}$, we can deduce from (44) that

$$\int_s^{s+1}\Big(\int_{\mathcal{O}}|w_m(\sigma)|^{2(\frac{N(r)}{N(r)-2})^{k+1}}\mathrm{d}x\Big)^{\frac{N(r)-2}{N(r)}}\mathrm{d}\sigma \leqslant c^2_{N(r)}C'_{\tilde{M}_k(t),l,N(r),k} \tag{46}$$

for any $s - \tau \geqslant \tilde{T}_k(t, \hat{D}, \hat{v}) + 1$. Therefore, set

$$\tilde{T}_{k+1}(t, \hat{D}, \hat{v}) = \tilde{T}_k(t, \hat{D}, \hat{v}) + 1 \quad \text{and} \quad \tilde{M}_{k+1}(t) = \max\{C_{\tilde{M}_k(t),l,N(r),k},\, c^2_{N(r)}C'_{\tilde{M}_k(t),l,N(r),k}\},$$

from (43) and (46) we know that (A_{k+1}) and (B_{k+1}) hold.

Based on the a priori estimates Theorem 4.1, we establish the following estimate for the weak solution of equation (1):

Theorem 4.2 *Let $\hat{D}=\{D(\tau):\tau\in\mathbb{R}\}\in\mathscr{D}_\lambda$ and $\hat{v}$ be the fixed complete trajectory given in (25). Then for each $t\in\mathbb{R}$ and each $k=0,1,2\ldots$, there exist two positive constants $T_k(t,\hat{D},\hat{v})$ (which depends only on k,t, $|D(\tau)|_\tau$ and $|v(\tau)|_\tau$) and $\bar{M}_k(t)$ (which depends only on $t,k,N(r)$ and $\int_{-\infty}^t e^{\lambda s}|f(s)|_2^2\mathrm{d}s$) such that*

$$\int_{\mathcal{O}}|U(t,\tau)u_\tau-v(t)|^{2(\frac{N(r)}{N(r)-2})^k}\mathrm{d}x\leqslant\bar{M}_k(t)$$

for any $t-\tau\geqslant T_k(t,\hat{D},\hat{v})$ and any $u_\tau\in D(\tau)$.

Proof For each fixed $t\in\mathbb{R}$ and $k\in\{0,1,2,\ldots\}$.

Take $T_k(t,\hat{D},\hat{v})=\tilde{T}_k(t,\hat{D},\hat{v})+1$, where $\tilde{T}_k(t,\hat{D},\hat{v})$ is just the constant given in Theorem 4.1 corresponding to the pair t,k.

Set $T=t+1$ and for any (fixed) τ satisfying $\tau\leqslant t-T_k(t,\hat{D},\hat{v})$.

For the interval $[\tau,T]$ defined above, choose two sequences $(u_{\tau m},f_m)$ and $(v_{\tau m},f_m)$ satisfying all of the conditions in (26), (29). Then, it follows from Theorem 4.1 (A_k) that

$$\int_{\mathcal{O}}|u_m(t)-v_m(t)|^{2(\frac{N(r)}{N(r)-2})^k}\mathrm{d}x\leqslant\tilde{M}_k(t),\tag{47}$$

where u_m and v_m are the weak solutions of (11) corresponding to the regular data $(u_{\tau m},f_m)$ and $(v_{\tau m},f_m)$ on interval $[\tau,T]$, respectively.

By Lemma 2.9, for weak solutions $u(t),v(t)$ of equation (1), we know that there are two subsequences $\{u_{m_j}(t)\}\subset\{u_m(t)\}$ and $\{v_{m_j}(t)\}\subset\{v_m(t)\}$ satisfying that

$$v_{m_j}(t)\to u(t)=U(t,\tau)u_\tau\ \text{ and }\ v_{m_j}(t)\to v(t)\ \text{ a.e., on }\ \mathcal{O}\ \text{ as }\ j\to\infty.$$

Hence, by taking $\bar{M}_k(t)=\tilde{M}_k(t)$ and applying the Fatou's lemma, we have

$$\begin{aligned}\int_{\mathcal{O}}|U(t,\tau)u_\tau-v(t)|^{2\left(\frac{N(r)}{N(r)-2}\right)^k}\mathrm{d}x&\leqslant\liminf_{j\to\infty}\int_{\mathcal{O}}|u_{m_j}(t)-v_{m_j}(t)|^{2\left(\frac{N(r)}{N(r)-2}\right)^k}\mathrm{d}x.\\&\leqslant\bar{M}_k(t).\end{aligned}$$

We are now ready to prove our main result Theorem 1.1:

Proof For each $\delta\in[0,\infty)$, there is a unique $k\in\{1,2,3,\ldots\}$ such that

$$2+\delta+1\in\big(2(\frac{N(r)}{N(r)-2})^{k-1},\,2(\frac{N(r)}{N(r)-2})^k\big].\tag{48}$$

Then, in Theorem 2.4, let $X=L^2(\mathcal{O})$, $Y=L^{2+\delta}(\mathcal{O})$ and $Z=L^{2(\frac{N(r)}{N(r)-2})^k}(\mathcal{O})$, $\mathscr{D}=\mathscr{D}_\lambda$, $\hat{\mathscr{A}}$ be the pullback $\mathscr{D}_\lambda$-attractor in $L^2(\mathcal{O})$ obtained in Theorem 2.10, $\hat{v}\in\hat{\mathscr{A}}$ be the complete trajectory given in (25), and for each $t\in\mathbb{R}$, define

$$B_0(t)=\big\{\phi\in L^{2(\frac{N(r)}{N(r)-2})^k}(\mathcal{O}):\ \|\phi\|^{2(\frac{N(r)}{N(r)-2})^k}_{L^{2(\frac{N(r)}{N(r)-2})^k}(\mathcal{O})}\leqslant\bar{M}_k(t)\big\},\tag{49}$$

where the constant $\bar{M}_k(t)$ is given in Theorem 4.2.

We know that all of assumptions in Theorem 2.4 are satisfied, consequently, the $L^{2+\delta}$-pullback $\mathscr{D}_\lambda$-attraction follows from (10), and the a priori bound (6) follows from (49) with the constants $M_\delta(t) := \bar{M}_k(t)$ and $T(t,\delta,\hat{D},\hat{\mathscr{A}}) := T_k(t,\hat{D},\hat{v})$ (where the constant k is fixed by (48)).

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