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The Signless Laplacian Spectral Characterization of Strongly Connected Bicyclic Digraphs

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Abstract Let $\overrightarrow{G}$ be a digraph and $A(\overrightarrow{G})$ be the adjacency matrix of $\overrightarrow{G}$. Let $D(\overrightarrow{G})$ be the diagonal matrix with outdegrees of vertices of $\overrightarrow{G}$ and $Q(\overrightarrow{G}) = D(\overrightarrow{G}) + A(\overrightarrow{G})$ be the signless Laplacian matrix of $\overrightarrow{G}$. The spectral radius of $Q(\overrightarrow{G})$ is called the signless Laplacian spectral radius of $\overrightarrow{G}$. In this paper, we determine the unique digraph which attains the maximum (or minimum) signless Laplacian spectral radius among all strongly connected bicyclic digraphs. Furthermore, we prove that any strongly connected bicyclic digraph is determined by the signless Laplacian spectrum.

Keywords the signless Laplacian spectral radius; ∞-digraph; θ-digraphn; bicyclic digraph

MR(2010) Subject Classification 05C50; 15A18

1. Introduction

All digraphs considered in this paper are finite simple strongly connected digraphs, i.e., without loops and multiple arcs. In the following, we just define some terminologies and notations which will be used then, for other terminology and notation, we refer the reader to [1] for an extensive treatment of digraphs.

Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a digraph with vertex set $V(\overrightarrow{G}) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(\overrightarrow{G})$. For a digraph $\overrightarrow{G}$, if two vertices are connected by an arc, then they are called adjacent. If there is an arc from v_i to v_j, we indicate this by writing (v_i, v_j), call v_j the head of (v_i, v_j), and v_i the tail of (v_i, v_j), respectively. The digraph $\overrightarrow{G}$ is strongly connected if for every pair of vertices $v_i, v_j \in V(\overrightarrow{G})$, there exists a directed path from v_i to v_j and a directed path from v_j to v_i. For any vertex v_i, let $N_i^+ = \{v_j \in V(\overrightarrow{G}) \mid (v_i, v_j) \in E(\overrightarrow{G})\}$ and $N_i^- = \{v_j \in V(\overrightarrow{G}) \mid (v_j, v_i) \in E(\overrightarrow{G})\}$ denote the out-neighbors and in-neighbors of v_i, respectively. Let $d_i^+ = |N_i^+|$ denote the outdegree of the vertex v_i, and $d_i^- = |N_i^-|$ denote the indegree of the vertex v_i in the digraph $\overrightarrow{G}$. Let $\overrightarrow{P_n}$ and $\overrightarrow{C_n}$ denote the directed path and the directed cycle on n vertices, respectively. Suppose $\overrightarrow{P_k} = v_1 v_2 \ldots v_k$. We call v_1 the initial vertex of the directed path $\overrightarrow{P_k}$, v_k the terminal vertex of the directed path $\overrightarrow{P_k}$, respectively. A digraph $\overrightarrow{G}$ is called a strongly connected bicyclic digraph if $\overrightarrow{G}$ is strongly connected and $|E(\overrightarrow{G})| = |V(\overrightarrow{G}| + 1$.

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For a digraph $\overrightarrow{G}$, let $A(\overrightarrow{G}) = (a_{ij})_{n\times n}$ be the adjacency matrix of $\overrightarrow{G}$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(\overrightarrow{G})$ and $a_{ij} = 0$ otherwise. Let $D(\overrightarrow{G})$ be the diagonal matrix with outdegrees of the vertices of $\overrightarrow{G}$. Then the matrix $Q(\overrightarrow{G}) = D(\overrightarrow{G}) + A(\overrightarrow{G})$ is called the signless Laplacian matrix of $\overrightarrow{G}$. The matrix $Q(\overrightarrow{G})$ is nonnegative and irreducible when $\overrightarrow{G}$ is strongly connected. The spectral radius of $Q(\overrightarrow{G})$, i.e., the largest modulus of the eigenvalues of $Q(\overrightarrow{G})$, is called the signless Laplacian spectral radius of $\overrightarrow{G}$, denoted by $q(\overrightarrow{G})$. The polynomial $\phi(\overrightarrow{G}, \lambda) = \det(\lambda I_n - Q(\overrightarrow{G}))$, where I_n is an $n \times n$ identity matrix, is defined as the characteristic polynomial with respect to the signless Laplacian matrix $Q(\overrightarrow{G})$. The collection of eigenvalues of $Q(\overrightarrow{G})$ together with multiplicates is called the Q-spectrum of $\overrightarrow{G}$. Two nonisomorphic digraphs are said to be Q-cospectral if they have the same signless Laplacian spectrum. A digraph is said to be determined by Q-spectrum if there is no other nonisomorphic digraph with the same signless Laplacian spectrum, we denote these digraphs as DQS digraphs. There are many articles on the topic which undirected graphs are DQS [2–4]. For additional remarks on this topic we refer the reader to see two excellent surveys [5] and [6]. However, there is not much known about digraphs.

It follows from the Perron-Frobenius Theorem [7] that $q(\overrightarrow{G})$ is an eigenvalue of the signless Laplacian matrix $Q(\overrightarrow{G})$ and there is a positive unit eigenvector corresponding to $q(\overrightarrow{G})$ when $\overrightarrow{G}$ is strongly connected. The positive unit eigenvector corresponding to $q(\overrightarrow{G})$ is called the Perron vector of $Q(\overrightarrow{G})$. The signless Laplacian spectral radius of digraphs has been studied in the literature [8–10]. So far, which digraphs have the maximum or minimum signless Laplacian spectral radius among all the strongly connected bicyclic digraphs has not been determined.

The rest of this paper is organized as follows. In Section 2, we characterize the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among θ-digraphs. In Section 3, we characterize the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among ∞-digraphs. In Section 4, we determine the extremal digraphs which attain the maximum and minimum signless Laplacian spectral radius among all the strongly connected bicyclic digraphs. Furthermore, we prove that any strongly connected bicyclic digraph is determined by the signless Laplacian spectrum, i.e., any strongly connected bicyclic digraph is DQS.

2. The signless Laplacian spectral radius of θ-digraphs

Let θ-graph be a graph consisting of three paths which have the same end-vertices. In [11], the authors defined the θ-digraph as follows. The θ-digraph consists of three directed paths P_{a+2}, P_{b+2}, and P_{c+2} such that the initial of P_{a+2} and P_{b+2} is the terminal vertex of P_{c+2}, and the initial vertex of P_{c+2} is the terminal of P_{a+2} and P_{b+2}, denoted by $\theta(a, b, c)$. In the following, we suppose that $a \leq b$ and $a + b + c + 2 = n$.

In this section, we will prove that $\theta(0, n-2, 0)$ is the unique digraph which attains the maximum signless Laplacian spectral radius among all $\theta(a, b, c)$-digraphs on n vertices and $\theta(0, 1, n-3)$ is the unique digraph which attains the minimum signless Laplacian spectral radius among all $\theta(a, b, c)$-digraphs on n vertices.

Lemma 2.1 ([12]) *Let A be a nonnegative irreducible matrix with the largest eigenvalue $\varrho(A)$ and row sums $s_1, s_2, \ldots, s_n$. Then*

$$\min_{1\le i\le n} s_i \le \varrho(A) \le \max_{1\le i\le n} s_i.$$

Moreover, one of the equalities holds if and only if the row sums of A are all equal.

Lemma 2.2 *If $a \ge 1$, then $q(\theta(a-1, b+1, c) > q(\theta(a, b, c))$.*

Proof Let $\theta(a, b, c)$ be a digraph shown in Figure 1. Suppose $\mathbf{X} = (x_u, x_v, x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b, z_1, z_2, \ldots, z_c)$ is the Perron vector of $Q(\theta(a, b, c))$ corresponding to $q(\theta(a, b, c))$, where x_u and x_v correspond to u and v, respectively, and x_i, y_j and z_k ($i = 1, 2, \ldots, a; j = 1, 2, \ldots, b; k = 1, 2, \ldots, c$) correspond to w_i, w_j^1 and w_k^2, respectively.

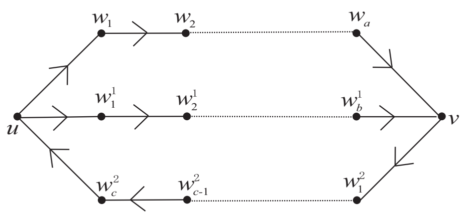

Figure 1 The digraph $\theta(a, b, c)$.

Since $Q(\theta(a, b, c))\mathbf{X} = q(\theta(a, b, c))\mathbf{X}$, one can easily see that

$$\begin{cases} q(\theta(a,b,c))x_i = x_i + x_{i+1}, & i = 1, 2, \ldots, a-1, \\ q(\theta(a,b,c))y_j = y_j + y_{j+1}, & j = 1, 2, \ldots, b-1, \\ q(\theta(a,b,c))z_k = z_k + z_{k+1}, & k = 1, 2, \ldots, c-1, \\ q(\theta(a,b,c))x_u = 2x_u + x_1 + y_1, & \\ q(\theta(a,b,c))x_v = x_v + z_1, & \\ q(\theta(a,b,c))x_a = x_a + x_v, & \\ q(\theta(a,b,c))y_b = y_b + x_v, & \\ q(\theta(a,b,c))z_c = z_c + x_u. & \end{cases}$$

Then

$$\begin{cases} x_a = (q(\theta(a,b,c)) - 1)^{a-1} x_1, \\ y_b = (q(\theta(a,b,c)) - 1)^{b-1} y_1, \\ z_c = (q(\theta(a,b,c)) - 1)^{c-1} z_1, \\ x_v = (q(\theta(a,b,c)) - 1)^{a} x_1 = (q(\theta(a,b,c)) - 1)^{b} y_1. \end{cases}$$

Furthermore,

$$x_u = (q(\theta(a,b,c)) - 1)^c z_1 = (q(\theta(a,b,c)) - 1)^{c+1} x_v = (q(\theta(a,b,c)) - 1)^{c+b+1} y_1.$$

Thus we deduce that

$$(q(\theta(a,b,c)) - 2)(q(\theta(a,b,c)) - 1)^{c+b+1} y_1 = (q(\theta(a,b,c)) - 1)^{b-a} y_1 + y_1.$$

By Perron-Frobenius Theorem, we have $y_1 > 0$, therefore

$$(q(\theta(a,b,c)) - 2)(q(\theta(a,b,c)) - 1)^{n-1} = (q(\theta(a,b,c)) - 1)^b + (q(\theta(a,b,c)) - 1)^a.$$

Similarly, we have

$$\begin{aligned}&(q(\theta(a-1,b+1,c)) - 2)(q(\theta(a-1,b+1,c)) - 1)^{n-1}\\ &\quad = (q(\theta(a-1,b+1,c)) - 1)^{b+1} + (q(\theta(a-1,b+1,c)) - 1)^{a-1}.\end{aligned}$$

Let $f(x) = (x-2)(x-1)^{n-1} - (x-1)^b - (x-1)^a$ and $g(x) = (x-2)(x-1)^{n-1} - (x-1)^{b+1} - (x-1)^{a-1}$. It is not difficult to see that $q(\theta(a,b,c))$ is the largest real root of $f(x) = (x-2)(x-1)^{n-1} - (x-1)^b - (x-1)^a = 0$. Similarly, $q(\theta(a-1,b+1,c))$ is the largest real root of $g(x) = (x-2)(x-1)^{n-1} - (x-1)^{b+1} - (x-1)^{a-1} = 0$. $f(x) - g(x) = (x-2)((x-1)^b - (x-1)^{a-1}) > 0$, for all $x > 2$. Since the minimum row sum of $Q(\theta(a,b,c))$ is 2, and the row sums of $Q(\theta(a,b,c))$ are not all equal, then by Lemma 2.1, we have $q(\theta(a,b,c)) > 2$. Then we have $q(\theta(a,b,c)) < q(\theta(a-1,b+1,c))$.

Lemma 2.3 ([9]) *Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a simple digraph on n vertices, u,v,w distinct vertices of $V(\overrightarrow{G})$, $(u,v) \in E(\overrightarrow{G})$ and $X = (x_1, x_2, \ldots, x_n)$ be the unique positive unit eigenvector corresponding to the signless Laplacian spectral radius $q(\overrightarrow{G})$, where x_i corresponds to the vertex i. Let $H = \overrightarrow{G} - \{(u,v)\} + \{(u,w)\}$ (Noting that if $(u,w) \in E(\overrightarrow{G})$, then H has multiple arc (u,w)). If $x_w \geq x_v$, then $q(H) \geq q(\overrightarrow{G})$. Furthermore, if H is strongly connected and $x_w > x_v$, then $q(H) > q(\overrightarrow{G})$.*

Lemma 2.4 *If $c \geq 1$, $b \geq 1$, then $q(\theta(a,b+1,c-1) > q(\theta(a,b,c)) > q(\theta(a,b-1,c+1))$.*

Proof Let $\theta(a,b,c)$ be a digraph shown in Figure 1 and $\mathbf{X} = (x_u, x_v, x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b, z_1, z_2, \ldots, z_c)$ be the Perron vector of $Q(\theta(a,b,c))$ corresponding to $q(\theta(a,b,c))$, where x_u and x_v correspond to u and v, respectively, and x_i, y_j and z_k $(i = 1,2,\ldots,a; j = 1,2,\ldots,b; k = 1,2,\ldots,c)$ correspond to w_i, w_j^1 and w_k^2, respectively. It is not difficult to see that $\theta(a,b+1,c-1) = \theta(a,b,c) - \{(w_a,v)\} + \{(w_a,w_1^2)\}$. Since $(q(\theta(a,b,c)) - 1)x_v = z_1$, $q(\theta(a,b,c)) > 2$, we have $z_1 > x_v$. By Lemma 2.3, we have $q(\theta(a,b+1,c-1)) > q(\theta(a,b,c))$. Similarly, we have $q(\theta(a,b,c)) > q(\theta(a,b-1,c+1))$. $\square$

Lemma 2.5 *If $c \geq 1$, $a \geq 1$, then $q(\theta(a+1,b,c-1) > q(\theta(a,b,c)) > q(\theta(a-1,b,c+1))$.*

Proof Let $\theta(a,b,c)$ be a digraph shown in Figure 1 and $\mathbf{X} = (x_u, x_v, x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b, z_1, z_2, \ldots, z_c)$ be the Perron vector of $Q(\theta(a,b,c))$ corresponding to $q(\theta(a,b,c))$, where x_u and x_v correspond to u and v, respectively, and x_i, y_j and z_k $(i = 1,2,\ldots,a; j = 1,2,\ldots,b; k = 1,2,\ldots,c)$ correspond to w_i, w_j^1 and w_k^2, respectively. It is not difficult to see that $\theta(a+1,b,c-1) = \theta(a,b,c) - \{(w_b^1,v)\} + \{(w_b^1,w_1^2)\}$. Since $(q(\theta(a,b,c)) - 1)x_v = z_1$, $q(\theta(a,b,c)) > 2$, we have $z_1 > x_v$. By Lemma 2.3, we have $q(\theta(a+1,b,c-1)) > q(\theta(a,b,c))$. Similarly, we have $q(\theta(a,b,c)) > q(\theta(a-1,b,c+1))$. $\square$

Combining Lemmas 2.2, 2.4, and 2.5, we have the following theorem.

Theorem 2.6 *Among all θ-digraphs, the digraph $\theta(0,n-2,0)$ is the unique digraph which* *attains the maximum signless Laplacian spectral radius and the digraph* $\theta(0,1,n-3)$ *is the unique digraph which attains the minimum signless Laplacian spectral radius.*

3. The signless Laplacian spectral radius of ∞-digraphs

Let an ∞-digraph be a digraph on n vertices obtained from two directed cycles $\overrightarrow{C_k}$ and $\overrightarrow{C_l}$ by identifying a vertex of $\overrightarrow{C_k}$ with a vertex of $\overrightarrow{C_l}$, denoted by $\infty(k,l)$, $k \le l$ and $k+l=n+1$. In this section, we will prove that $\infty(2,n-1)$ attains the maximum signless Laplacian spectral among all digraphs in $\infty(k,l)$ and $\infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ attains the minimum signless Laplacian spectral among all $\infty(k,l)$-digraphs for fixed n (see Figure 2).

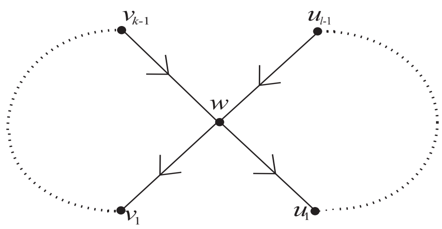

Figure 2 The digraph $\infty(k,l)$.

Lemma 3.1 *If* $k \ge 3$, *then* $q(\infty(k-1,l+1)) > q(\infty(k,l))$.

Proof Suppose that $\mathbf{X}=(x_w,x_1,x_2,\ldots,x_{k-1},y_1,y_2,\ldots,y_{l-1})$ is the Perron vector of $Q(\infty(k,l))$ corresponding to $q(\infty(k,l))$, where x_w corresponds to w, x_i and y_j $(i=1,2,\ldots,k-1; j=1,2,\ldots,l-1)$ correspond to v_i and u_j, respectively. Since $Q(\infty(k,l))\mathbf{X}=q(\infty(k,l))\mathbf{X}$, it is not difficult to see that

$$\begin{cases} q(\infty(k,l))x_i = x_i + x_{i+1}, & i=1,2,\ldots,k-2,\\ q(\infty(k,l))y_j = y_j + y_{j+1}, & j=1,2,\ldots,l-2,\\ q(\infty(k,l))x_{k-1} = x_w + x_{k-1}, & \\ q(\infty(k,l))y_{l-1} = x_w + y_{l-1}, & \\ q(\infty(k,l))x_w = 2x_w + x_1 + y_1. & \end{cases}$$

Then

$$\begin{cases} x_w = (q(\infty(k,l))-1)x_{k-1} = (q(\infty(k,l))-1)^{k-1}x_1,\\ x_w = (q(\infty(k,l))-1)y_{l-1} = (q(\infty(k,l))-1)^{l-1}y_1. \end{cases}$$

Thus we have

$$(q(\infty(k,l))-2)(q(\infty(k,l))-1)^{l-1}y_1 = (q(\infty(k,l))-1)^{l-k}y_1 + y_1.$$

By Perron-Frobenius Theorem, we have $y_1>0$, therefore

$$(q(\infty(k,l))-2)(q(\infty(k,l))-1)^{n-1} = (q(\infty(k,l))-1)^{l-1} + (q(\infty(k,l))-1)^{k-1}.$$

Similarly, we have

$$(q(\infty(k-1,l+1))-2)(q(\infty(k-1,l+1))-1)^{n-1}$$

$$= (q(\infty(k-1,l+1))-1)^l + (q(\infty(k-1,l+1))-1)^{k-2}.$$

Let $f(x) = (x-2)(x-1)^{n-1} - (x-1)^{l-1} - (x-1)^{k-1}$ and $g(x) = (x-2)(x-1)^{n-1} - (x-1)^l - (x-1)^{k-2}$. One can easily see that $q(\infty(k,l))$ is the largest real root of $f(x) = 0$ and $q(\infty(k-1,l+1))$ is the largest real root of $g(x) = 0$. $f(x) - g(x) = (x-2)((x-1)^{l-1} - (x-1)^{k-2}) > 0$, for all $x > 2$. Since the minimum row sum of $Q(\infty(k,l))$ is 2, and the row sums of $Q(\infty(k,l))$ are not all equal, by Lemma 2.1, we have $q(\infty(k,l)) > 2$. Then we have $q(\infty(k-1,l+1)) > q(\infty(k,l))$. Thus the proof is complete. □

By Lemma 3.1, we immediately get the following theorem.

Theorem 3.2 *Among all $\infty(k,l)$-digraphs on n vertices, the digraph $\infty(2,n-1)$ is the unique digraph which attains the maximum signless Laplacian spectral radius, and the digraph $\infty(\lfloor\frac{n+1}{2}\rfloor, \lceil\frac{n+1}{2}\rceil)$ is the unique digraph which attains the minimum signless Laplacian spectral radius.*

4. The maximum (or minimum) signless Laplacian spectral radius and the signless Laplacian spectral characterization of strongly connected bicyclic digraphs

We can know that each strongly connected bicyclic digraph is either a θ-digraph or an ∞-digraph. In the following, we will first determine the digraphs which attain the maximum and minimum signless Laplacian spectral radius among all strongly connected bicyclic digraphs, respectively.

Lemma 4.1 *Let $\theta(a,b,c)$ and $\infty(k,l)$-digraph be a θ-digraph as shown in Figure 1 and an ∞-digraph as shown in Figure 2, respectively. Then $q(\theta(a,b,c)) < q(\infty(b+1,a+c+2))$.*

Proof Suppose $\mathbf{X} = (x_u, x_v, x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b, z_1, z_2, \ldots, z_c)$ is the Perron vector of $Q(\theta(a,b,c))$ corresponding to $q(\theta(a,b,c))$, where x_u and x_v correspond to u and v, respectively, and x_i, y_j and z_k $(i = 1,2,\ldots,a; j = 1,2,\ldots,b; k = 1,2,\ldots,c)$ correspond to w_i, w_j^1 and w_k^2, respectively. By the proof of Lemma 2.2, we know that $x_u > x_v$. One can easily see that $\infty(b+1,a+c+2) = \theta(a,b,c) - \{(w_b^1, v)\} + \{(w_b^1, u)\}$. Then by Lemma 2.3, we have $q(\theta(a,b,c)) < q(\infty(b+1,a+c+2))$.

By Lemma 4.1, we know that the digraph that attains the maximum signless Laplacian spectral radius among all the strongly connected bicyclic digraphs must be in ∞-digraphs, and the digraph that attains the minimum signless Laplacian spectral radius among all the strongly connected bicyclic digraphs must be in θ-digraphs. Combining Theorems 2.6 and 3.2, we get the following theorem.

Theorem 4.2 *Among all the strongly connected bicyclic digraphs with order n, the digraph $\infty(2,n-1))$ is the unique digraph which attains the maximum signless Laplacian spectral radius, and the digraph $\theta(0,1,n-3)$ attains the minimum signless Laplacian spectral radius.*

Next, we will prove that each strongly connected bicyclic digraph is determined by their signless Laplacian spectrum. By Lemmas 2.2 and 3.1, it is not difficult to see that

$$\phi(\theta(a,b,c),\lambda) = (\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^b - (\lambda-1)^a,$$

$$\phi(\infty(k,l),\lambda) = (\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^{l-1} - (\lambda-1)^{k-1}.$$

Lemma 4.3 ([5]) *For $n \times n$ matrices A and B, the following are equivalent:*

(i) A and B are cospectral;

(ii) A and B have the same characteristic polynomial;

(iii) $\mathrm{tr}(A^i) = \mathrm{tr}(B^i)$ for $i = 1, 2, \ldots, n$.

Let $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ be two digraphs. If the signless Laplacian spectrum of them are the same, i.e., $\mathrm{Spec}_Q(\overrightarrow{G_1}) = \mathrm{Spec}_Q(\overrightarrow{G_2})$, then the number of vertices and arcs in $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are equal, respectively.

Lemma 4.4 *A digraph Q-cospectral to a θ-digraph is either a θ-digraph or an ∞-digraph.*

Proof Let D be Q-cospectral to $\theta(a,b,c)$. Then by Lemma 4.3, they have the same number of vertices and arcs. Therefore D is a strongly connected bicyclic digraph. Note that each strongly connected bicyclic digraph is either a θ-digraph or an ∞-digraph. So D is either a θ-digraph or an ∞-digraph.

Lemma 4.5 *No two nonisomorphic θ-digraphs are Q-cospectral.*

Proof Suppose that $\overrightarrow{G_1} = \theta(a,b,c)$ and $\overrightarrow{G_2} = \theta(a',b',c')$ are Q-cospectral. By convection, $a \le b$ and $a' \le b'$. Since $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ have the same number of vertices, we have

$$a + b + c = a' + b' + c',$$

and $\phi(\overrightarrow{G_1},\lambda) = \phi(\overrightarrow{G_2},\lambda)$, that is

$$(\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^a - (\lambda-1)^b = (\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^{a'} - (\lambda-1)^{b'}.$$

Therefore, we have either $a = a'$ and $b = b'$, or $a = b'$ and $b = a'$.

If $a = a'$ and $b = b'$, then $c = c'$. Thus $\overrightarrow{G_1} \cong \overrightarrow{G_2}$.

If $a = b'$ and $b = a'$, then $b = a' \le b' = a$. Since $a \le b$, we have $b' = a = b = a'$. Then we also have $a = a'$, $b = b'$ and $c = c'$, thus $\overrightarrow{G_1} \cong \overrightarrow{G_2}$. Therefore the proof is completed. □

Lemma 4.6 *No two nonisomorphic ∞-digraphs are Q-cospectral.*

Proof Suppose that $\overrightarrow{G_1} = \infty(k,l)$ and $\overrightarrow{G_2} = \infty(k',l')$ are Q-cospectral. By convention, $k \le l$ and $k' \le l'$. Since $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ have the same number of vertices, we have

$$k + l = k' + l',$$

and $\phi(\overrightarrow{G_1},\lambda) = \phi(\overrightarrow{G_2},\lambda)$, that is

$$(\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^{l-1} - (\lambda-1)^{k-1} = (\lambda-2)(\lambda-1)^{n-1} - (\lambda-1)^{l'-1} - (\lambda-1)^{k'-1}.$$

Therefore, we have either $k = k'$ and $l = l'$, or $k = l'$ and $l = k'$.

If $k=k'$ and $l=l'$, then we have $\overrightarrow{G_1}\cong\overrightarrow{G_2}$.

If $k=l'$ and $l=k'$, then $l=k'\le l'=k$. Since $k\le l$, we have $l'=k=l=k'$. Then we also have $k=k'$ and $l=l'$, thus $\overrightarrow{G_1}\cong\overrightarrow{G_2}$. Therefore the proof is completed. $\square$

Lemma 4.7 *There is no θ-digraph Q-cospectral with an ∞-digraph.*

Proof Suppose that $\overrightarrow{G_1}=\theta(a,b,c)$ and $\overrightarrow{G_2}=\infty(k,l)$ are Q-cospectral. By convention, $a\le b$ and $k\le l$. Since they have the same signless Lpalacian characteristic polynomials, that is $\phi(\overrightarrow{G_1},\lambda)=\phi(\overrightarrow{G_2},\lambda)$, therefore $a=k-1$ and $b=l-1$ or $a=l-1$ and $b=k-1$.

If $a=k-1$ and $b=l-1$, then $a+b=k+l-2=n-1$. Since $a+b+c+2=n$, we have $n-1+2+c=n+1+c=n$, a contradiction.

If $a=l-1$ and $b=k-1$, then $a+b=k+l-2=n-1$. Since $a+b+c+2=n$, we have $n-1+2+c=n+1+c=n$, a contradiction. Therefore there is no θ-digraph Q-cospectral with an ∞-digraph. Thus the proof is completed. $\square$

By Lemmas 4.4–4.7, we finally get our main result in this section.

Theorem 4.8 *Any strongly connected bicyclic digraph is DQS.*

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