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Decomposing Complete 3-Uniform Hypergraphs into Cycles

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Abstract The problem of decomposing a complete 3-uniform hypergraph into Hamilton cycles was introduced by Bailey and Stevens using a generalization of Hamiltonian chain to uniform hypergraphs by Katona and Kierstead. Decomposing the complete 3-uniform hypergraphs $K_n^{(3)}$ into k-cycles ($3 \le k < n$) was then considered by Meszka and Rosa. This study investigates this problem using a difference pattern of combinatorics and shows that $K_{n\cdot 5^m}^{(3)}$ can be decomposed into 5-cycles for $n \in \{5, 7, 10, 11, 16, 17, 20, 22, 26\}$ using computer programming.

Keywords uniform hypergraph; 5-cycle; cycle decomposition

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1. Introduction

The problem of decomposing a graph into Hamiltonian cycles is a long-standing and well-known issue in graph theory. Decomposing the graph $G = (V, E)$ is a partition of the edge-set E and the Hamiltonian decomposition of G is a type of decomposition into Hamiltonian cycles.

Hypergraphs are subset systems of a finite set that can be regarded as one of the most general structures in discrete mathematics. With the emergence of informational science, hypergraphs serve as useful mathematical models for a broad range of applications such as networks, database theory, clustering, and chemistry. As a natural generalization of the Hamiltonian decomposition of graphs, the Hamiltonian decomposition of hypergraphs has emerged. The definition of a Hamiltonian cycle was provided in [1]. A Hamiltonian cycle in a complete k-uniform hypergraph is a cyclic ordering of its vertices such that each consecutive k-tuple of vertices is an edge. A Hamiltonian decomposition of a complete k-uniform hypergraph is a partition of the set of its edges into disjoint Hamiltonian cycles.

Many scholars have considered the decomposition into cycles (not Hamiltonian) of some fixed length in recent decades, but this problem has not been fully resolved. The case where

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all the cycles have the same length has been completely solved by Alspach and Gavlas in [2]. This problem can also be generalized into hypergraphs. In particular, the decomposition of a complete k-uniform hypergraphs into Hamiltonian cycles has been described in detail in [3,4] and [8,9]. This paper mainly investigates the decomposition of complete 3-uniform hypergraphs into 5-cycles. A necessary condition for the existence of the decomposition of a complete 3-uniform hypergraphs into 5-cycles is that $n \equiv$1,2,5,7,10 or 11 (mod 15). The decomposition of $K_n^{(3)}$ into 5-cycles occurs for all admissible $n \leq 17$, and for all $n = 4^m + 1$, m is a positive integer. This study investigates the decomposition of the complete 3-uniform hypergraph $K_n^{(3)}$ into 5-cycles. Results show that $K_{n\cdot 5^m}^{(3)}$ can be decomposed into 5-cycles for $n \in \{5, 7, 10, 11, 16, 17, 20, 22, 26\}$.

2. Preliminaries

We present some definitions and lemmas.

Definition 2.1 ([5]) *A hypergraph $H = (V, E)$ is a set of vertices $V = V(H) = \{v_1, v_2, \dots, v_n\}$ and a set of (hyper)edges $E = E(H) = \{e_1, e_2, \dots, e_m\}$, where $e_i \subseteq V$ and $|e_i| \geq 0$, $1 \leq i \leq m$.*

Definition 2.2 ([1]) *A hypergraph $H = (V, E)$ consists of a finite set $V = \{v_1, v_2, \dots, v_n\}$ of vertices. If each (hyper)edge has size k, we say that H is a k-uniform hypergraph. In particular, the complete k-uniform hypergraph on n vertices has all k-subsets of V as edges, denoted by $K_n^{(k)}$.*

Definition 2.3 ([6]) *Let $H = (V, E)$ be a k-uniform hypergraph. A cycle of ℓ in H is referred to as ℓ-cycle. An ℓ-cycle in H is an ordered cyclic sequence $(v_0, v_1, \dots, v_{\ell-1})$ of V where each consecutive k-tuple of vertices is an edge of H.*

Definition 2.4 ([6]) *An ℓ-cycle decomposition of H is a partition of the set of (hyper)edges of H into mutually-disjoint ℓ-cycles.*

Definition 2.5 *Let $CS = \{(c_{i,0}, c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}) \mid i \in \mathbb{Z}_{\frac{n(n-1)(n-2)}{30}}\}$ be a 5-cycles decomposition of $K_n^{(3)}$. If cycle $C_{t,i} = \{k\lfloor\frac{c_{t,0}}{k}\rfloor + (c_{t,0} + i) \pmod k, k\lfloor\frac{c_{t,1}}{k}\rfloor + (c_{t,1} + i) \pmod k, k\lfloor\frac{c_{t,2}}{k}\rfloor + (c_{t,2} + i) \pmod k, k\lfloor\frac{c_{t,3}}{k}\rfloor + (c_{t,3} + i) \pmod k, k\lfloor\frac{c_{t,4}}{k}\rfloor + (c_{t,4} + i) \pmod k\} \in CS$ for all $i \in \mathbb{Z}_k$, then $C_t = \{c_{t,0}, c_{t,1}, c_{t,2}, c_{t,3}, c_{t,4}\}$ is called a k-base 5-cycle.*

Lemma 2.6 ([3]) *$K_7^{(3)}$ can be decomposed into 5-cycles.*

Proof The edges of $K_7^{(3)}$ can be decomposed into 7 5-cycles.

$$(0, 1, 2, 3, 4), (0, 1, 3, 4, 5), (0, 2, 3, 5, 6), (0, 3, 6, 2, 5), (0, 4, 2, 1, 6), (1, 3, 6, 4, 5), (1, 5, 2, 4, 6). \quad \square$$

Lemma 2.7 ([3]) *$K_{11}^{(3)}$ can be decomposed into 5-cycles.*

Proof The edges of $K_{11}^{(3)}$ can be decomposed into 33 5-cycles produced by 3 11-base 5-cycles as follows:

$$(0, 2, 1, 7, 4),\ (0, 5, 3, 6, 7),\ (0, 6, 8, 5, 4). \quad \square$$

Lemma 2.8 ([3]) *$K_{16}^{(3)}$ can be decomposed into 5-cycles.*

Proof The edges of $K_{16}^{(3)}$ can be decomposed into 112 5-cycles produced by 7 16-base 5-cycles as follows:

$$(0,1,2,4,5),\quad (0,2,4,7,8),\quad (0,3,5,6,9),\quad (0,4,6,15,9),$$
$$(0,5,11,4,8),\quad (0,6,1,14,5),\quad (0,7,14,6,12).\ \square$$

Lemma 2.9 ([6]) *$K_{20}^{(3)}$ can be decomposed into 5-cycles.*

Proof The edges of $K_{20}^{(3)}$ can be decomposed into 220 5 -cycles produced by 11 20-base 5-cycles as follows:

$$(0,2,8,12,18),\quad (0,6,4,16,14),\quad (0,1,4,16,19),\quad (0,3,18,2,17),$$
$$(0,5,12,8,15),\quad (0,7,6,14,13),\quad (0,9,2,18,11),\quad (0,19,2,11,5),$$
$$(0,19,8,3,10),\quad (0,15,9,18,1),\quad (0,10,17,12,1),$$

and 8 5-cycles as follows:

$$(0,4,8,12,16),\quad (1,5,9,13,17),\quad (2,6,10,14,18),\quad (3,7,11,15,19),$$
$$(0,8,16,4,12),\quad (1,9,17,5,13),\quad (2,10,18,6,14),\quad (3,11,19,7,15).\ \square$$

Lemma 2.10 ([7]) *$K_{22}^{(3)}$ can be decomposed into 5-cycles.*

Proof The edges of $K_{22}^{(3)}$ can be decomposed into 308 5-cycles produced by 14 22-base 5-cycles as follows:

$$(0,1,4,18,21),\quad (0,3,10,12,19),\quad (0,5,12,10,17),\quad (0,9,10,12,13),\quad (0,9,6,13,5),$$
$$(0,6,8,16,4),\quad (0,9,8,2,19),\quad (0,16,21,20,9),\quad (0,4,2,14,8),\quad (0,18,6,14,16),$$
$$(0,13,2,1,6),\quad (0,17,9,16,13),\quad (0,3,20,14,13),\quad (0,7,18,4,15).\ \square$$

3. Main results

In this section, we shall prove the following theorems.

Theorem 3.1 *$K_{17}^{(3)}$ can be decomposed into 5-cycles.*

Proof We can decompose the edges of $K_{17}^{(3)}$ into 136 5-cycles produced by 8 17-base 5-cycles as follows:

$$(0,1,2,8,11),\quad (0,1,3,5,6),\quad (0,2,6,10,2),\quad (0,3,4,7,12),$$
$$(0,2,11,1,8),\quad (0,3,6,1,5),\quad (0,6,12,2,10),\quad (0,4,12,3,7).\ \square$$

Theorem 3.2 *$K_{26}^{(3)}$ can be decomposed into 5-cycles.*

Proof We can decompose the edges of $K_{26}^{(3)}$ into 520 5-cycles produced by 20 26-base 5-cycles as follows:

$$(0,1,2,4,5),\quad (0,1,4,5,10),\quad (0,1,7,3,9),\quad (0,1,8,3,11),\quad (0,1,12,3,13),$$
$$(0,1,14,3,16),\quad (0,1,15,3,18),\quad (0,1,19,9,21),\quad (0,2,4,7,9),\quad (0,2,8,5,12),$$
$$(0,2,10,5,18),\quad (0,4,21,9,16),\quad (0,3,19,9,15),\quad (0,3,20,6,12),\quad (0,4,12,8,17),$$
$$(0,4,14,21,11),\quad (0,4,15,21,10),\quad (0,4,18,12,16),\quad (0,2,20,13,17),\quad (0,5,20,13,7).\ \square$$

Theorem 3.3 *If $K_n^{(3)}$ can be decomposed into 5-cycles, then $K_{5n}^{(3)}$ can also be decomposed into 5-cycles.*

Proof For $i \in \mathbb{Z}_n$, let $V(K_{5n}^{(3)}) = \bigcup_{i\in\mathbb{Z}_n} V_i = \bigcup_{i\in\mathbb{Z}_n}\{5i+j|j\in\mathbb{Z}_5\}$ be a partition of the set of vertices of $K_{5n}^{(3)}$. Let $E(K_{5n}^{(3)}) = E_1 \cup E_2 \cup E_3$ be a partition of the set of edges of $K_{5n}^{(3)}$, where

$$\begin{aligned}
E_1 &= \{(v_{t_1}, v_{t_2}, v_{t_3}) \mid v_{t_1}, v_{t_2}, v_{t_3} \in V_i,\ i \in \mathbb{Z}_n\},\\
E_2 &= \{(v_{t_1}, v_{t_2}, v_{t_3}) \mid v_{t_1}, v_{t_2} \in V_i,\ v_{t_3} \in V_j,\ i, j \in \mathbb{Z}_n,\ i \neq j\},\\
E_3 &= \{(v_{t_1}, v_{t_2}, v_{t_3}) \mid v_{t_1} \in V_i, v_{t_2} \in V_j, v_{t_3} \in V_k,\ i < j < k \in \mathbb{Z}_n\}.
\end{aligned}$$

Let $C_0^1 = (0,1,2,3,4), C_1^1 = (0,2,4,1,3)$. Then $K_5^{(3)}$ can be decomposed into two 5-cycles, namely C_0^1 and C_1^1. For $t = 0,1$, let $C_t^1 = (c_{t,0}^1, c_{t,1}^1, c_{t,2}^1, c_{t,3}^1, c_{t,4}^1)$, $C_{t,i}^1 = (5i + c_{t,0}^1, 5i + c_{t,1}^1, 5i + c_{t,2}^1, 5i + c_{t,3}^1, 5i + c_{t,4}^1)$ for $i \in \mathbb{Z}_n$. Then E_1 can be decomposed into $2n$ 5-cycles, namely $C_{t,i}^1$ ($t = 0,1$ and $i \in \mathbb{Z}_n$) (See Table 1 for n=5).

i	$C_{0,i}^1$	$C_{1,i}^1$
0	(0, 1, 2, 3, 4)	(0, 2, 4, 1, 3)
1	(5, 6, 7, 8, 9)	(5, 7, 9, 6, 8)
2	(10, 11, 12, 13, 14)	(10, 12, 14, 11, 13)
3	(15, 16, 17, 18, 19)	(15, 17, 19, 16, 18)
4	(20, 21, 22, 23, 24)	(20, 22, 24, 21, 23)

Table 1 A list of $C_{t,i}^1$ for $n = 5, t = 0, 1,\ i \in \mathbb{Z}_5$

For $t \in \mathbb{Z}_{20}$, let

$$\begin{array}{llll}
C_0^2 = (0,1,5,2,6), & C_1^2 = (1,2,6,3,7), & C_2^2 = (2,3,7,4,8), & C_3^2 = (3,4,8,0,9),\\
C_4^2 = (4,0,9,1,5), & C_5^2 = (0,2,5,3,7), & C_6^2 = (1,3,6,4,8), & C_7^2 = (2,4,7,0,9),\\
C_8^2 = (3,0,8,1,5), & C_9^2 = (4,1,9,2,6), & C_{10}^2 = (0,6,3,5,8), & C_{11}^2 = (1,7,4,6,9),\\
C_{12}^2 = (2,8,0,7,5), & C_{13}^2 = (3,9,1,8,6), & C_{14}^2 = (4,5,2,9,7), & C_{15}^2 = (0,6,4,5,9),\\
C_{16}^2 = (1,7,0,6,5), & C_{17}^2 = (2,8,1,7,6), & C_{18}^2 = (3,9,2,8,7), & C_{19}^2 = (4,5,3,9,8).
\end{array}$$

Then $K_{(5,5)}^{(3)}$ can be decomposed into twenty 5-cycles, namely $c_t^2,\ t \in \mathbb{Z}_{20}$. Let

$$\begin{aligned}
&C_t^2 = (c_{t,0}^2, c_{t,1}^2, c_{t,2}^2, c_{t,3}^2, c_{t,4}^2),\\
&C_{t,i,j}^2 = (c_{t,i,j,0}^2, c_{t,i,j,1}^2, c_{t,i,j,2}^2, c_{t,i,j,3}^2, c_{t,i,j,4}^2),\quad 0 \le i < j \le n-1,
\end{aligned}$$

where

$$c_{t,i,j,s}^2 = \begin{cases} c_{t,s}^2, & \pmod 5 + 5i;\\ c_{t,s}^2, & \pmod 5 + 5j \end{cases}$$

for $t \in \mathbb{Z}_{20}, 0 \le i < j \le n-1, s \in \mathbb{Z}_5$. Then E_2 can be decomposed into $20\binom{n}{2}$ 5-cycles, namely $C_{t,i,j}^2 (t \in \mathbb{Z}_{20}, 0 \le i < j \le n-1)$ (See Table 2 for $n = 5, t = 0$).

i	j	$C^2_{0,i,j}$
0	1	(0, 1, 5, 2, 6)
0	2	(0, 1, 10, 2, 11)
0	3	(0, 1, 15, 2, 16)
0	4	(0, 1, 20, 2, 21)
1	2	(5, 6, 10, 7, 11)
1	3	(5, 6, 15, 7, 16)
1	4	(5, 6, 20, 7, 21)
2	3	(10, 11, 15, 12, 16)
2	4	(10, 11, 20, 12, 21)
3	4	(15, 16, 20, 17, 21)

Table 2 A list of $C^2_{t,i,j}$ for $n = 5, t = 0,\ 0 \le i < j \le 4$

Let $C^3_t = (c^3_{t,0}, c^3_{t,1}, c^3_{t,2}, c^3_{t,3}, c^3_{t,4})$, $(t \in \mathbb{Z}_{\frac{n(n-1)(n-2)}{30}})$ be the $\frac{n(n-1)(n-2)}{30}$ 5-cycles $K^{(3)}_n$ decomposed, and $C^3_{t,i,j,k} = (i + 5c^3_{t,0}, (i + j) \pmod 5 + 5c^3_{t,1}, (i + k) \pmod 5 + 5c^3_{t,2}, (i + 2j + 2) \pmod 5 + 5c^3_{t,3}, (i + j + k + 1) \pmod 5 + 5c^3_{t,4})$, where $i < j < k \in \mathbb{Z}_5$. Then E_3 can be decomposed into $125 \times \frac{n(n-1)(n-2)}{30}$ 5-cycles, namely $C^3_{t,i,j,k}$ $(t \in \mathbb{Z}_{\frac{n(n-1)(n-2)}{30}},\ i, j, k \in \mathbb{Z}_5)$ (For $n = 5$, $C^3_0 = (0, 1, 2, 3, 4), C^3_1 = (0, 2, 4, 1, 3)$. $C^3_{t,i,j,k}$ for $n = 5, t = 0, i = 0$ are listed in Table 3).

j	k	$C^3_{0,0,j,k}$
0	0	(0, 5, 10, 17, 21)
0	1	(0, 5, 11, 17, 22)
0	2	(0, 5, 12, 17, 2)
0	3	(0, 5, 13, 17, 24)
0	4	(0, 5, 14, 17, 20)
1	0	(0, 6, 10, 19, 22)
1	1	(0, 6, 11, 19, 23)
1	2	(0, 6, 12, 19, 24)
1	3	(0, 6, 13, 19, 20)
1	4	(0, 6, 14, 19, 21)
2	0	(0, 7, 10, 16, 23)
2	1	(0, 7, 11, 16, 24)
2	2	(0, 7, 12, 16, 20)
2	3	(0, 7, 13, 16, 21)

2	4	$(0,7,14,16,22)$
3	0	$(0,8,10,18,24)$
3	1	$(0,8,11,18,20)$
3	2	$(0,8,12,18,21)$
3	3	$(0,8,13,18,22)$
3	4	$(0,8,14,18,23)$
4	0	$(0,9,10,15,20)$
4	1	$(0,9,11,15,21)$
4	2	$(0,9,12,15,22)$
4	3	$(0,9,13,15,23)$
4	4	$(0,9,14,15,24)$

Table 3 A list of $C^3_{t,i,j,k}$ for $n=5,\ t=0,\ i=0,\ j,k\in\mathbb{Z}_5$

Corollary 3.4 *$K^{(3)}_{n\cdot 5^m}$ can be decomposed into 5-cycles for $n\in\{5,7,10,11,16,17,20,22,\ 26\}$.*

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