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G-Inner Actions Equivalence of G-Crossed Products

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Abstract We give the definition of G-inner action of two semilattice graded weak Hopf algebras with the same semilattice Y, and the necessary and sufficient conditions for two G-crossed products to be isomorphic.

Keywords semilattice graded weak Hopf algebra; G-crossed product; G-inner actions

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1. Introduction

Because of the important role of Hopf algebras in the theory of quantum groups and related notions in mathematical physics, along with the deepening of the research, the meanings of some weaker concepts within Hopf-algebra theory have come under closer attention and are becoming better understood. A well-known example is the weak Hopf algebra, which was introduced in [1] in studying the non-invertible solution of the Yang-Baxter Equation based on this class of bialgebras. One has only two ways to produce examples, through semigroup algebras of regular monoids (in particular, Clifford monoids) and the weak quantum algebras $\mathrm{wsl}_q(2)$ and $\mathrm{vsl}_q(2)$ (see [2]). The term "weak Hopf algebra" was also used as another generalization of Hopf algebras in [3–5] where comultiplication is no longer required to preserve the unit (equivalently, the counit is not required to be an algebra homomorphism). We must point out that these two generalizations are completely distinct as the only common subclass just consists of Hopf algebras [6]. The initial motivation of the latter was its connection with the theory of algebra extensions.

Semilattice graded weak Hopf algebras were introduced in [7] and a singular solution of the quantum Yang-Baxter equation has been obtained by the quantum G-double. The necessary and sufficient conditions for two crossed products to be isomorphic was shown by Doi in [8]. Our focus here is to characterize the isomorphism of G-crossed products, which were introduced in [9]. To do this, we first need some definitions.

Definition 1.1 ([7]) *A weak Hopf algebra H with weak antipode T is called a semilattice graded weak Hopf algebra if $H = \bigoplus_{\alpha\in Y} H_\alpha$ is a semilattice grading sum where H_α are Hopf subalgebras of H with antipodes $T|_{H_\alpha}$ for all $\alpha \in Y$ and there are Hopf-algebra homomorphisms $\varphi_{\alpha,\beta}$ from*

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H_α to H_β if $\alpha\beta = \beta$, such that for any $a \in H_\alpha$ and $b \in H_\beta$, the multiplication $a * b$ in H can be given by $a * b = \varphi_{\alpha,\alpha\beta}(a)\varphi_{\beta,\alpha\beta}(b)$.

Similarly to the discussion on semilattice graded weak Hopf algebras, we obtain the following results for semilattice graded algebras A:

(1) $\{1_{A_\alpha}\}_{\alpha\in Y} \subset C(A)$, the center of A;

(2) A is a semilattice graded algebra if and only if $A = \bigoplus_{\alpha\in Y} A_\alpha$ and $A_\alpha A_\beta \subseteq A_{\alpha\beta}$ for any $\alpha, \beta \in Y$.

Definition 1.2 ([9]) *Let $H = \bigoplus_{\alpha\in Y} H_\alpha$ be a semilattice graded weak Hopf algebra and $A = \bigoplus_{\alpha\in Y} A_\alpha$ be a semilattice graded algebra with the same semilattice Y. Then $B = \bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha)$ is called a G-crossed product if it satisfies:*

(1) There is a k-linear map $H \otimes A \to A$, given by $h \otimes a \mapsto h \cdot a$, such that for any $h \in H_\alpha$, $a \in A_\beta$, $b \in A_\gamma$, $h \cdot a \in A_{\alpha\beta}$, $h \cdot 1_{A_\beta} = \varepsilon(h)1_{A_{\alpha\beta}}$ and $h \cdot (ab) = \sum_{(h)}(h' \cdot a)(h'' \cdot b) \in A_{\alpha\beta}A_{\alpha\gamma} \in A_{\alpha\beta\gamma}$, where $\Delta(h) = \sum_{(h)} h' \otimes h''$, and a map σ from $H \otimes H$ to A which satisfies $\sigma(h, k) \in A_{\alpha\beta}$ for any $h \in H_\alpha$ and $k \in H_\beta$;

(2) For any $\alpha \in Y$, H_α measures A_α and $\sigma|_{H_\alpha\otimes H_\alpha}$ is a (convolution) invertible map from $H_\alpha \otimes H_\alpha$ to A_α; for any $\alpha \in Y$, A_α is a twisted H_α-module and $\sigma|_{H_\alpha\otimes H_\alpha}$ is a cocycle.

Definition 1.3 ([9]) *Let $A = \bigoplus_{\alpha\in Y} A_\alpha \subset B = \bigoplus_{\alpha\in Y} B_\alpha$ be semilattice graded k-algebras and $H = \bigoplus_{\alpha\in Y} H_\alpha$ a semilattice graded weak Hopf algebra with the same semilattice Y.*

(1) $A \subset B$ is a (right) H-G-extension if B is a right H-comodule algebra with $\rho : B \to B \otimes H$ such that B_α is a right H_α-comodule algebra with $\rho|_{B_\alpha} = \rho_\alpha : B_\alpha \to B_\alpha \otimes H_\alpha$ and $B_\alpha^{\mathrm{co}H_\alpha} = A_\alpha$ for any $\alpha \in Y$.

(2) The H-G-extension $A \subset B$ is an H-G-cleft if there exists a right H-comodule graded map $\gamma : H \to B$ which is regular (convolution) invertible with $\gamma^{-1} : H \to B$ satisfies: $\gamma(H_\alpha) \subset B_\alpha$ and $\gamma_\alpha = \gamma|_{H_\alpha} : H_\alpha \to B_\alpha$ is an invertible right H_α-comodule map with inverse $\gamma_\alpha^{-1} = \gamma^{-1}|_{H_\alpha}$.

2. G-inner actions

Before looking at the general case of isomorphism of G-crossed products, we consider the special case of the so-called G-inner actions. In this situation the G-crossed product can be replaced by another one in which the action becomes trivial but the cocycle has been changed.

Definition 2.1 *Let $H = \bigoplus_{\alpha\in Y} H_\alpha$ be a semilattice graded weak Hopf algebra and $B = \bigoplus_{\alpha\in Y} B_\alpha \subset A = \bigoplus_{\alpha\in Y} A_\alpha$ be semilattice graded algebras with the same semilattice Y. Consider an action $H \otimes B \to A$ given by $h \otimes b \mapsto h \cdot b$ which satisfies:*

(1) $h \cdot (ab) = \sum_{(h)}(h' \cdot a)(h'' \cdot b)$ for $h \in H$, $a, b \in B$;

(2) $h_\alpha \cdot 1_{A_\beta} = \varepsilon(h)1_{A_{\alpha\beta}}$ for $h \in H_\alpha$, $\alpha \in Y$.

Then this action is called G-inner if there exists a convolution regular invertible map $\mu \in \mathrm{Hom}(H, A)$ and $\mu|_{H_\alpha} \in \mathrm{Hom}(H_\alpha, A_\alpha)$ is invertible for any $\alpha \in Y$, such that for all $h \in H$, $b \in B$,

$$h \cdot b = \sum_{(h)} \mu(h')b\mu^{-1}(h''),$$

where μ^{-1} is the unique regular inverse of μ.

Proposition 2.2 *Let $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ be a G-crossed product such that the action of H on A is G-inner via some regular invertible $\mu\in \mathrm{Hom}(H,A)$. Define $\tau\in\mathrm{Hom}(H\otimes H,A)$ by*

$$\tau(h,k)=\sum_{(h),(k)}\mu^{-1}(k')\mu^{-1}(h')\sigma(h'',k'')\mu(h'''k''')$$

for any $h\in H_\alpha$, $k\in H_\beta$. Then τ is a cocycle for any H_α and $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\cong\bigoplus_{\alpha\in Y}A_{\alpha\tau}[H_\alpha]$ and $A\#_\sigma H\cong A_\tau[H]$, a twist product with trivial action, via an algebra isomorphism which is also a left A-module and right H-comodule map.

Proof For any $h\in H_\alpha$, $k\in H_\beta$, $\mu^{-1}(k')\in A_\beta$, $\mu^{-1}(h')\in A_\alpha$, $\sigma(h'',k'')\in A_{\alpha\beta}$ and $\mu(h'''k''')\in A_\alpha$, hence $\tau(h,k)\in A_{\alpha\beta}$. Moreover, if $\alpha=\beta$, then $\tau(h,k)\in A_\alpha$, that is $\tau|_{H_\alpha\otimes H_\alpha}\in\mathrm{Hom}(H_\alpha\otimes H_\alpha,A_\alpha)$.

Define $\phi:\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\to\bigoplus_{\alpha\in Y}A_{\alpha\tau}[H_\alpha]$ by $a\#h\mapsto\sum_{(h)}a\mu(h')\otimes h''$. Then ϕ has an inverse $\psi:\bigoplus_{\alpha\in Y}A_{\alpha\tau}[H_\alpha]\to\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ by $a\otimes h\mapsto\sum_{(h)}a\mu^{-1}(h')\#h''$ for any $a\in A_\alpha$ and $h\in H_\alpha$ with $\alpha\in Y$. It is straightforward to check that ϕ and ψ are inverse. For any $a\in A_\alpha$ and $h\in H_\alpha$ with $\alpha\in Y$,

$$\begin{aligned}\phi\psi(a\otimes h)&=\phi(\sum_{(h)}a\mu^{-1}(h')\#h'')=\sum_{(h)}a\mu^{-1}(h')\mu(h'')\otimes h'''\\&=\sum_{(h)}a\varepsilon(h')1_{A_\alpha}\otimes h''=a1_{A_\alpha}\otimes h=a\otimes h\end{aligned}$$

and

$$\begin{aligned}\psi\phi(a\#h)&=\psi(\sum_{(h)}a\mu(h')\otimes h'')=\sum_{(h)}a\mu(h')\mu^{-1}(h'')\#h'''\\&=\sum_{(h)}a\varepsilon(h')1_{A_\alpha}\#h''=a\#h.\end{aligned}$$

Then, ϕ and ψ are inverses and $\phi(A_\alpha\#_\sigma H_\alpha)\subset A_{\alpha\tau}[H_\alpha]$, that is, ϕ is a semilattice graded isomorphism.

Next, we check that ϕ is an algebra map: for $a\#h\in A_\alpha\#_\sigma H_\alpha$, $b\#k\in A_\beta\#_\sigma H_\beta$ with $\alpha,\beta\in Y$,

$$\begin{aligned}\phi((a\#h)(b\#k))&=\phi(\sum_{(h),(k)}a(h'\cdot b)\sigma(h'',k')\#h'''k'')=\sum_{(h),(k)}a(h'\cdot b)\sigma(h'',k')\mu(h'''k'')\otimes h^{(4)}h'''\\&=\sum_{(h),(k)}a\mu(h')b\mu^{-1}(h'')\sigma(h''',k')\mu(h^{(4)}k'')\otimes h^{(5)}k'''\\&=\sum_{(h),(k)}a\mu(h')b1_{A_\beta}\mu^{-1}(h'')\sigma(h''',k')\mu(h^{(4)}k'')\otimes h^{(5)}k'''\\&=\sum_{(h),(k)}a\mu(h')b\mu(k')\mu^{-1}(k'')\mu^{-1}(h'')\sigma(h''',k''')\mu(h^{(4)}k^{(4)})\otimes h^{(5)}k^{(5)}\\&=\sum_{(h),(k)}a\mu(h')b\mu(k')\tau(h'',k'')\otimes h'''k'''\\&=(\sum_{(h)}a\mu(h')\otimes h'')(\sum_{(k)}b\mu(k')\otimes k'')=\phi(a\#h)\phi(b\#k).\end{aligned}$$

Since, $\bigoplus_\alpha A_{\alpha\tau}[H_\alpha] \cong \bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha)$ as algebras, and $\bigoplus_\alpha A_{\alpha\tau}[H_\alpha]$ is an associative algebra, thus τ is a cocycle.

Next, we define the left A-module action on $\bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha)$ via $a\cdot(b\#h) = (a\#1_{H_\alpha})(b\#h)$ for any $a \in A_\alpha$ and $b\#h \in A_\beta\#H_\beta$ with $\alpha, \beta \in Y$, the left A-module action on $\bigoplus_{\alpha\in Y} A_{\alpha\tau}[H_\alpha]$ can be defined similarly. It is clear that ϕ is a left A-module, right H-comodule map since for any $a \in A_\alpha$ and $b\#h \in A_\beta\#H_\beta$ with $\alpha, \beta \in Y$,

$$\begin{aligned}\phi(a\cdot(b\#h)) &= \phi((a\#1_{H_\alpha})(b\#h)) = \phi(a\#1_{H_\alpha})\phi(b\#h) = (a\mu(1_{H_\alpha})\otimes 1_{H_\alpha})\phi(b\#h)\\ &= (a\otimes 1_{H_\alpha})\phi(b\#h) = a\cdot\phi(b\#h)\end{aligned}$$

and

$$\begin{aligned}\rho(\phi(a\#h)) &= \rho(\sum_{(h)} a\mu(h')\otimes h'') = \sum_{(h)} a\mu(h')\otimes h''\otimes h''' = (\phi\otimes \mathrm{id})(\sum_{(h)} a\#h'\otimes h'')\\ &= (\phi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)(a\#h). \quad \Box\end{aligned}$$

The converse of this proposition is also true: that is, if $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha) \cong A_{\alpha\tau}[H_\alpha]$ for some as semilattice graded algebras by an isomorphism ϕ which is a left A-module and right H-comodule map, then the original action must have been G-inner via some $\mu \in \mathrm{Hom}(H, A)$ such that ϕ is given by the above proposition.

More generally, one can give necessary and sufficient conditions for two G-crossed products to be isomorphic.

Theorem 2.3 *Let $A = \bigoplus_{\alpha\in Y} A_\alpha$ be a semilattice graded algebra and $H = \bigoplus_{\alpha\in Y} H_\alpha$ be a semilattice graded weak Hopf algebra with the same semilattice Y, with two G-crossed product actions $h\otimes a \mapsto h\cdot a$ and $h\otimes a \mapsto h\cdot' a$ with respect to two cocycles $\sigma, \sigma' : H\otimes H \to A$, respectively. Assume that*

$$\phi : \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha) \to \bigoplus_{\alpha\in Y}(A_\alpha\#'_{\sigma'} H_\alpha)$$

is a semilattice graded algebra isomorphism, which is also a left A-module and right H-comodule map. Then, there exists an regular invertible map $\mu \in \mathrm{Hom}(H, A)$ and $\mu|_{H_\alpha} \in \mathrm{Hom}(H_\alpha, A_\alpha)$ for any $\alpha \in Y$ is invertible, such that for all $a \in A_\alpha$, $h \in H_\alpha$ and $k \in H_\beta$ with $\alpha, \beta \in Y$,

(1) $\phi(a\#h) = \sum_{(h)} a\mu(h')\#'h''$;

(2) $h\cdot' a = \sum_{(h)} \mu^{-1}(h')(h''\cdot a)\mu(h''')$;

(3) $\sigma'(h,k) = \sum_{(h),(k)} \mu^{-1}(h')(h''\cdot\mu^{-1}(k'))\sigma(h''',k'')\mu(h^{(4)}k''')$.

Conversely, given a map $\mu \in \mathrm{Hom}(H, A)$ with $\mu_\alpha = \mu|_{H_\alpha} \in \mathrm{Hom}(H_\alpha, A_\alpha)$ for any $\alpha \in Y$ such that (2) and (3) hold, then the map ϕ in (1) is a semilattice graded isomorphism.

Proof (1) Define $\mu \in \mathrm{Hom}(H, A)$ by $\mu(h) = (\mathrm{id}\otimes\varepsilon)\phi(1_{A_\alpha}\#h)$ for all $h \in H_\alpha$ with $\alpha \in Y$, then $\mu \in \mathrm{Hom}(H, A)$ and $\mu_\alpha = \mu|_{H_\alpha} \in \mathrm{Hom}(H_\alpha, A_\alpha)$. Since ϕ is a left A-module map, we have for any $a \in A_\alpha$,

$$(a\#'1_{H_\alpha})\phi(1_{A_\alpha}\#h) = \phi((a\#1_{A_\alpha})(1_{A_\alpha}\#h)) = \phi(\sum_{(h)} a(1_{H_\alpha}\cdot 1_{A_\alpha})\sigma(1_{H_\alpha}, h')\#1_{H_\alpha}h'')$$

$$= \phi(\sum_{(h)} a\varepsilon(h')1_{A_\alpha}\#h'') = \phi(a\#h).$$

Then, $(\mathrm{id}\otimes\varepsilon)\phi(a\#h) = (\mathrm{id}\otimes\varepsilon)[(a\#'1_{H_\alpha})\phi(1_{A_\alpha}\#h)] = (\mathrm{id}\otimes\varepsilon)(a\#'1_{H_\alpha})(\mathrm{id}\otimes\varepsilon)(\phi(1_{H_\alpha}\#h)) = a\varepsilon(1_{H_\alpha})\mu(h)$. Since ϕ is a right H-comodule map, we have

$$(\mathrm{id}\otimes\Delta)\circ\phi = (\phi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta).$$

Apply $\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}$ to both sides of the equation. The left side becomes $(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ\phi = \phi$ and the right side becomes

$$(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\circ((\phi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta))(a\#h) = (\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\circ(\phi\otimes\mathrm{id})(\sum_{(h)} a\#h'\otimes h'')$$
$$= (\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})(\sum_{(h)}\phi(a\#h')\otimes h'') = \sum_{(h)}(\mathrm{id}\otimes\varepsilon)\phi(a\#h')\otimes h'' = \sum_{(h)} a\mu(h')\#'h''$$

for any $a\#h \in \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$. Hence, $\phi(a\#h) = \sum_{(h)} a\mu(h')\#'h''$.

(2) Similarly, as $\phi^{-1} : \bigoplus_{\alpha\in Y}(A_\alpha\#'_{\sigma'}H_\alpha) \to \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is an isomorphism satisfying the same hypothesis as ϕ, we may set $\nu(h) = (\mathrm{id}\otimes\varepsilon)\phi^{-1}(1_{A_\alpha}\#'h)$ for all $h\in H_\alpha$ with any $\alpha\in Y$ and conclude as above that $\phi^{-1}(a\#'h) = \sum_{(h)} a\nu(h')\#h''$ for any $a\in A_\alpha$ and $h\in H_\alpha$ with $\alpha\in Y$. We claim that $\nu|_{H_\alpha} = \nu_\alpha = \mu_\alpha^{-1}$ and ν is a regular inverse of μ. Since for any $h\in H_\alpha$ with $\alpha\in Y$,

$$1_{A_\alpha}\#h = (\phi^{-1}\circ\phi)(1_{A_\alpha}\#h) = \phi^{-1}(\sum_{(h)} 1_{A_\alpha}\mu(h')\#'h'') = \phi^{-1}(\sum_{(h)}\mu(h')\#'h'')$$
$$= \sum_{(h)}\mu(h')\nu(h'')\#h'''.$$

Applying $\mathrm{id}\otimes\varepsilon$ to both sides, we see that the left side becomes $(\mathrm{id}\otimes\varepsilon)(1_{A_\alpha}\#h) = \varepsilon(h)1_{A_\alpha}$, and the right side becomes $(\mathrm{id}\otimes\varepsilon)(\sum_{(h)}\mu(h')\nu(h'')\#h''') = \sum_{(h)}\mu(h')\nu(h'')\varepsilon(h''') = \sum_{(h)}\mu(h')\nu(h'')$, hence, $\sum_{(h)}\mu(h')\nu(h'') = \varepsilon(h)1_{A_\alpha}$ for any $h\in H_\alpha$ with $\alpha\in Y$. Similarly, we see $\sum_{(h)}\nu(h')\mu(h'') = \varepsilon(h)1_{A_\alpha}$, and thus for any $\alpha\in Y$, $\nu_\alpha = \mu_\alpha^{-1}$. Moreover,

$$(\mu*\nu*\mu)(h) = \sum_{(h)}\mu(h')\nu(h'')\mu(h''') = \sum_{(h)}\varepsilon(h')1_{A_\alpha}\mu(h'') = \mu(h)$$

and

$$(\nu*\mu*\nu)(h) = \sum_{(h)}\nu(h')\mu(h'')\nu(h''') = \sum_{(h)}\varepsilon(h')1_{A_\alpha}\nu(h'') = \nu(h).$$

Hence, $\nu*\mu*\nu = \nu$ and $\mu*\nu*\mu = \mu$, thus ν is a regular inverse of μ such that $\nu_\alpha = \mu_\alpha^{-1}$.

Now for any $a\#'h \in A_\alpha\#'_{\sigma'}H_\alpha$, $b\#'k\in A_\beta\#'_{\sigma'}H_\beta$ with $\alpha,\beta\in Y$, the equation $\phi^{-1}((a\#'h)(b\#'k)) = \phi^{-1}(a\#'h)\phi^{-1}(b\#k)$ becomes

$$\phi^{-1}(\sum_{(h),(k)} a(h'\cdot' b)\sigma'(h'',k')\#'h'''k'') = \sum_{(h),(k)} a(h'\cdot' b)\sigma'(h'',k')\nu(h'''k'')\#h^{(4)}k'''$$
$$= \sum_{(h),(k)} a\nu(h')(h''\cdot(b\nu(k')))\sigma(h''',k'')\#h^{(4)}k'''$$
$$= (\sum_{(h)} a\nu(h')\#h'')(\sum_{(k)} b\nu(k')\#k'').$$

Setting $a = 1_{A_\alpha}$, $b = 1_{A_\beta}$ and applying $\mathrm{id} \otimes \varepsilon$ to both sides yields

$$
\begin{aligned}
&(\mathrm{id} \otimes \varepsilon)\Big(\sum_{(h),(k)} 1_{A_\alpha}(h' \cdot' 1_{A_\beta})\sigma'(h'', k')\nu(h'''k'')\#h^{(4)}k''')\\
&= \sum_{(h),(k)} 1_{A_\alpha}\varepsilon(h')1_{A_{\alpha\beta}}\sigma'(h'', k')\nu(h'''k'')\#\varepsilon(h^{(4)})\epsilon(k''')\\
&= \sum_{(h),(k)} \sigma'(h', k')\nu(h''k'')\#1_{H_{\alpha\beta}}\\
&= \sum_{(h),(k)} \nu(h')(h'' \cdot \nu(k'))\sigma(h''', k'')\#1_{H_{\alpha\beta}}\\
&= \sum_{(h),(k)} 1_{A_\alpha}\nu(h')(h'' \cdot \nu(k'))\sigma(h''', k'')\#\varepsilon(h^{(4)})\varepsilon(k''')\\
&= (\mathrm{id} \otimes \varepsilon)\Big(\sum_{(h),(k)} 1_{A_\alpha}\nu(h')(1'' \cdot (1_{A_\beta}\nu(k')))\sigma(h''', k'')\#h^{(4)}k''').
\end{aligned}
$$

Hence,

$$\sum_{(h),(k)} \sigma'(h', k')\nu(h''k'') = \sum_{(h),(k)} \nu(h')(h'' \cdot \nu(k'))\sigma(h'''k'').$$

Then

$$
\begin{aligned}
\sigma'(h, k) &= \sum_{(h),(k)} \nu(h')(h'' \cdot \nu(k'))\sigma(h''', k'')\nu^{-1}(h^{(4)}k''')\\
&= \sum_{(h),(k)} \mu^{-1}(h')(h'' \cdot \mu^{-1}(k'))\sigma(h''', k'')\mu(h^{(4)}k''')
\end{aligned}
$$

by using $\nu_\alpha = \mu_\alpha^{-1}$ for any $\alpha \in Y$. This proves (3).

(3) Using the above equation again with $a = 1_{A_\alpha}$, $k = 1_{H_\alpha}$ and applying $\mathrm{id} \otimes \varepsilon$ to both sides gives

$$
\begin{aligned}
&(\mathrm{id} \otimes \varepsilon)\Big(\sum_{(h)} 1_{A_\alpha}(h' \cdot' b)\sigma'(h'', 1_{H_\alpha})\nu(h'''1_{H_\alpha}))\#h^{(4)}1_{H_\alpha}\\
&= \sum_{(h)}(h' \cdot' b)\varepsilon(h'')1_{A_\alpha}\nu(h''')\#\varepsilon(h^{(4)})1_{H_\alpha}\\
&= \sum_{(h)}(h' \cdot' b)\nu(h'')\#1_{H_\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{id} \otimes \varepsilon)\Big(\sum_{(h)} 1_{A_\alpha}\nu(h')(h'' \cdot (b\nu(1_{H_\alpha})))\sigma(h''', 1_{H_\alpha})\#h^{(4)}1_{H_\alpha})\\
&= \sum_{(h)} 1_{A_\alpha}\nu(h')(h'' \cdot b1_{A_\alpha})\varepsilon(h''')1_{A_\alpha}\#\varepsilon(h^{(4)})1_{H_\alpha}\\
&= \sum_{(h)} \nu(h')(h'' \cdot b)\#1_{H_\alpha}.
\end{aligned}
$$

Hence,

$$\sum_{(h)}(h' \cdot' b)\nu(h'') = \sum_{(h)} \nu(h')(h'' \cdot b),$$

thus

$$h \cdot' b = \sum_{(h)} \nu(h')(h'' \cdot b)\nu^{-1}(h''') = \sum_{(h)} \mu^{-1}(h')(h'' \cdot b)\mu(h''').$$

The converse follows as in the proof of Proposition 2.2. $\square$

Corollary 2.4 *Let $H = \bigoplus_{\alpha\in Y} H_\alpha$ be a semilattice graded weak Hopf algebra and $A = \bigoplus_{\alpha\in Y} A_\alpha \subset B = \bigoplus_{\alpha\in Y} B_\alpha$ be a right H-G-extension which is H-G-cleft via $\gamma, \gamma' : H \to B$ with $\gamma(1_{H_\alpha}) = \gamma'(1_{H_\alpha}) = 1_{A_\alpha}$ for any $\alpha \in Y$. Let $\bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha)$ and $\bigoplus_{\alpha\in Y}(A_\alpha \#'_{\sigma'} H_\alpha)$ be the two representations of B as G-crossed products over A and H with the two cocycles σ, σ' and actions as defined in Proposition 2.4.5 and define $\mu = \gamma * (\gamma')^{-1}$ in $\mathrm{Hom}(H, B)$. Then the actions and cocycles are related as in the above proposition (2) and (3).*

Proof Let

$$\Phi : \bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha) \to B \quad \text{by} \quad a\#h \mapsto a\gamma(h)$$

and

$$\Phi : \bigoplus_{\alpha\in Y}(A_\alpha \#'_{\sigma'} H_\alpha) \to B \quad \text{by} \quad a\#'h \mapsto a\gamma'(h).$$

Since Φ and Φ' are semilattice graded algebra isomorphisms, left A-module and right H-comodule maps, so is

$$\Theta = (\Phi')^{-1}\Phi : \bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha) \to \bigoplus_{\alpha\in Y}(A_\alpha \#'_{\sigma'} H_\alpha)$$
$$a\#h \mapsto \sum_{(h)} a\mu(h')\#'h''.$$

Applying Φ' to both sides, we see that $a\gamma(h) = \sum_{(h)} a\mu(h')\gamma'(h'')$. Setting $a = 1_{A_\alpha}$ gives $\gamma = \mu * \gamma'$. The result follows. $\square$

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