Existence and Uniqueness of Solutions for a Nonlinear Fractional Integrodifferential Equation with Three-Point Fractional Boundary Conditions

Zhenyu GUO, Min LIU*, Zhijing WANG

School of Sciences, Liaoning Shihua University, Liaoning 113001*, P. R. China*

Abstract A nonlinear fractional integrodifferential equation with three-point fractional boundary conditions is studied in this paper, and some sufficient conditions for existence and uniqueness of solutions for the equation are established by Krasnoselskii fixed point theorem and Banach contraction principle, respectively.

Keywords   fractional integrodifferential equations; Krasnoselskii fixed point theorem; Banach contraction principle; boundary fractional conditions

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1. Introduction

This article is concerned with the following nonlinear fractional integrodifferential equation with three-point fractional boundary conditions:

$$D^q x(t) = f\big(t, x(t), (\phi x)(t), (\psi x)(t)\big), \quad 0 < t < 1, \ 1 < q \le 2,$$
$$D^{(q-1)/3} x(0) = 0, \quad D^{(q-1)/3} x(1) + ax(\eta) = 0, \ 0 < \eta < 1, \tag{1.1}$$

where D denotes the standard Riemann-Liouville fractional derivative, $f : [0,1] \times X \times X \times X \to X$ is a given continuous function, $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1], X)$ is the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$, $a \in \mathbb{R}$ satisfies $\Gamma(q) + a\eta^{q-1}\Gamma(\frac{2q+1}{3}) \ne 0$, for $\gamma, \delta : [0,1] \times [0,1] \to [0,+\infty)$,

$$(\phi x)(t) = \int_0^t \gamma(t,s)x(s)\mathrm{d}s, \quad (\psi x)(t) = \int_0^t \delta(t,s)x(s)\mathrm{d}s. \tag{1.2}$$

Recently, fractional order differential equations and systems have been of great interest. For detailed discussion on this topic, refer to the monographs of Kilbas et al. [1], and the papers by Anguraj et al. [2], Ahmad and Alsaedi [3,4], Cui [5], Guo and Liu [6–8], Kosmatov [9], Lakshmikantham and Vatsala [10], Li and Deng [11], Li [12], Li and Guérékata [13], Mao et al. [14] and the references therein.

Applying Krasnoselskii fixed point theorem and Banach contraction principle, we obtain theorems of existence and uniqueness of solutions for equation (1.1).

2. Preliminaries

Let us recall some basic definitions on fractional calculus, which can be found in the literature.

Definition 2.1 *The Riemann-Liouville fractional integral of order q is defined by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \mathrm{d}s, \quad q > 0, \tag{2.1}$$

provided the integral exists.

Definition 2.2 *The Riemann-Liouville fractional derivaltive of order q is defined by*

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-q-1} f(s) \mathrm{d}s, \quad n-1 < q \le n, \ q > 0, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 *For $q > 0$, let $x, D^q x \in C(0,1) \cap L(0,1)$. Then*

$$I^q D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 1, 2, \ldots, n$ (n is the smallest integer such that $n \ge q$).

Lemma 2.4 *Let $x \in L(0,1)$. Then*

(i) *$D^p I^q x(t) = I^{q-p} x(t), \ q > p > 0$;*

(ii) *$D^q t^{a-1} = (\Gamma(a)/\Gamma(a-q)) t^{a-q-1}, \ q > 0, a > 0$.*

Theorem 2.5 (Krasnoselskii fixed point theorem) *Let D be a closed convex and nonempty subset of a Banach space X, and A, B be two operators such that*

(i) *$Ax + By \in D$ whenever $x, y \in D$;*

(ii) *A is compact and continuous;*

(iii) *B is a contraction mapping.*

Then there exists $z \in D$ such that $z = Az + Bz$.

Lemma 2.6 *Given $f \in C(0,1) \cap L(0,1)$, the unique solution of*

$$D^q x(t) = f(t), \quad 0 < t < 1, \ 1 < q \le 2,$$

$$D^{(q-1)/3} x(0) = 0, \quad D^{(q-1)/3} x(1) + ax(\eta) = 0, \ 0 < \eta < 1 \tag{2.4}$$

is

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) \mathrm{d}s - \frac{\Gamma((2q+1)/3) t^{q-1}}{\Gamma(q) + a\eta^{q-1} \Gamma((2q+1)/3)} \times$$
$$\left(\int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} f(s) \mathrm{d}s + a \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s) \mathrm{d}s \right). \tag{2.5}$$

Proof It follows from Lemma 2.3 that the fractional differential equation in (2.4) is equivalent to the integral equation

$$x(t) = I^q f(t) + c_1 t^{q-1} + c_2 t^{q-2} = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + c_1 t^{q-1} + c_2 t^{q-2}, \qquad (2.6)$$

where $c_1, c_2 \in \mathbb{R}$. From the boundary conditions for (2.4), we have $c_2 = 0$ and

$$c_1 = -\frac{\Gamma((2q+1)/3)}{\Gamma(q) + a\eta^{q-1}\Gamma((2q+1)/3)} \Big(\int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} f(s)ds + a \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s)ds \Big), \quad (2.7)$$

which completes the proof. $\square$

Now list the following hypotheses for convenience:

(H1) There exist positive functions $L_1(t), L_2(t), L_3(t)$ such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \le L_1(t)\|x_1 - x_2\| + L_2(t)\|y_1 - y_2\| + L_3(t)\|z_1 - z_2\|,$$

$$\forall t \in [0,1], x_1, x_2, y_1, y_2, z_1, z_2 \in X. \qquad (2.8)$$

Further,

$$\gamma_0 = \sup_{t \in [0,1]} \Big| \int_0^t \gamma(t,s)ds \Big|, \quad \delta_0 = \sup_{t \in [0,1]} \Big| \int_0^t \delta(t,s)ds \Big|,$$

$$I^{(2q+1)/3}L(1) = \max\{|I^{(2q+1)/3}L_1(1)|, |I^{(2q+1)/3}L_2(1)|, |I^{(2q+1)/3}L_3(1)|\},$$

$$I^q L(\eta) = \max\{|I^q L_1(\eta)|, |I^q L_2(\eta)|, |I^q L_3(\eta)|\},$$

$$I_L^q = \sup_{t \in [0,1]} \{|I^q L_1(t)|, |I^q L_2(t)|, |I^q L_3(t)|\},$$

$$\lambda = \frac{\Gamma((2q+1)/3)}{\Gamma(q) + a\eta^{q-1}\Gamma((2q+1)/3)}. \qquad (2.9)$$

(H2) $\|f(t,x,y,z)\| \le \mu(t)$, for all $(t,x,y,z) \in [0,1] \times X \times X \times X, \mu \in L^1([0,1], \mathbb{R}^+)$.

3. Main results

In this section, the theorems of existence and uniqueness of a solution for equation (1.1) will be given.

Theorem 3.1 *Assume that $f : [0,1] \times X \times X \times X \to X$ is jointly continuous and satisfies (H1) and (H2). If*

$$\lambda(1 + \gamma_0 + \delta_0)\big(I^{(2q+1)/3}L(1) + |a|I^q L(\eta)\big) < 1, \qquad (3.1)$$

then the fractional integrodifferential equation (1.1) has at least one solution.

Proof Consider $B_r = \{x \in C : \|x\| \le r\}$, where

$$r \ge \Big(\frac{1 + \lambda|a|\eta^q}{\Gamma(q+1)} + \frac{\lambda}{\Gamma((2q+4)/3)} \Big) \|\mu\|_{L^1}. \qquad (3.2)$$

Define two mappings A and B on B_r by

$$(Ax)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big)ds,$$

$$(Bx)(t) = -\frac{\Gamma((2q+1)/3)t^{q-1}}{\Gamma(q) + a\eta^{q-1}\Gamma((2q+1)/3)} \times$$
$$\left(\int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s + \right.$$
$$\left. a \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s \right). \tag{3.3}$$

For $x, y \in B_r$, by (H2), we obtain

$$\|(Ax)(t) + (By)(t)\| \le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \big\| \mathrm{d}s +$$
$$\left| \frac{\Gamma((2q+1)/3)t^{q-1}}{\Gamma(q) + a\eta^{q-1}\Gamma((2q+1)/3)} \right| \times$$
$$\left(\int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big\| f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big) \big\| \mathrm{d}s + \right.$$
$$\left. |a| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \big\| f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big) \big\| \mathrm{d}s \right)$$
$$\le \|\mu\|_{L^1} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \mathrm{d}s + \lambda \|\mu\|_{L^1} \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \mathrm{d}s +$$
$$\lambda |a| \cdot \|\mu\|_{L^1} \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \mathrm{d}s$$
$$= \|\mu\|_{L^1} \left(\frac{t^q}{\Gamma(q+1)} + \frac{\lambda}{\Gamma((2q+4)/3)} + \frac{\lambda|a|\eta^q}{\Gamma(q+1)} \right)$$
$$\le \|\mu\|_{L^1} \left(\frac{1 + \lambda|a|\eta^q}{\Gamma(q+1)} + \frac{\lambda}{\Gamma((2q+4)/3)} \right) \le r, \tag{3.4}$$

which means $Ax + By \in B_r$.

It is claimed that A is compact and continuous. Continuity of f implies that $(Ax)(t)$ is continuous. $(Ax)(t)$ is uniformly bounded on B_r as

$$\|Ax\| \le \frac{\|\mu\|_{L^1}}{\Gamma(q+1)}. \tag{3.5}$$

Since f is bounded on the compact set $[0,1] \times B_r \times B_r \times B_r$, let

$$\sup_{(t,x,\phi x,\psi x) \in [0,1] \times B_r \times B_r \times B_r} \|f(t, x, \phi x, \psi x)\| = f_{\max}.$$

Then, for $t_1, t_2 \in [0,1]$, we get

$$\|(Ax)(t_2) - (Ax)(t_1)\| = \frac{1}{\Gamma(q)} \left\| \int_0^{t_1} \big((t_2-s)^{q-1} - (t_1-s)^{q-1}\big) f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s + \right.$$
$$\left. \int_{t_1}^{t_2} (t_2-s)^{q-1} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s \right\|$$
$$\le \frac{f_{\max}}{\Gamma(q)} \left| -\frac{(t_2-t_1)^q}{q} + \frac{t_2^q}{q} - \frac{t_1^q}{q} + \frac{(t_2-t_1)^q}{q} \right|$$
$$\le \frac{f_{\max}}{\Gamma(q+1)} |t_2^q - t_1^q|, \tag{3.6}$$

which is independent of x. Therefore, A is relatively compact on B_r. By Arzela-Ascoli's Theorem,

A is compact on B_r.

For $x, y \in B_r$ and $t \in [0, 1]$, by (H1), we have

$$\|(Bx)(t) - (By)(t)\|$$
$$\leq \lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) - f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big) \big\| \mathrm{d}s +$$
$$\lambda |a| \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) - f\big(s, y(s), (\phi y)(s), (\psi y)(s)\big) \big\| \mathrm{d}s$$
$$\leq \lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big(L_1(s)\|x - y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|\big) \mathrm{d}s +$$
$$\lambda |a| \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} \big(L_1(s)\|x - y\| + L_2(s)\|\phi x - \phi y\| + L_3(s)\|\psi x - \psi y\|\big) \mathrm{d}s$$
$$\leq \lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\big)\|x - y\| \mathrm{d}s +$$
$$\lambda |a| \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} \big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\big)\|x - y\| \mathrm{d}s$$
$$\leq \lambda \big(I^{(2q+1)/3} L_1(1) + \gamma_0 I^{(2q+1)/3} L_2(1) + \delta_0 I^{(2q+1)/3} L_3(1)\big)\|x - y\| +$$
$$\lambda |a| \big(I^q L_1(\eta) + \gamma_0 I^q L_2(\eta) + \delta_0 I^q L_3(\eta)\big)\|x - y\|$$
$$\leq \lambda(1 + \gamma_0 + \delta_0)\big(I^{(2q+1)/3} L(1) + |a| I^q L(\eta)\big)\|x - y\|. \tag{3.7}$$

It follows from (3.1) that B is a contraction mapping. Thus, by Krasnoselskii fixed point theorem, (1.1) has at least one solution.

Theorem 3.2 *Assume that $f : [0, 1] \times X \times X \times X \to X$ is jointly continuous and satisfies (H1). If*

$$\Lambda = (1 + \gamma_0 + \delta_0)\big(I_L^q + \lambda I^{(2q+1)/3} L(1) + \lambda |a| I^q L(\eta)\big) < 1, \tag{3.8}$$

then the fractional integrodifferential equation (1.1) has a unique solution.

Proof Define a mapping $F : C \to C$ by

$$(Fx)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s -$$
$$\frac{\Gamma((2q+1)/3)t^{q-1}}{\Gamma(q) + a\eta^{q-1}\Gamma((2q+1)/3)} \times$$
$$\Big(\int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s +$$
$$a \int_0^\eta \frac{(\eta - s)^{q-1}}{\Gamma(q)} f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \mathrm{d}s \Big), \quad t \in [0, 1]. \tag{3.9}$$

Let $\sup_{t \in [0,1]} |f(t, 0, 0, 0)| = M$, and choose

$$r \geq \frac{(1 + \lambda |a| \eta^q)\Gamma((2q+4)/3) + \lambda \Gamma(q+1)}{(1 - \Lambda)\Gamma(q+1)\Gamma((2q+4)/3)} M. \tag{3.10}$$

It is claimed that $FB_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. In fact, for $x \in B_r$, by (3.8), (3.10)

and (H1), we obtain

$$
\|(Fx)(t)\| \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \big\| \mathrm{d}s +
$$

$$
\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \big\| \mathrm{d}s +
$$

$$
\lambda|a| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) \big\| \mathrm{d}s
$$

$$
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big(\big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) - f(s,0,0,0) \big\| + \| f(s,0,0,0) \| \big) \mathrm{d}s +
$$

$$
\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big(\big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) - f(s,0,0,0) \big\| + \| f(s,0,0,0) \| \big) \mathrm{d}s +
$$

$$
\lambda|a| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \big(\big\| f\big(s, x(s), (\phi x)(s), (\psi x)(s)\big) - f(s,0,0,0) \big\| + \| f(s,0,0,0) \| \big) \mathrm{d}s
$$

$$
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big(L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M \big) \mathrm{d}s +
$$

$$
\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big(L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M \big) \mathrm{d}s +
$$

$$
\lambda|a| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \big(L_1(s)\|x(s)\| + L_2(s)\|(\phi x)(s)\| + L_3(s)\|(\psi x)(s)\| + M \big) \mathrm{d}s
$$

$$
\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big(L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M \big) \mathrm{d}s +
$$

$$
\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)} \big(L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M \big) \mathrm{d}s +
$$

$$
\lambda|a| \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} \big(L_1(s)\|x(s)\| + \gamma_0 L_2(s)\|x(s)\| + \delta_0 L_3(s)\|x(s)\| + M \big) \mathrm{d}s
$$

$$
\leq \big(I^q L_1(t) + \gamma_0 I^q L_2(t) + \delta_0 I^q L_3(t) \big) r + \frac{M t^q}{\Gamma(q+1)} +
$$

$$
\lambda \big(I^{(2q+1)/3} L_1(1) + \gamma_0 I^{(2q+1)/3} L_2(1) \delta_0 I^{(2q+1)/3} L_3(1) \big) r + \frac{\lambda M}{\Gamma((2q+4)/3)} +
$$

$$
\lambda|a| \big(I^q L_1(\eta) + \gamma_0 I^q L_2(\eta) + \delta_0 I^q L_3(\eta) \big) r + \frac{\lambda|a|\eta^q M}{\Gamma(q+1)}
$$

$$
\leq I_L^q (1 + \gamma_0 + \delta_0) r + \lambda I^{(2q+1)/3} L(1)(1 + \gamma_0 + \delta_0) r + \lambda|a| I^q L(\eta)(1 + \gamma_0 + \delta_0) r +
$$

$$
\frac{(1 + \lambda|a|\eta^q) M}{\Gamma(q+1)} + \frac{\lambda M}{\Gamma((2q+4)/3)}
$$

$$
\leq (1 + \gamma_0 + \delta_0) \big(I_L^q + \lambda I^{(2q+1)/3} L(1) + \lambda|a| I^q L(\eta) \big) r +
$$

$$
\frac{(1 + \lambda|a|\eta^q)\Gamma((2q+4)/3) + \lambda\Gamma(q+1)}{\Gamma(q+1)\Gamma((2q+4)/3)} M
$$

$$
\leq \Lambda r + (1 - \Lambda) r = r. \tag{3.11}
$$

It is declared that F is a contraction mapping. For $x, y \in C$ and $t \in [0,1]$, by (3.8) and (H1), we have

$$
\|(Fx)(t) - (Fy)(t)\|
$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big\| f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) - f\big(s,y(s),(\phi y)(s),(\psi y)(s)\big)\big\|\mathrm{d}s+$$

$$\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)}\big\| f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) - f\big(s,y(s),(\phi y)(s),(\psi y)(s)\big)\big\|\mathrm{d}s+$$

$$\lambda|a|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\big\| f\big(s,x(s),(\phi x)(s),(\psi x)(s)\big) - f\big(s,y(s),(\phi y)(s),(\psi y)(s)\big)\big\|\mathrm{d}s$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big(L_1(s)\|x-y\| + L_2(s)\|\phi x-\phi y\| + L_3(s)\|\psi x-\psi y\|\big)\mathrm{d}s+$$

$$\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)}\big(L_1(s)\|x-y\| + L_2(s)\|\phi x-\phi y\| + L_3(s)\|\psi x-\psi y\|\big)\mathrm{d}s+$$

$$\lambda|a|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\big(L_1(s)\|x-y\| + L_2(s)\|\phi x-\phi y\| + L_3(s)\|\psi x-\psi y\|\big)\mathrm{d}s$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\big)\|x-y\|\mathrm{d}s+$$

$$\lambda \int_0^1 \frac{(1-s)^{(2q-2)/3}}{\Gamma((2q+1)/3)}\big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\big)\|x-y\|\mathrm{d}s+$$

$$\lambda|a|\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}\big(L_1(s) + \gamma_0 L_2(s) + \delta_0 L_3(s)\big)\|x-y\|\mathrm{d}s$$

$$\leq \big(I^q L_1(t) + \gamma_0 I^q L_2(t) + \delta_0 I^q L_3(t)\big)\|x-y\|+$$

$$\lambda\big(I^{(2q+1)/3}L_1(1) + \gamma_0 I^{(2q+1)/3}L_2(1) + \delta_0 I^{(2q+1)/3}L_3(1)\big)\|x-y\|+$$

$$\lambda|a|\big(I^q L_1(\eta) + \gamma_0 I^q L_2(\eta) + \delta_0 I^q L_3(\eta)\big)\|x-y\|$$

$$\leq (1+\gamma_0+\delta_0)\big(I_L^q + \lambda I^{(2q+1)/3}L(1) + \lambda|a|I^q L(\eta)\big)\|x-y\|$$

$$= \Lambda\|x-y\|. \tag{3.16}$$

$\Lambda < 1$ ensures that F is contractive. Therefore, the conclusion of the theorem follows from the contraction mapping principle. $\square$

4. An example

Consider the following fractional integrodifferential equation

$$D^{3/2}x(t) = tx(t) + t^2\int_0^t s^5 x(s)\mathrm{d}s + t\int_0^t s^7 x(s)\mathrm{d}s, \quad 0 < t < 1,$$

$$D^{1/6}x(0) = 0, \quad D^{1/6}x(1) + 2x(1/2) = 0. \tag{4.1}$$

Comparing (4.1) and (1.1), we see that $q = 3/2, f(t,x,y,z) = tx + ty + tz, \gamma(t,s) = ts^5, \delta(t,s) = s^7, a = 2$, and $\eta = 1/2$. If we choose $L_1(t) = L_2(t) = L_3(t) = t$, then (H1) holds. From (2.9), it is easy to get that $\gamma_0 = 1/6, \delta_0 = 1/8, I^{(2q+1)/3}L(1) = \frac{9}{28\Gamma(4/3)}, I^q L(\eta) = \frac{1}{15\sqrt{2}\Gamma(3/2)}, I_L^q = \frac{4}{15\Gamma(3/2)}$, and

$$\lambda = \frac{\Gamma(4/3)}{\Gamma(3/2) + 2/\sqrt{2}\Gamma(4/3)} < \frac{\Gamma(4/3)}{\Gamma(4/3) + \sqrt{2}\Gamma(4/3)} = \frac{1}{1+\sqrt{2}}. \tag{4.2}$$

Noting that $\Gamma(3/2) > 4/5, \Gamma(4/3) > 1$ and (4.2), we have

$$(1+\gamma_0+\delta_0)\big(I_L^q + \lambda I^{(2q+1)/3}L(1) + \lambda|a|I^q L(\eta)\big)$$

$$= \frac{31}{24}\Big(\frac{4}{15\Gamma(3/2)} + \lambda\frac{9}{28\Gamma(4/3)} + \lambda\frac{8}{15\Gamma(3/2)}\Big)$$
$$< \frac{31}{72}\Big(1 + \frac{3}{1+\sqrt{2}}\Big) < 1,$$

that is, (3.8) holds. Thus, by Theorem 3.2, we obtain that (1.1) has a unique solution.

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