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Nontrivial Solutions for a Class of Quasilinear Elliptic Equations

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Abstract The main purpose of this paper is to establish the existence results of one nontrivial solution (infinitely many nontrivial solutions) for a class of p-Laplacian equation with subcritical polynomial growth and subcritical exponential growth by using a linking theorem and the symmetric mountain pass theorem.

Keywords Linking theorem; Adams-type inequality; subcritical polynomial growth; subcritical exponential growth

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$. We consider the following quasilinear elliptic boundary problem

$$\begin{cases} -\Delta_p u(x) - \mu\triangle u = lu + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $2 < p < \infty$, $\triangle_p$ denotes the p-Laplacian operator defined by $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, μ, $l \geq 0$ are real parameters and $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$.

It is known that the nontrivial solutions of problem (1.1) are equivalent to the corresponding nonzero critical points of the C^1- energy functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{\mu}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{l}{2}\int_\Omega |u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \tag{1.2}$$

for all $u \in W_0^{1,p}(\Omega)$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

For the case of $p > 2$, $l = 0$ and $\mu > 0$, there has been an increasing interest in looking for the existence of solutions of (1.1). Using the following conditions

$$\lambda_m < f'(x,0) < \lambda_{m+1}, \quad F(x,t) < \frac{\mu_1}{p}|t|^p + C, \ x \in \Omega,$$

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where $m \geq 1$ and C is a constant, the authors in [1,2] proved that (1.1) has at least two nontrivial solutions by the three critical point theorems, here and in the sequel $0 < \lambda_1 < \lambda_2 < \cdots$, λ_i $(i = 1, 2, \ldots)$ denotes the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$, and μ_1 is the first eigenvalue of $-\triangle_p$ in $W_0^{1,p}(\Omega)$ (see [3]). For Eq. (1.1) with right-hand side having p-linear growth at infinity, i.e., $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda \notin \sigma(-\triangle_p)$, the spectrum of $-\triangle_p$ in $W_0^{1,p}(\Omega)$, the papers [4,5] get the existence of nontrivial solution. In [6], the author extended the results in [1,2] under the general asymptotically linear condition (compared with the previous paper [7]).

The main purpose of this paper is to establish existence and multiplicity result for problem (1.1) with $2 < p \leq N$ when the nonlinear term f satisfies a weaker condition (a new kind of subcritical polynomial growth or subcritical exponential growth) but not satisfying the Ambrosetti-Rabinowitz condition. It is worth observing that the right-hand side of equation (1.1) possibly has the resonant term. This leads us to adapt linking theorem to study problem (1.1) and obtain one nontrivial solution. We will also obtain infinitely many nontrivial solutions by using an improved symmetric mountain pass theorem if the nonlinearity term f is odd.

When $2 < p < N$, there have been substantial amount of works to study the existence of nontrivial solution for (1.1). Nevertheless, almost all of the works involve the nonlinear term $f(x, u)$ of a subcritical (polynomial) growth, say,

(SCP): There exist positive constants c_1 and c_2 and $q_0 \in (p-1, p^*-1)$ such that

$$|f(x,t)| \leq c_1 + c_2|t|^{q_0} \text{ for all } t \in \mathbb{R} \text{ and } \ x \in \Omega,$$

where $p^* = Np/(N-p)$ denotes the critical Sobolev exponent. One of the main reasons to assume this condition (SCP) is that they can use the Sobolev compact embedding $W_0^{1,p} \hookrightarrow L^q(\Omega)$, $1 \leq q < p^*$.

In this paper, we always assume that $\mu = 1$ in (1.1). Under the motivation of Lam and Lu [8], our first main results will be to study problem (1.1) in the improved subcritical polynomial growth

$$(\text{SCPI}): \lim_{t\to\infty} \frac{f(x,t)}{t^{p^*-1}} = 0 \text{ uniformly on } x \in \Omega,$$

which is weaker than (SCP). Note that in this case, we do not have the Sobolev compact embedding anymore. Our work again is to study problem (1.1) without the (AR)-condition. In fact, this condition was studied by Liu and Wang in [9] in the case of Laplacian (i.e., $p = 2$ and $\mu, l = 0$) by the Nehari manifold approach. However, we will use the linking theorem (or an improved symmetric mountain pass theorem) to get the one nontrivial solution (or infinitely many nontrivial solutions) to problem (1.1) in the general case $2 < p < N$.

Let us now state our results: Suppose that $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$ and satisfies:

(H_1) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly for all $x \in \Omega$;

(H_2) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \infty$ uniformly for all $x \in \Omega$;

(H_3) There is a constant $\theta \geq 1$ such that for all $(x,t) \in \Omega \times R$ and $s \in [0,1]$,

$$\theta(f(x,t)t - pF(x,t)) \geq (sf(x,st)t - pF(x,st));$$

Theorem 1.1 *Let $2 < p < N$ and assume that f has the improved subcritical polynomial* *growth on Ω (condition (SCPI)) and satisfies (H_1)–(H_3). If $l = \lambda_i$ $(i \geq 2)$, then problem (1.1) has at least a nontrivial solution.*

Theorem 1.2 *Let $2 < p < N$ and assume that f has the improved subcritical polynomial growth on Ω (condition (SCPI)) and satisfies (H_1)–(H_3). If $f(x,t)$ is odd in t and $l = \lambda_i$ $(i \geq 1)$, then problem (1.1) has infinitely many nontrivial solutions.*

In case of $p = N$, we have $p^* = +\infty$. In this case, every polynomial growth is admitted, but one knows easy examples that $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to R^+$ with maximal growth such that

$$\sup_{u \in W_0^{1,N}, \|u\| \leq 1} \int_\Omega g(u) \mathrm{d}x < \infty.$$

It was shown by Trudinger [10] and Moser [11] that the maximal growth is of exponential type. So, we must redefine the subcritical (exponential) growth in this case as follows:

(SCE): f has subcritical (exponential) growth on Ω, i.e., $\lim_{t \to \infty} \frac{|f(x,t)|}{\exp(\alpha |t|^{\frac{N}{N-1}})} = 0$ uniformly on $x \in \Omega$ for all $\alpha > 0$.

When $p = N$ and f has the subcritical (exponential) growth (SCE), our work is still to study problem (1.1) without the (AR)-condition. To our knowledge, this case is completely new. Our results are as follows:

Theorem 1.3 *Let $p = N$ and assume that f has the subcritical exponential growth on Ω (condition (SCE)) and satisfies (H_1)–(H_3). If $l = \lambda_i$ $(i \geq 2)$, then problem (1.1) has at least a nontrivial solution.*

Theorem 1.4 *Let $p = N$ and assume that f has the subcritical exponential growth on Ω (condition (SCE)) and satisfies (H_1)–(H_3). If $f(x,t)$ is odd in t and $l = \lambda_i$ $(i \geq 1)$, then problem (1.1) has infinitely many nontrivial solutions.*

2. Preliminaries and some lemmas

Let X be a Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

Consider two sequences of subspaces:

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1, X_0^2 \subset X_1^2 \subset \cdots \subset X^2$$

such that

$$X^j = \cup_{n \in N} X_n^j, \quad j = 1, 2.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1}^1 \oplus X_{\alpha_2}^2$. We know that

$$\alpha \leq \beta \Leftrightarrow \alpha_1 \leq \beta_1, \ \alpha_2 \leq \beta_2.$$

A sequence $(\alpha_n) \subset N^2$ is admissible if for every $\alpha \in N^2$, there is $m \in N$ such that $n \geq m \Rightarrow \alpha_n \geq \alpha$. For every $I : X \to R$, we denote by I_α the function I restricted on X_α.

Definition 2.1 *Let I be locally Lipschitz on X and $c \in R$. The functional I satisfies the $(C)_c^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and*

$$u_{\alpha_n} \in X_{\alpha_n},\ I(u_{\alpha_n}) \to c,\ (1+\|u_{\alpha_n}\|)I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

Definition 2.2 *Let I be locally Lipschitz on X and $c \in R$. The functional I satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and*

$$u_{\alpha_n} \in X_{\alpha_n},\ \sup_n I(u_{\alpha_n}) \le c,\ (1+||u_{\alpha_n}||)I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

Remark 2.3 (1) The $(C)^*$ condition implies the $(C)_c^*$ condition for every $c \in R$.

(2) When the $(C)_c^*$ sequence is bounded, then the sequence is a $(PS)_c^*$ sequence [12].

(3) Without loss of generality, we assume that the norm in X satisfies

$$\|u_1+u_2\|^2 = \|u_1\|^2 + \|u_2\|^2,\quad u_j \in X_j, j=1,2.$$

Definition 2.4 *Let X be a Banach space with a direct sum decomposition*

$$X = X^1 \oplus X^2.$$

The function $I \in C^1(X,R)$ has a local linking at 0, with respect to (X^1, X^2), if , for some $r>0$,

$$I(u) \ge 0,\ u \in X^1,\ \|u\| \le r,$$

$$I(u) \le 0,\ u \in X^2,\ \|u\| \le r.$$

Lemma 2.5 ([13]) *Suppose that $I \in C^1(X,R)$ satisfies the following assumptions:*

(B_1) I has a local linking at 0 and $X^1 \ne \{0\}$;

(B_2) I satisfies $(PS)^*$;

(B_3) I maps bounded sets into bounded sets;

(B_4) for every $m \in N$, $I(u) \to -\infty, \|u\| \to \infty, u \in X = X_m^1 \oplus X^2$. Then I has at least two critical points.

Remark 2.6 Assume I satisfies the $(C)_c^*$ condition. Then this theorem still holds.

Lemma 2.7 ([10,11]) *Let $u \in W_0^{1,N}(\Omega)$. Then $\exp(|u|^{\frac{N}{N-1}}) \in L^q(\Omega)$ for all $1 \le q < \infty$. Moreover,*

$$\sup_{u\in W_0^{1,N}(\Omega),\|u\|\le 1} \int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}})\mathrm{d}x \le C(\Omega) \text{ for } \alpha \le \alpha_N.$$

The inequality is optimal: for any growth $\exp(\alpha|u|^{\frac{N}{N-1}})$ with $\alpha > \alpha_N$ the corresponding supremum is $+\infty$.

3. Proofs of the main results

Proof of Theorem 1.1 (1) Since $p>2$, we shall apply Lemma 2.5 to the functional $I(u)$. Let

$$H^- = \oplus_{i\le m-1}\ker(-\triangle - \lambda_i),$$

$$H^0 = \ker(-\triangle - \lambda_m),$$

$$H^+ = \overline{\oplus_{j\geq m+1}\ker(-\triangle - \lambda_j)}.$$

Then we have

$$W_0^{1,2}(\Omega) = H^- \oplus H^0 \oplus H^+.$$

Set $X^2 = H^-$. Since $p > 2$, by the regularity theory (see[14]) we have

$$X^2 \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and $W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ continuously. Let $X^1 = (H^+ \cup H^0) \cap W_0^{1,p}(\Omega)$. Then we get the splitting

$$W_0^{1,p}(\Omega) = X^1 \oplus X^2.$$

Now, we choose a Hilbertian basis $e_n (n \geq 0)$ for X^1 and define

$$X_n^1 = \mathrm{span}(e_0, e_1, \dots, e_n),\ n \in N;$$

$$X_n^2 = X^2,\ n \in N;$$

$$X^1 = \overline{\bigcup_{n\in N} X_n^1}.$$

By the condition (H_1) and the Sobolev inequalities, it is easy to see that the functional I belongs to $C^1(X, R)$ and maps bounded sets to bounded sets.

(2) We claim that I has a local linking at 0 with respect to (X^1, X^2). It follows from (H_1) that, for any $\epsilon > 0$ small enough,

$$|F(x,u)| \leq \epsilon |u|^p, \text{ as } \|u\| \text{ is small.}$$

So, we have

$$\begin{aligned} I(u) &\leq \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x - \frac{l}{2}\int_\Omega |u|^2 \mathrm{d}x - \frac{\epsilon}{p}\int_\Omega |u|^p \mathrm{d}x \\ &\leq -C^*\|u\|^2 + C^{**}\|u\|^p, \end{aligned}$$

where C^* and C^{**} are positive constants. Hence, for $r > 0$ small enough,

$$I(u) \leq 0,\ u \in X^2,\ \|u\| \leq r.$$

By conditions (H_1) and (SCPI), for any $\epsilon > 0$ small enough, there exists C_ϵ such that

$$F(x,u) \leq \frac{\epsilon}{p}|u|^p + C_\epsilon |u|^{p^*},\ u \in R,\ x \in \Omega.$$

Then for $u \in X^1$ we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x - \frac{1}{2}\int_\Omega l u^2 - \frac{\epsilon}{p}\int_\Omega |u|^p \mathrm{d}x - C_\epsilon \int_\Omega |u|^{p^*} \mathrm{d}x \\ &\geq C_3 ||u||^p - C_4 \|u\|^{p^*}, \end{aligned}$$

which implies that

$$I(u) \geq 0,\ \forall u \in X^1 \text{ with } ||u|| \leq r$$

for $0 < r$ small enough.

(3) We claim that I satisfies $(C)^*_c$. Consider a sequence (u_{α_n}) such that (α_n) is admissible, $\|u_{\alpha_n}\| \to \infty$ and

$$u_{\alpha_n} \in X_{\alpha_n},\ I(u_{\alpha_n}) \to c,\ (1+||u_{\alpha_n}||)I'(u_{\alpha_n}) \to 0 \tag{3.1}$$

and

$$\lim_{n\to\infty} \Big\{(\frac{1}{2}-\frac{1}{p})\int_\Omega |\nabla u_{\alpha_n}|^2 \mathrm{d}x + (\frac{l}{2}-\frac{l}{p})\int_\Omega |u_{\alpha_n}|^2 \mathrm{d}x + \int_\Omega (\frac{1}{p} f(x,u_{\alpha_n})u_{\alpha_n} - F(x,u_{\alpha_n}))\mathrm{d}x\Big\} = c. \tag{3.2}$$

Let $w_{\alpha_n} = ||u_{\alpha_n}||^{-1}u_{\alpha_n}$. Up to a subsequence, we have

$$w_{\alpha_n} \rightharpoonup w \text{ in } X,\ w_{\alpha_n} \to w \text{ in } L^p,\ w_{\alpha_n}(x) \to w(x) \text{ a.e. } x \in \Omega.$$

If $w = 0$, we choose a sequence $\{t_n\} \subset [0,1]$ such that

$$I(t_n u_{\alpha_n}) = \max_{t\in[0,1]} I(tu_{\alpha_n}).$$

For any $m > 0$, let $v_{\alpha_n} = (2pm)^{\frac{1}{p}} w_{\alpha_n}$. By the Sobolev imbedded theory, we have

$$\lim_{n\to\infty} \int_\Omega F(x, v_{\alpha_n})\mathrm{d}x = 0.$$

So for n large enough, $(2pm)^{\frac{1}{p}}||u_{\alpha_n}||^{-1} \in (0,1)$, we have

$$I(t_n u_{\alpha_n}) \ge I(v_{\alpha_n}) \ge 2m - \epsilon \ge m, \tag{3.3}$$

where ϵ is a small enough constant. That is, $I(t_n u_{\alpha_n}) \to \infty$. Now, $I(0) = 0$, $I(u_{\alpha_n}) \to c$, we know that $t_n \in [0,1]$ and

$$\int_\Omega |\nabla(t_n u_{\alpha_n})|^p \mathrm{d}x + \int_\Omega |\nabla(t_n u_{\alpha_n})|^2 \mathrm{d}x + \int_\Omega |t_n u_{\alpha_n}|^2 \mathrm{d}x - \int_\Omega f(x, t_n u_{\alpha_n}) t_n u_{\alpha_n} \mathrm{d}x = t_n \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_n} I(tu_{\alpha_n}) = 0. \tag{3.4}$$

Therefore, using (H_3), we have

$$\begin{aligned}&(\frac{1}{2}-\frac{1}{p})\int_\Omega |\nabla u_{\alpha_n}|^2 \mathrm{d}x + (\frac{l}{2}-\frac{l}{p})\int_\Omega |u_{\alpha_n}|^2 \mathrm{d}x + \int_\Omega \frac{1}{p} f(x,u_{\alpha_n})u_{\alpha_n} - F(x,u_{\alpha_n})\mathrm{d}x \\ &\ge (\frac{1}{2}-\frac{1}{p})\int_\Omega |\nabla u_{\alpha_n}|^2 \mathrm{d}x + (\frac{l}{2}-\frac{l}{p})\int_\Omega |u_{\alpha_n}|^2 \mathrm{d}x + \\ &\quad \frac{1}{\theta}\int_\Omega (\frac{1}{p} f(x, t_n u_{\alpha_n}) t_n u_{\alpha_n} - F(x, t_n u_{\alpha_n}))\mathrm{d}x \to +\infty.\end{aligned}$$

This contradicts (3.2). If $w \ne 0$, then the set $\ominus = \{x \in \Omega : w(x) \ne 0\}$ has a positive Lebesgue measure. For $x \in \ominus$, we have $|u_{\alpha_n}(x)| \to \infty$. Hence, by ($\mathrm{H}_3$), we have

$$\frac{f(x, u_{\alpha_n}(x))u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^p} |w_{\alpha_n}(x)|^p \mathrm{d}x \to \infty. \tag{3.5}$$

From (3.1), we obtain

$$1 - \circ(1) \ge \Big(\int_{w\ne 0} + \int_{w=0}\Big) \frac{f(x, u_{\alpha_n}(x))u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^p} |w_{\alpha_n}(x)|^p \mathrm{d}x. \tag{3.6}$$

By (3.5), the right-hand side of (3.6) $\to +\infty$. This is a contradiction.

In any case, we obtain a contradiction. Therefore, $\{u_{\alpha_n}\}$ is bounded.

Now, we prove that $\{u_n\}$ ($=\{u_{\alpha_n}\}$) has a convergence subsequence. In fact, we can suppose that

$$u_n \rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega),$$

$$u_n \to u \text{ in } L^q(\Omega),\ \forall 1 \le q < p^*,$$

$$u_n(x) \to u(x) \text{ a.e. } x \in \Omega.$$

Now, since f has the subcritical growth on Ω, for every $\epsilon > 0$, we can find a constant $C(\epsilon) > 0$ such that

$$f(x,s) \le C(\epsilon) + \epsilon|s|^{p^*-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned}
&\Big|\int_\Omega f(x,u_n)(u_n-u)\mathrm{d}x\Big| \\
&\le C(\epsilon)\int_\Omega |u_n-u|\mathrm{d}x + \epsilon\int_\Omega |u_n-u||u_n|^{p^*-1}\mathrm{d}x \\
&\le C(\epsilon)\int_\Omega |u_n-u|\mathrm{d}x + \epsilon\Big(\int_\Omega (|u_n|^{p^*-1})^{\frac{p^*}{p^*-1}}\mathrm{d}x\Big)^{\frac{p^*-1}{p^*}}\Big(\int_\Omega |u_n-u|^{p^*}\Big)^{\frac{1}{p^*}} \\
&\le C(\epsilon)\int_\Omega |u_n-u|\mathrm{d}x + \epsilon C(\Omega).
\end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $\int_\Omega |u_n-u|\mathrm{d}x \to 0$. Since $\epsilon > 0$ is arbitrary, we can conclude that

$$\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)\mathrm{d}x \to 0 \text{ as } n\to\infty. \tag{3.10}$$

By (3.2), we have

$$\langle I'(u_n) - I'(u), (u_n-u)\rangle \to 0 \text{ as } n \to \infty. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u) \to 0 \text{ as } n \to \infty.$$

Using an elementary inequality

$$2^{2-p}|b-a|^p \le \langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle, \forall a,b\ \in \mathbb{R}^N,$$

we can get

$$\nabla u_n \to \nabla u \text{ in } L^p(\Omega).$$

So we have $u_n \to u$ in $W_0^{1,p}(\Omega)$ which means that I satisfies $(\mathrm{C})_c^*$.

Finally, we claim that for every $m \in N$,

$$I(u) \to -\infty \text{ as } \|u\| \to \infty,\ u \in X_m^1 \oplus X^2.$$

By (H_2), there exists an M large enough such that

$$F(x,t) \ge Mt^p - C_5, \quad x \in \Omega,\ t \in \mathbb{R}.$$

So, for any $u \in X_m^1 \oplus X^2$, we have

$$\begin{aligned} I(tu) &\le \frac{t^p}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{t^2}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x,tu)\mathrm{d}x \\ &\le \frac{1}{p}t^p\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{t^2}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - Mt^p\int_\Omega |u|^p\mathrm{d}x + C_5|\Omega| \to -\infty \text{ as } t\to +\infty. \end{aligned}$$

Hence, our claim holds. □

Proof of Theorem 1.2 Let $X = W_0^{1,p}(\Omega)$. It follows from the assumptions that I is even. Obviously, $I \in C^1(X,\mathbb{R})$ and $I(0) = 0$. By the proof of Theorem 1.1, we easily prove that $I(u)$ satisfies the Cerami condition $(C)_c$ $(c > 0)$ (see [15]). Now, we can prove the theorem by using the symmetric mountain pass Theorem in [15,16].

Step 1. We claim that condition (i) holds in [16, Theorem 9.12]. Let $V_1 = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_{m-1}}$, $V_2 = X \setminus V_1$. For all $u \in V_2$, by (SCPI) and (H_1), similarly to the proof of the step 2 in Theorem 1.1, we have

$$I(u) \ge \alpha$$

for $\|u\| = \rho$ small enough, where $\alpha > 0$.

Step 2. We claim condition (ii) holds in [16, Theorem 9.12]. By the last proof of Theorem 1.1, for every finite dimension subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E})$ such that

$$I(u) \le 0, \quad u \in \tilde{E}\backslash B_R(\tilde{E})$$

and our claim holds. □

Proof of Theorem 1.3 Similarly to the proof of Theorem 1.1, we only need to prove that $I(u) \ge 0$ for $u \in X^1$ with $\|u\| \le r$ and $r > 0$ small enough and bounded sequence $\{u_n\}$ has a strong convergence subsequence.

First, we we claim that $I(u) \ge 0$ for $u \in X^1$ with $||u|| \le r$ and $r > 0$ small enough. By (SCE) and (H_1), for any $\varepsilon > 0$, there exist $A_1 = A_1(\varepsilon)$, $\kappa > 0$ and $q > N$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \le \frac{1}{N}(\epsilon)|s|^N + A_1 \exp(\kappa|s|^{\frac{N}{N-1}})|s|^q.$$

Choose $\varepsilon > 0$ such that $\varepsilon < \lambda_1$. By above inequality, the Hölder inequality and the Moser-Trudinger embedding inequality, we get

$$\begin{aligned} I(u) &\ge \frac{1}{N}\|u\|^N - \frac{\epsilon}{N}|u|_N^N - A_1\int_\Omega \exp(\kappa|u|^{\frac{N}{N-1}})|u|^q\mathrm{d}x \\ &\ge \frac{1}{N}\big(1-\frac{\epsilon}{\lambda_1}\big)\|u\|^N - A_1\Big(\int_\Omega \exp(\kappa r\|u\|^{\frac{N}{N-1}}(\frac{|u|}{\|u\|})^{\frac{N}{N-1}})\mathrm{d}x\Big)^{\frac{1}{r}}\Big(\int_\Omega |u|^{r'q}\mathrm{d}x\Big)^{\frac{1}{r'}} \\ &\ge \frac{1}{N}\big(1-\frac{\epsilon}{\lambda_1}\big)\|u\|^N - C_6\|u\|^q, \end{aligned}$$

where $r > 1$ sufficiently close to 1, $\|u\| \le \sigma$ and $\kappa r\sigma^{\frac{N}{N-1}} < \alpha_N$. So, we get

$$I(u) \ge 0, \ \forall u \in X^1 \text{ with } \|u\| \le r$$

for $0 < r$ small enough.

Next, we show that bounded sequence $\{u_n\}$ has a strong convergence subsequence. Without loss of generality, suppose that

$$\|u_n\| \le \beta,$$

$$u_n \rightharpoonup u \ \text{ in } W_0^{1,N}(\Omega),$$

$$u_n \to u \text{ in } L^q(\Omega),\ \forall q \ge 1,$$

$$u_n(x) \to u(x) \text{ a.e. } x \in \Omega.$$

Now, since f has the subcritical exponential growth (SCE) on Ω, we can find a constant $C_\beta > 0$ such that

$$|f(x,t)| \le C_\beta \exp\big(\frac{\alpha_N}{2\beta^{\frac{N}{N-1}}}|t|^{\frac{N}{N-1}}\big),\ \forall (x,t) \in \Omega \times \mathbb{R}.$$

Thus, by the Moser-Trudinger inequality (see Lemma 2.7),

$$\begin{aligned}\Big|\int_\Omega f(x,u_n)(u_n-u)\mathrm{d}x\Big| &\le C_7\Big(\int_\Omega \exp\big(\frac{\alpha_N}{\beta^{\frac{N}{N-1}}}|u_n|^{\frac{N}{N-1}}\big)\mathrm{d}x\Big)^{\frac12}|u_n-u|_2\\ &\le C_7\Big(\int_\Omega \exp\big(\frac{\alpha_N}{\beta^{\frac{N}{N-1}}}\|u_n\|^{\frac{N}{N-1}}|\frac{u_n}{\|u_n\|}|^{\frac{N}{N-1}}\big)\mathrm{d}x\Big)^{\frac12}|u_n-u|_2\\ &\le C_8|u_n-u|_2 \to 0.\end{aligned}$$

Similarly to the last proof of Theorem 1.1, we have $u_n \to u$ in $W_0^{1,N}(\Omega)$ which means that I satisfies $(\mathrm{C})_\mathrm{c}$. □

Proof of Theorem 1.4 Combining the proofs of Theorems 1.2 and 1.3, we easily prove it. We omit it here. □

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