Journal of Mathematical Research with Applications
Mar., 2016, Vol. 36, No. 2, pp. 127–139
DOI:10.3770/j.issn:2095-2651.2016.02.001
Http://jmre.dlut.edu.cn

Classification of Flag-Transitive Primitive Symmetric (v,k,λ) Designs with $\mathrm{PSL}(2,q)$ as Socle

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Abstract Let $\mathcal{D}$ be a nontrivial symmetric (v,k,λ) design, and G be a subgroup of the full automorphism group of $\mathcal{D}$. In this paper we prove that if G acts flag-transitively, point-primitively on $\mathcal{D}$ and $\mathrm{Soc}(G)=\mathrm{PSL}(2,q)$, then $\mathcal{D}$ has parameters $(7,3,1)$, $(7,4,2)$, $(11,5,2)$, $(11,6,3)$ or $(15,8,4)$.

Keywords symmetric design; flag-transitive; primitive group

MR(2010) Subject Classification 05B05; 05B25; 20B25

1. Introduction

A 2-(v,k,λ) design $\mathcal{D}$ is a set P of v points together with a set $\mathcal{B}$ of b blocks, such that every block contains k points and every pair of points is in exactly λ blocks. The design $\mathcal{D}$ is symmetric if $b=v$, and is non-trivial if $2<k<v-1$. In this paper we only study non-trivial symmetric 2-(v,k,λ) designs, and for brevity we call such a design a symmetric (v,k,λ) design. A flag in a design is an incident point-block pair. The complement of $\mathcal{D}$, denoted by $\overline{\mathcal{D}}$, is a symmetric $(v,v-k,v-2k+\lambda)$ design whose set of points is the same as the set of points of $\mathcal{D}$, and whose blocks are the complements of the blocks of $\mathcal{D}$. The automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ consists of all permutations of P which leave $\mathcal{B}$ invariant. For $G\le \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D}$ is called point-primitive if G is primitive on P, and flag-transitive if G is transitive on the set of flags. The socle of a group G, denoted by $\mathrm{Soc}(G)$, is the subgroup generated by its minimal normal subgroups.

The classification program for symmetric (v,k,λ) designs has been studied by several researchers. In 1985, Kantor [1] classified all symmetric (v,k,λ) designs admitting 2-transitive automorphism groups. In [2], Dempwolff determined all symmetric (v,k,λ) designs which admit an automorphism group G such that G has a nonabelian socle and is a primitive rank three group on points (and blocks). In [3,4], we classified flag-transitive point-primitive symmetric (v,k,λ) designs admitting an automorphism group G such that $\mathrm{Soc}(G)$ is a sporadic simple group. This paper is devoted to the complete classification of flag-transitive point-primitive

Received Marh 27, 2015; Accepted September 14, 2015
Supported by the National Natural Science Foundation of China (Grant Nos. 11471123; 11426066) and the Natural Science Foundation of Guangdong Province (Grant No. S2013010011928).
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symmetric (v,k,λ) designs which admit an automorphism group G with $\mathrm{Soc}(G)=\mathrm{PSL}(2,q)$, and extend the result of symmetric designs with $\lambda=4$ in [5] to the general case.

Theorem 1.1 *Let $\mathcal{D}=(P,\mathcal{B})$ be a symmetric (v,k,λ) design which admits a flag-transitive, point-primitive automorphism group G, and x be a point of P. If G is an almost simple group and $X=\mathrm{Soc}(G)=\mathrm{PSL}(2,q)$, where $q=p^f$ and p is a prime, then $\mathcal{D}$ is one of the following:*

(i) a $(7,3,1)$ *design with* $X=\mathrm{PSL}(2,7)$ *and* $X_x=S_4$;

(ii) a $(7,4,2)$ *design with* $X=\mathrm{PSL}(2,7)$ *and* $X_x=S_4$;

(iii) a $(11,5,2)$ *design with* $X=\mathrm{PSL}(2,11)$ *and* $X_x=A_5$;

(iv) a $(11,6,3)$ *design with* $X=\mathrm{PSL}(2,11)$ *and* $X_x=A_5$;

(v) a $(15,8,4)$ *design with* $X=\mathrm{PSL}(2,9)$ *and* $X_x=\mathrm{PGL}(2,3)$.

Corollary 1.2 *For $\lambda\geq 5$, there is no symmetric (v,k,λ) design admitting a flag-transitive, point-primitive almost simple automorphism group with socle* $\mathrm{PSL}(2,q)$.

2. Preliminaries

In this section we state some preliminary results which will be needed later in this paper. From [6,7] we get the following:

Lemma 2.1 *Let $\mathcal{D}$ be a flag-transitive symmetric (v,k,λ) design. Then the following hold:*

(i) $k(k-1)=\lambda(v-1)$, *and in particular* $k^2>v$;

(ii) $k\mid\lambda d_i$, *where d_i is any non-trivial subdegree of G;*

(iii) $k\mid|G_x|$ *and* $|G_x|^3>|G|$, *where G_x is the stabilizer in G of a point $x\in P$.*

Lemma 2.2 ([8]) *Let $\mathcal{D}$ be a symmetric (v,k,λ) design and $G\leq\mathrm{Aut}(\mathcal{D})$. Then*

(i) G has as many orbits on points as on blocks;

(ii) if G is a transitive automorphism group, then G has the same rank whether considered as a permutation group on points or on blocks.

Lemma 2.3 *Let $\mathcal{D}=(P,\mathcal{B})$ be a symmetric (v,k,λ) design and G be a subgroup of* $\mathrm{Aut}(\mathcal{D})$. *Then G is 2-transitive on P if and only if G is flag-transitive on both the design $\mathcal{D}$ and the complement design $\overline{\mathcal{D}}$ of $\mathcal{D}$.*

Proof Note that when G is transitive on P and $x\in P$, G is flag-transitive on $\mathcal{D}$ if and only if G_x is transitive on the set of all blocks in B containing x. Suppose G is 2-transitive on P. Then for any $x\in P$, G_x has two orbits on P. By (i) of Lemma 2.2, G_x has two orbits on B. But $x\in P$ has at least 2 orbits on B, the set of blocks containing x and the set of blocks not containing x. Therefore the set of blocks containing x must be a single orbit of G_x on B and G is flag-transitive on $\mathcal{D}$. Similarly, since G is also 2-transitive on P for $\overline{\mathcal{D}}=(P,\overline{B})$, G is also flag-transitive on $\overline{\mathcal{D}}$.

Conversely, if $\overline{\mathcal{D}}$ is flag-transitive, then G_C is transitive on the points of C for every block C of $\overline{\mathcal{D}}$. Let $B=P-C$. Then B is one of the blocks of $\mathcal{D}$ and $G_B=G_C$. Since $\mathcal{D}$ is also flag-transitive, G_B is transitive on B. Thus G_B has two orbits acting on points, which implies that the point-stabilizer G_x has two orbits acting on P by Lemma 2.2. Hence G is 2-transitive on P. $\square$

Lemma 2.4 *Let G be a transitive group on P, and let $X \trianglelefteq G$. Then each orbit of G_x acting on P is the union of some orbits of X_x which have the same cardinality.*

Proof This is well known, and follows from the $\frac{1}{2}$-transitivity of X_x since $X_x \trianglelefteq G_x$ and G_x is transitive on each G_x-orbit. $\square$

Lemma 2.5 *Let $\mathcal{D} = (P, \mathcal{B})$ be a symmetric (v,k,λ) design admitting a flag-transitive, point-primitive automorphism group G with socle X. If the non-trivial subdegree t of X appears with multiplicity s, then $k \mid \lambda st$.*

Proof Suppose that $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ are all orbits of X_x with cardinality t, where $x \in P$. By Lemma 2.4, the group G_x acts on $\Gamma = \bigcup_{i=1}^{s} \Gamma_i$, and the cardinalities of orbits of G_x are

$$(a_1 t)^{s_1}, (a_2 t)^{s_2}, \ldots, (a_r t)^{s_r},$$

where a^b means that a appears with multiplicity b, and r, a_i, s_i $(1 \le i \le r)$ are all positive integers such that $\sum_{i=1}^{r} a_i s_i = s$, and $a_i \ne a_j$ if and only if $i \ne j$ for $1 \le i, j \le r$. Let $c = \gcd(a_1, a_2, \ldots, a_r)$. Then $c \mid s$. Lemma 2.1 (ii) shows that $k \mid \lambda(a_i t)$, $i = 1, 2, \ldots, r$. So $k \mid \lambda ct$, and hence $k \mid \lambda st$. $\square$

The subgroups of PSL$(2,q)$ are well-known and given by Huppert [9].

Lemma 2.6 ([9]) *The subgroups of the group* PSL$(2,q)$ $(q = p^f)$ *are as follows.*

(i) *An elementary abelian group C_p^{ℓ}, where $\ell \le f$;*

(ii) *A cyclic group C_z, where $z \mid \frac{p^f \pm 1}{d}$ and $d = \gcd(2, q-1)$;*

(iii) *A dihedral group D_{2z}, where z is the same as in (ii);*

(iv) *The alternating group A_4 when $p > 2$ or $p = 2$ and $2 \mid f$;*

(v) *The symmetric group S_4 when $p^{2f} \equiv 1 \pmod{16}$;*

(vi) *The alternating group A_5 when $p = 5$ or $p^{2f} \equiv 1 \pmod{5}$;*

(vii) *$C_p^{\ell} : C_t$, where $t \mid \gcd(p^{\ell} - 1, \frac{p^f - 1}{d})$ and $d = \gcd(2, q-1)$;*

(viii) PSL$(2, p^{\ell})$ *when $\ell \mid f$ and* PGL$(2, p^{\ell})$ *when $2\ell \mid f$.*

The following lemma is a combination of Theorems 1.1, 2.1 and 2.2 in [10].

Lemma 2.7 *Let $X = \mathrm{PSL}(2,q) \le G \le \mathrm{P\Gamma L}(2,q)$ and let M be a maximal subgroup of G which does not contain X. Then either $M \cap X$ is maximal in X, or G and M are given in Table 1. The maximal subgroups of X appear in Tables 2 and 3.*

G	M	$\|G:M\|$
$\mathrm{PGL}(2,7)$	$N_G(D_6)=D_{12}$	28
$\mathrm{PGL}(2,7)$	$N_G(D_8)=D_{16}$	21
$\mathrm{PGL}(2,9)$	$N_G(D_{10})=D_{20}$	36
$\mathrm{PGL}(2,9)$	$N_G(D_8)=D_{16}$	45
M_{10}	$N_G(D_{10})=C_5\rtimes C_4$	36
M_{10}	$N_G(D_8)=C_8\rtimes C_2$	45
$\mathrm{P\Gamma L}(2,9)$	$N_G(D_{10})=C_{10}\rtimes C_4$	36
$\mathrm{P\Gamma L}(2,9)$	$N_G(D_8)=C_8.\mathrm{Aut}(C_8)$	45
$\mathrm{PGL}(2,11)$	$N_G(D_{10})=D_{20}$	66
$\mathrm{PGL}(2,q), q=p\equiv\pm 11,19 \pmod{40}$	$N_G(A_4)=S_4$	$\frac{q(q^2-1)}{24}$

Table 1 G and M of Lemma 2.7

Structure	*Conditions*	*Order*	*Index*
$C_p^f:C_{(q-1)/2}$		$\frac{q(q-1)}{2}$	$q+1$
D_{q-1}	$q\geq 13$	$q+1$	$\frac{q(q-1)}{2}$
D_{q+1}	$q\neq 7,9$	$q-1$	$\frac{q(q+1)}{2}$
$\mathrm{PGL}(2,q_0)$	$q=q_0^2$	$q_0(q_0^2-1)$	$\frac{q_0(q_0^2+1)}{2}$
$\mathrm{PSL}(2,q_0)$	$q=q_0^r$, *r odd prime*	$\frac{q_0(q_0^2-1)}{2}$	$\frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$
A_5	$q=p\equiv\pm 1 \pmod 5$, *or* $q=p^2\equiv -1 \pmod 5)$	120	$\frac{q(q^2-1)}{120}$
A_4	$q=p\equiv\pm 3 \pmod 8$, *and* $q\not\equiv\pm 1 \pmod{10}$	12	$\frac{q(q^2-1)}{24}$
S_4	$q=p\equiv\pm 1 \pmod 8$	24	$\frac{q(q^2-1)}{24}$

Table 2 Maximal subgroups of $\mathrm{PSL}(2,q)$ *with* $q=p^f\geq 5$, *p odd prime*

Structure	*Conditions*	*Order*	*Index*
$C_2^f:C_{q-1}$		$q(q-1)$	$q+1$
$D_{2(q-1)}$		$2(q+1)$	$\frac{q(q-1)}{2}$
$D_{2(q+1)}$		$2(q-1)$	$\frac{q(q+1)}{2}$
$\mathrm{PSL}(2,q_0)$	$q=q_0^r$, *r prime*, $q_0\neq 2$	$q_0(q_0^2-1)$	$\frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$

Table 3 Maximal subgroups of $\mathrm{PSL}(2,q)$ *with* $q=2^f\geq 4$

Now we state the following algorithm, which will be useful to search for symmetric designs which satisfy the condition "$k\mid u$". The output of the algorithm is the list Designs of parameter sequences (v,k,λ) of potential symmetric designs.

Algorithm 2.8 *(*Designs*)*

INPUT: u, v.
OUTPUT: *The list* DESIGNS $:= S$.
set S *:= an empty list;*
for each k *dividing* u *with* $2 < k < v-1$
$\lambda := k*(k-1)/(v-1)$;
if λ *is an integer*
Add (v,k,λ) *to the list* S;
return S.

3. Proof of Theorem 1.1

Let $\mathcal{D} = (P, \mathcal{B})$ be a symmetric (v,k,λ) design admitting a flag-transitive, point-primitive automorphism group G with $X \trianglelefteq G \leq \mathrm{Aut}(X)$, where $X = \mathrm{PSL}(2,q)$ with $q = p^f$ and p prime. As a maximal subgroup of G, the point stabilizer G_x does not contain X since X is transitive on P. Thus Lemma 2.7 shows that either $X \cap G_x$ is maximal in X, or G and G_x are given in Table 1. We will prove Theorem 1.1 by the following three subsections.

3.1. Cases in Table 1

In these cases, we may view the maximal subgroup M as the point stabilizer G_x. We get the 3-tuples $(|G|, u, v)$ in Table 1 where v is the index $|G : G_x|$ and $u = |G_x|$. For each case except the last one, we can obtain all potential symmetric designs using Algorithm 2.8 implemented in GAP [11]. There exists only one potential $(21,16,12)$ design with $G = \mathrm{PGL}(2,7)$ and $G_x = D_{16}$. The subdegrees of PGL$(2,7)$ acting on the cosets of D_{16} are $1,4,8$ and 8 (Throughout this paper, we apply Magma [12] to calculate the subdegrees of G and the number of the conjugacy class of subgroups). Then by using the Magma-command Subgroups (G: OrderEqual:= n) where $n = |G|/v$, we obtain the fact that G has only one conjugacy class of subgroups with index 21. Thus G_x is conjugate to G_B for any $x \in \mathcal{P}, B \in \mathcal{B}$ which forces that there exists a block B_0 such that $G_x = G_{B_0}$. The flag-transitivity of G implies that G_{B_0} is transitive on the block B_0. So B_0 should be an orbit of G_x, but there is no such orbit of size $k = 16$, a contradiction.

Now we consider the last case. Here $G = \mathrm{PGL}(2,q)$ with $q = p \equiv \pm 11, 19 \pmod{40}$, $G_x = S_4$, and $v = \frac{q(q^2-1)}{24}$. Since $|G_x|^3 > |G|$, we have $24^3 > q(q^2-1)$, and so $q = p = 11$ or 19. If $q = 11$, then $v = 55$. There exist two potential symmetric designs with parameters $(55,27,13)$ and $(55,28,14)$, but neither of them satisfies the condition that $k \mid |G_x|$. If $q = 19$, then $v = 285$, and so $(k,\lambda) = (72,18)$ or $(213,159)$. However, every case $k > 24$ contradicts the fact that k divides $|G_x|$.

3.2. Odd characteristic

In this subsection, we consider the cases that G has odd characteristic p and $X \cap G_x$ is maximal in X. The structure of $X \cap G_x$ comes from Table 2.

Case 1 $X \cap G_x = C_p^f : C_{(q-1)/2}$.

Here $v=q+1$, so $k(k-1)=\lambda(v-1)=\lambda q=\lambda p^f$. If $p\,|\,k$, then from $\gcd(p,k-1)=1$ we have $p^f\,|\,k$, that is, $v-1\,|\,k$ which is impossible. Then $p\nmid k$, and so $p^f\,|\,k-1$ implies $v-1\,|\,k-1$, which contradicts $k<v-1$.

Case 2 $X\cap G_x=D_{q-1}$ $(q\geq 13)$.

In this case, $v=\frac{1}{2}q(q+1)$, $|\mathrm{Out}(X)|=2f$, $|G|=\frac{1}{2}eq(q^2-1)$ and $|G_x|=e(q-1)$, where e is a positive integer and $e\,|\,2f$.

From $k\,|\,|G_x|$ we get $k\,|\,e(q-1)$. So there exists a positive integer m such that $k=\frac{e(q-1)}{m}$. The equality $k(k-1)=\lambda(v-1)$ implies that $\frac{e(q-1)}{m}(\frac{e(q-1)}{m}-1)=\frac{1}{2}\lambda(q+2)(q-1)$, and hence

$$(2e^2-m^2\lambda)q=2m^2\lambda+2e^2+2em>0.$$

This implies that $m<2e$. From $p^f=q=\frac{2m^2\lambda+2e^2+2em}{2e^2-m^2\lambda}=\frac{6e^2+2em}{2e^2-m^2\lambda}-2$, we get

$$p^f<6e^2+2em<6e^2+(2e)^2=10e^2\leq 40f^2.$$

It follows that the 3-tuples (q,p,f) are

$$\begin{array}{llllll}(27,3,3), & (81,3,4), & (243,3,5), & (729,3,6), & (25,5,2), & (125,5,3),\\ (625,5,4), & (49,7,2), & (343,7,3), & (121,11,2), & (13,13,1), & (17,17,1),\\ (19,19,1), & (23,23,1), & (29,29,1), & (31,31,1), & (37,37,1). & \end{array}$$

We call each of these 3-tuples a subcase. Since $k\,|\,e(q-1)$ and $e\,|\,2f$, it follows that $k\,|\,u$, where $u=2f(q-1)$. It is easy to compute the values of u and v for every subcase. However, for every subcase, there is no such symmetric design satisfying the condition that $k\,|\,u$ by Algorithm 2.8 calculated with GAP.

Case 3 $X\cap G_x=D_{q+1}$ $(q\neq 7,9)$.

Now $v=\frac{1}{2}q(q-1)$, $|\mathrm{Out}(X)|=2f$, $|G|=\frac{1}{2}eq(q^2-1)$, and $|G_x|=e(q+1)$, where e is a positive integer and $e\,|\,2f$.

Since $k\,|\,|G_x|=e(q+1)$, it follows that there exists a positive integer m such that $k=\frac{e(q+1)}{m}$. Then $\frac{e(q+1)}{m}(\frac{e(q+1)}{m}-1)=\frac{1}{2}\lambda(q-2)(q+1)$. So we have

$$(m^2\lambda-2e^2)q=2e^2-2em+2m^2\lambda=2(e-\frac{1}{2}m)^2+(2\lambda-\frac{1}{2})m^2>0.$$

Thus $p^f=q=\frac{2e^2-2em+2m^2\lambda}{m^2\lambda-2e^2}=\frac{6e^2-2em}{m^2\lambda-2e^2}+2$ which gives

$$p^f<6e^2+2\leq 24f^2+2.$$

Combining this with $q\neq 7,9$, we obtain all possible 3-tuples (q,p,f):

$$\begin{array}{lllllll}(27,3,3), & (81,3,4), & (243,3,5), & (729,3,6), & (5,5,1), & (25,5,2), & (125,5,3),\\ (49,7,2), & (11,11,1), & (13,13,1), & (17,17,1), & (19,19,1), & (23,23,1). & \end{array}$$

Since $k\,|\,e(q+1)$ and $e\,|\,2f$, $k\,|\,u=2f(q+1)$. The values of v and u can be calculated easily for each 3-tuple (p,q,f). In fact, we get no such symmetric design satisfying $k\,|\,u$ by Algorithm 2.8 calculated with GAP.

Case 4 $X\cap G_x=\mathrm{PGL}(2,q^{\frac{1}{2}})=\mathrm{PGL}(2,q_0)$.

Here $v=\frac{q_0(q_0^2+1)}{2}$, $|X_x|=|X\cap G_x|=q_0(q_0^2-1)$, $|\mathrm{Out}(X)|=2f$, $|G|=\frac{1}{2}eq(q^2-1)$, and $|G_x|=eq_0(q_0^2-1)$, where $e\,|\,2f$ and f is even.

The subdegrees of $\mathrm{PSL}(2,q)$ on the cosets of $\mathrm{PGL}(2,q_0)$ are

$$1,\ \frac{q_0(q_0-\varepsilon)}{2},\ \ q_0^2-1,\ (q_0(q_0-1))^{\frac{q_0-4-\varepsilon}{4}},\ (q_0(q_0+1))^{\frac{q_0-2+\varepsilon}{4}},$$

where $q_0\equiv\varepsilon \pmod 4$ with $\varepsilon=\pm1$ (see [13]). Recall that here a^b means the subdegree a appears with multiplicity b. We consider two subcases in the following.

Subcase 4.1 $\varepsilon=-1$. Then there exists a positive integer s such that $q_0=4s-1$, and the subdegrees here are: 1, $\frac{q_0(q_0+1)}{2}$, q_0^2-1, $(q_0(q_0-1))^{\frac{q_0-3}{4}}$, $(q_0(q_0+1))^{\frac{q_0-3}{4}}$. By Lemma 2.5, we get

$$k\,|\,\lambda\gcd\big(\frac{q_0(q_0+1)}{2},q_0^2-1,\frac{q_0(q_0-1)(q_0-3)}{4},\frac{q_0(q_0+1)(q_0-3)}{4}\big).$$

Since $q_0=4s-1$, it follows that $k\,|\,2\lambda$. Then from $k>\lambda$ we get $k=2\lambda$. By Lemma 2.1 (i), $\lambda=\frac{v+1}{4}=\frac{q_0^3+q_0+2}{8}$ and $k=\frac{q_0^3+q_0+2}{4}$. Since $k\,|\,|G_x|$ and $e\,|\,2f$, $k\,|\,u=2fq_0(q_0^2-1)$. It follows that $q_0^3+q_0+2\,|\,8fq_0(q_0^2-1)$. Note that $\gcd(q_0^3+q_0+2,q_0)=1$ and $\gcd(q_0^2-q_0+2,q_0-1)=2$, we get $q_0^2-q_0+2\,|\,16f$. So $q_0^2-q_0+2\le 16f$, i.e., $(p^{\frac{1}{2}f})^2-p^{\frac{1}{2}f}+2\le 16f$. It follows that $(f,p)=(2,3)$ or $(2,5)$ because f is even. If $(f,p)=(2,5)$, then q_0 is equal to p which contradicts $q_0=4s-1$. Suppose that $(f,p)=(2,3)$. Then $\mathcal{D}$ has parameters $(15,8,4)$ with $X=\mathrm{PSL}(2,9)\cong A_6$, $X_x=\mathrm{PGL}(2,3)\cong S_4$. The existence of this design has been discussed in [5].

Subcase 4.2 $\varepsilon=1$. Then $q_0=4s+1$ for some positive integer s. Let $q_0=p^a$. Then $f=2a$. The subdegrees are: 1, $\frac{q_0(q_0-1)}{2}$, q_0^2-1, $(q_0(q_0-1))^{\frac{q_0-5}{4}}$, $(q_0(q_0+1))^{\frac{q_0-1}{4}}$. Lemma 2.5 shows that

$$k\,|\,\lambda\gcd\big(\frac{q_0(q_0-1)}{2},q_0^2-1,\frac{q_0(q_0-1)(q_0-5)}{4},\frac{q_0(q_0+1)(q_0-1)}{4}\big).$$

Since $\gcd(\frac{q_0(q_0-1)}{2},\ q_0^2-1)=\frac{1}{2}(q_0-1)$, it follows that $k\,|\,\frac{1}{2}\lambda(q_0-1)$. Combining this with $k(k-1)=\lambda(v-1)=\frac{1}{2}\lambda(q_0-1)(q_0^2+q_0+2)$, we get

$$q_0^2+q_0+2\,|\,k-1,$$

which implies that k is odd.

The flag-transitivity of G implies that G_x acts transitively on $P(x)$, the set of all blocks which are incident with the point x. Therefore G_x has some subgroup L with index k. Since $X_x\trianglelefteq G_x$, we have $L/(L\cap X_x)\cong LX_x/X_x$. Let $H=L\cap X_x$, and $|LX_x:X_x|=c$ for some integer c. Then $c\,|\,e$ and $|H|=\frac{eq_0(q_0^2-1)}{ck}$, and hence

$$k=\frac{e_0q_0(q_0^2-1)}{|H|},$$

where $e_0=\frac{e}{c}$. The fact $e\,|\,2f=4a$ yields $e_0\,|\,4a$.

Since $\mathrm{PSL}(2,q_0)$ is the normal subgroup of $\mathrm{PGL}(2,q_0)$ with index 2, and $H\le X_x=\mathrm{PGL}(2,q_0)$, we get $|H:H\cap\mathrm{PSL}(2,q_0)|=|\mathrm{PSL}(2,q_0)H:\mathrm{PSL}(2,q_0)|=1$ or 2. Lemma 2.6 gives all the subgroups of $\mathrm{PSL}(2,q_0)$, and hence $|H|$ must be one of the following:

(i) p^ℓ or $2p^\ell$, where $\ell\le a$;

(ii) z or $2z$, where $z \mid \frac{q_0 \pm 1}{2}$;

(iii) $2z$ or $4z$, where $z \mid \frac{q_0 \pm 1}{2}$;

(iv) 12 or 24;

(v) 24 or 48 when $p^{2a} \equiv 1 \pmod{16}$;

(vi) 60 or 120 when $p = 5$ or $p^{2a} \equiv 1 \pmod 5$;

(vii) tp^ℓ or $2tp^\ell$, where $t \mid \gcd(p^\ell - 1, \frac{p^a - 1}{2})$;

(viii) $\frac{1}{2}p^\ell(p^{2\ell} - 1)$ or $p^\ell(p^{2\ell} - 1)$ when $\ell \mid a$, and $p^\ell(p^{2\ell} - 1)$ or $2p^\ell(p^{2\ell} - 1)$ when $2\ell \mid a$.

Recall that $q_0 = 4s + 1$, and so $8 \mid q_0^2 - 1$. Combining this with the fact that $k = \frac{e_0 q_0 (q_0^2 - 1)}{|H|}$ is odd, gives $8 \mid |H|$. It follows that $|H| \neq p^\ell, 2p^\ell, z$, 12 and 60, and we deal with the remaining possible values of $|H|$ in turn.

If $|H| = 2z$ where $z \mid \frac{q_0 \pm 1}{2}$ as in (ii) or (iii), then it is easily known from $k = \frac{e_0 q_0 (q_0^2 - 1)}{2z}$ that k is even, a contradiction.

If $|H| = 4z$ as in (iii), and in addition $z \mid \frac{q_0 + 1}{2}$, then k is even, a contradiction. Next suppose that $z \mid \frac{q_0 - 1}{2}$. Since $q_0^2 + q_0 + 2 \mid k - 1$, $q_0^2 + q_0 + 2$ divides

$$z(k-1) = \frac{e_0 q_0 (q_0^2 - 1)}{4} - z = \frac{e_0 (q_0 - 1)}{4}(q_0^2 + q_0 + 2) - \frac{e_0 (q_0 - 1)}{2} - z,$$

which implies that $q_0^2 + q_0 + 2 \mid \frac{e_0(q_0 - 1)}{2} + z$. Therefore, $q_0^2 + q_0 + 2 \leq \frac{e_0(q_0 - 1)}{2} + z$. It follows that $p^{2a} + p^a + 2 \leq 2a(p^a - 1) + \frac{p^a - 1}{2}$ because $e_0 \leq 4a$ and $z \leq \frac{q_0 - 1}{2}$, and hence $2p^{2a} + 2a + 5 \leq (2a - 1)p^a$ which is a contradiction.

If $|H| = 24$, then $k = \frac{e_0 q_0 (q_0^2 - 1)}{24}$. The fact that $q_0^2 + q_0 + 2 \mid k - 1$ implies that $q_0^2 + q_0 + 2$ divides

$$6(k-1) = \frac{e_0 q_0 (q_0^2 - 1)}{4} - 6 = \frac{e_0 (q_0 - 1)}{4}(q_0^2 + q_0 + 2) - \frac{e_0 (q_0 - 1)}{2} - 6.$$

Thus $q_0^2 + q_0 + 2 \mid \frac{e_0(q_0 - 1)}{2} + 6$, and so $q_0^2 + q_0 + 2 \leq \frac{e_0(q_0 - 1)}{2} + 6$. Since $e_0 \mid 4a$, we have $p^{2a} + p^a + 2 \leq \frac{e_0(q_0 - 1)}{2} + 6 \leq 2a(p^a - 1) + 6$, which is impossible since $p \geq 3$.

The case (v) $|H| = 48$, or (vi) $|H| = 120$ can be ruled out similarly.

For (vii), if $|H| = tp^\ell$ or $2tp^\ell$, then $k = \frac{e_0 q_0 (q_0^2 - 1)}{itp^\ell}$ where $i = 1$ or 2, and hence k is even because $t \mid \frac{p^a - 1}{2}$, a contradiction.

For (viii), suppose first that $\ell \mid a$ and $|H| = p^\ell(p^{2\ell} - 1)$ or $\frac{1}{2}p^\ell(p^{2\ell} - 1)$. Then $k = \frac{i e_0 p^a (p^{2a} - 1)}{p^\ell (p^{2\ell} - 1)}$ where $i = 1$ or 2. If $\ell = a$, then $k = ie_0$. From $v < k^2$ and $e_0 \mid 4a$, we see $\frac{p^a(p^{2a} + 1)}{2} < (ie_0)^2 \leq 16i^2a^2$. It follows that $(p, a) = (3, 1)$, and so $q_0 = 3$, contradicting $q_0 = 4s + 1$. Thus $\ell < a$, and so $a \geq 2$. It is easy to see that $p^\ell - 1 \mid p^a - 1$ because $\ell \mid a$. Since $q_0^2 + q_0 + 2 \mid k - 1$, we obtain that $q_0^2 + q_0 + 2$ divides

$$\begin{aligned} p^\ell(p^\ell + 1)(k - 1) &= \frac{i e_0 p^a (p^{2a} - 1)}{p^\ell - 1} - p^\ell(p^\ell + 1) \\ &= \frac{i e_0 (p^a - 1)}{p^\ell - 1}(p^{2a} + p^a + 2) - \frac{2 i e_0 (p^a - 1)}{p^\ell - 1} - p^\ell(p^\ell + 1). \end{aligned}$$

Thus $p^{2a} + p^a + 2 \mid \frac{2ie_0(p^a - 1)}{p^\ell - 1} + p^\ell(p^\ell + 1)$. Since $\ell \mid a$ and $\ell < a$, we have $2\ell \leq a$, and so $p^{2\ell} \leq p^a$. Then $p^{2a} + p^a + 2 < 8ia(p^a - 1) + 2p^a$. Combining this with $q_0 = p^a = 4s + 1$ and $a \geq 2$, gives $(p, a) = (3, 2)$ when $i = 1$, and $(p, a) = (3, 2)$ or $(5, 2)$ when $i = 2$. It follows that $\ell = 1$ and

$e_0 = 1, 2, 4$ or 8. For all these parameters e_0, p, a and ℓ, we can get all possible values of v and k. It is not hard to check that for all these pairs (v,k), there are no integer values of λ satisfying equation $k(k-1) = \lambda(v-1)$, a contradiction.

Now suppose that $2\ell \mid a$ and $|H| = p^\ell(p^{2\ell}-1)$. Then $a \geq 2$ and $k = \frac{e_0 p^a(p^a+1)(p^a-1)}{p^\ell(p^{2\ell}-1)}$ is even since $p^{2\ell}-1 \mid p^a-1$, a contradiction. Finally suppose that $2\ell \mid a$ and $|H| = 2p^\ell(p^{2\ell}-1)$ so that $k = \frac{e_0 p^a(p^{2a}-1)}{2p^\ell(p^{2\ell}-1)}$ and $a \geq 2$. Then by $q_0^2 + q_0 + 2 \mid k-1$, we get that $q_0^2+q_0+2$ divides

$$\begin{aligned} p^\ell(p^\ell+1)(k-1) &= \frac{e_0 p^a(p^{2a}-1)}{2(p^\ell-1)} - p^\ell(p^\ell+1) \\ &= \frac{e_0(p^a-1)}{2(p^\ell-1)}(p^{2a}+p^a+2) - \frac{e_0(p^a-1)}{p^\ell-1} - p^\ell(p^\ell+1), \end{aligned}$$

which yields $p^{2a}+p^a+2 \mid \frac{e_0(p^a-1)}{p^\ell-1} + p^\ell(p^\ell+1)$. By $p^{2\ell} \leq p^a$, we have $p^{2a}+p^a+2 < 4a(p^a-1)+2p^a$. It follows that $p^{2a} < (4a+1)(p^a-1)-1 < (4a+1)p^a$, and then $p^a < 4a+1$. This is impossible.

Case 5 $X \cap G_x = \mathrm{PSL}(2,q_0)$, for $q = q_0^r$ where r is an odd prime.

Here $v = \frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$, $|X_x| = \frac{1}{2}q_0(q_0^2-1)$, $|\mathrm{Out}(X)| = 2f$, $|G| = \frac{1}{2}eq(q^2-1)$, and $|G_x| = \frac{1}{2}eq_0(q_0^2-1)$, where $e \mid 2f$. Let $q_0 = p^a$. Then $f = ra$.

From $|G_x|^3 > |G|$, that is, $(\frac{1}{2}eq_0(q_0^2-1))^3 > \frac{1}{2}eq(q^2-1) = \frac{1}{2}eq_0^r(q_0^{2r}-1)$, we obtain

$$4f^2 \geq e^2 > 4q_0^{r-3}\frac{q_0^{2r}-1}{q_0^6-3q_0^4+3q_0^2-1}.$$

For an odd prime r, if $r \geq 5$, then

$$f^2 > q_0^{r-3}\frac{q_0^{2r}-1}{q_0^6-3q_0^4+3q_0^2-1} \geq q_0^{r-3}\frac{q_0^{10}-1}{q_0^6-3q_0^4+3q_0^2-1} > q_0^r = q = p^f,$$

where the third inequality holds because $q_0^{10}-1 > q_0^3(q_0^6-3q_0^4+3q_0^2-1) = q_0^9 - 3q_0^5(q_0^2-1) - q_0^3$. But it is easy to see that $\frac{p^f}{f^2} > 1$ when $p \geq 3$ and $f \geq r \geq 5$, a contradiction. Hence $r = 3$, and so $v = q_0^2(q_0^4+q_0^2+1)$ and $f = 3a$.

The subdegrees of $\mathrm{PSL}(2,q_0^3)$ on the cosets of $\mathrm{PSL}(2,q_0)$ are as follows [13]:

$$1,\ \left(\frac{q_0^2-1}{2}\right)^{2(q_0+1)},\ (q_0(q_0-1))^{\frac{q_0(q_0-1)}{2}},\ (q_0(q_0+1))^{\frac{q_0(q_0+1)}{2}},\ \left(\frac{q_0(q_0^2-1)}{2}\right)^{2(q_0^3+q_0-1)}.$$

By Lemma 2.5, we know that k divides λ times the greatest common divisor of the above non-trivial subdegrees, so that $k \mid 2\lambda$. Thus $k = 2\lambda$ follows from $k > \lambda$. The equation $k(k-1) = \lambda(v-1)$ forces $v = 4\lambda - 1$. Therefore $\lambda = \frac{v+1}{4} = \frac{q_0^6+q_0^4+q_0^2+1}{4}$ and $k = 2\lambda = \frac{q_0^6+q_0^4+q_0^2+1}{2}$. Then by Lemma 2.1 (iii), $k \mid |G_x| = \frac{1}{2}eq_0(q_0^2-1)$. This together with $e \mid 2f = 6a$ and $q_0 = p^a$, implies $\frac{p^{6a}+p^{4a}+p^{2a}+1}{2} \leq 3ap^a(p^{2a}-1)$ and so that $p^{6a} < 6a \cdot p^a \cdot p^{2a}$, i.e., $p^{3a} < 6a$, which is impossible.

Case 6 $X \cap G_x = A_5$, where $q = p \equiv \pm 1 \pmod 5$ or $q = p^2 \equiv -1 \pmod 5$.

Here $v = \frac{q(q^2-1)}{120}$, $|X_x| = |X \cap G_x| = 60$, $|\mathrm{Out}(X)| = 2f$, $|G| = \frac{1}{2}eq(q^2-1)$ and $|G_x| = 60e$, where $e \mid 2f$ and $f = 1$ or 2.

From the inequality $|G_x|^3 > |G|$ we have $(60e)^3 > \frac{1}{2}eq(q^2-1)$. This together with $e \mid 2f$, implies $2 \cdot 60^3 \cdot (2f)^2 \geq p^f(p^{2f}-1)$, i.e.,

$$120^3 f^2 \geq p^f(p^{2f}-1).$$

If $f=1$, then $q=p\equiv\pm1\ (\mathrm{mod}\,5)$ and $120^3\geq p(p^2-1)$, which force $q=11,19,29,31,41,59,61,71,79,89,101$ or 109. Now we compute the values of v by $v=\frac{q(q^2-1)}{120}$, and from $k\,|\,|G_x|$, $e=1$ or 2 we get $k\,|\,u=120$. We then check all possibilities for v by using Algorithm 2.8, and obtain three potential parameters: $(11,5,2)$, $(11,6,3)$ and $(57,8,1)$. If $(v,k,\lambda)=(57,8,1)$, then $X=\mathrm{PSL}(2,19)$. The subdegrees of X on the cosets of A_5 are $1,6,20$ and 30. By Lemma 2.4, the subdegrees of G are also $1,6,20$ and 30, contradicting Lemma 2.1 (ii). If $(v,k,\lambda)=(11,5,2)$, then $X=\mathrm{PSL}(2,11)$, and so $G=\mathrm{PSL}(2,11)$ or $\mathrm{PGL}(2,11)$. The GAP-command Transitivity(G,Ω) returns the degree t of transitivity of the action implied by the arguments; that is, the largest integer t such that the action is t-transitive. Thus we know that G acts as 2-transitive permutation group on the set P of 11 points by GAP. Then Lemma 2.3 shows that $\mathcal{D}$ is flag-transitive, as required. In fact, this design has been found in [6]. If $(v,k,\lambda)=(11,6,3)$, then Lemma 2.3 shows that $\mathcal{D}$ is also flag-transitive, as described in [7].

If $f=2$, then $q=p^2\equiv-1\ (\mathrm{mod}\,5)$ and $120^3\cdot4^2\geq p^2(p^4-1)$. Hence, the possible pairs (p,v) are $(3,6)$, $(7,980)$ and $(13,40222)$. Since $k\,|\,60e$ and $e\,|\,2f=4$, we have $k\,|\,u=240$. Running Algorithm 2.8 with $u=240$ and $v=6,980$ or 40222, returns an empty list Designs for every case, a contradiction.

Case 7 $X\cap G_x=A_4$, $q=p\equiv\pm3\ (\mathrm{mod}\,8)$ and $q\not\equiv\pm1\ (\mathrm{mod}\,10)$.

Here $v=\frac{q(q^2-1)}{24}$, $|X_x|=|X\cap G_x|=12$, $|\mathrm{Out}(X)|=2$, $|G|=\frac{1}{2}eq(q^2-1)$ and $|G_x|=12e$, where $e=1$ or 2.

The inequality $|G_x|^3>|G|$ gives $(12e)^3>\frac{1}{2}eq(q^2-1)$. Since $q\geq5$, $q=p\equiv\pm3\ (\mathrm{mod}\,8)$ and $q\not\equiv\pm1\ (\mathrm{mod}\,10)$, we get $q=5$ or 13. Thus $v=5$ or 91, respectively. It is not hard to see that there is no symmetric (v,k,λ) design with $v=5$. If $v=91$, then all possible parameters of (k,λ) are

$$(10,1),(36,14),(45,22),(46,23),(55,33)\text{ and }(81,72).$$

However, by $k\,|\,12e$ and $e=1$ or 2, we have $k\,|\,24$, the desired contradiction.

Case 8 $X\cap G_x=S_4$, $q=p\equiv\pm1\ (\mathrm{mod}\,8)$.

Now $v=\frac{q(q^2-1)}{48}$, $|X_x|=|X\cap G_x|=24$, $|\mathrm{Out}(X)|=2$, $|G|=\frac{1}{2}eq(q^2-1)$, $|G_x|=24e$, where $e=1$ or 2.

Since $q=p$, $e\leq2$ and $|G_x|^3>|G|$, that is, $(24e)^3>\frac{1}{2}eq(q^2-1)$, we get

$$q(q^2-1)<2\cdot24^3\cdot e^2\leq48^3.$$

Since $q\equiv\pm1\ (\mathrm{mod}\,8)$, we obtain that the possible pairs (q,v) are $(7,7)$, $(17,102)$, $(23,253)$, $(31,620)$, $(41,1435)$ and $(47,2162)$. Since $k\,|\,|G_x|=24e$ and $e=1$ or 2, we get $k\,|\,u=48$. Thus Algorithm 2.8 gives only two parameters: $(7,3,1)$ and $(7,4,2)$. If $(v,k,\lambda)=(7,3,1)$, then $X=\mathrm{PSL}(2,7)$, and so $G=\mathrm{PSL}(2,7)$ or $\mathrm{PGL}(2,7)$. Hence G acts as a 2-transitive permutation group on the set P of 7 points by GAP. Thus Lemma 2.3 shows that $\mathcal{D}$ is flag-transitive. If $(v,k,\lambda)=(7,4,2)$, then $\mathcal{D}$ is also flag-transitive by Lemma 2.3. This design has been discussed in [6].

3.3. Characteristic two

In this subsection, we suppose that G is of characteristic 2 and $X\cap G_x$ is maximal in X. The structure of $X\cap G_x$ is given in Table 2.

Case 1 $X\cap G_x = C_2^f : C_{q-1}$.

This can be ruled out as Case 1 of Section 3.2.

Case 2 $X\cap G_x = D_{2(q-1)}$.

Now $v=\frac{1}{2}q(q+1)$, $|\mathrm{Out}(X)|=f$, $|G|=eq(q^2-1)$ and $|G_x|=2e(q-1)$, where $e\,|\,f$.

From $q=2^f\geq 4$ we know that $v=\frac{1}{2}q(q+1)$ is even. So λ is also even since $k(k-1)=\lambda(v-1)$. Lemma 2.1 (iii) shows $k\,|\,2e(q-1)$. Then there exists a positive integer m such that $k=\frac{2e(q-1)}{m}$. Again by $k(k-1)=\lambda(v-1)$, we have $\frac{2e(q-1)}{m}(\frac{2e(q-1)}{m}-1)=\lambda(\frac{1}{2}q(q+1)-1)$, and so $(8e^2-m^2\lambda)q=2m^2\lambda+8e^2+4em$, which forces $8e^2-m^2\lambda>0$ and so $m<2e$. The fact that λ is even implies that $8e^2-m^2\lambda\geq 2$. So we have

$$2^f=q=\frac{24e^2+4em}{8e^2-m^2\lambda}-2\leq\frac{24e^2+4e\cdot 2e}{2}\leq 16f^2.$$

Hence $2\leq f\leq 10$. Since $k\,|\,2e(q-1)$ and $e\,|\,f$, we get $k\,|\,u=2f(q-1)$. The pairs (v,u), for $2\leq f\leq 10$, are $(10,12),(36,42),(136,120),(528,310),(2080,756),(8256,1778),(32896,4080),$ $(131328,9198)$ and $(524800,20460)$. Then Algorithm 2.8 gives only one possible set of parameters $(36,21,12)$. Suppose $(v,k,\lambda)=(36,21,12)$. Then $G=\mathrm{PSL}(2,8)$ or $\mathrm{P\Gamma L}(2,8)$. When $G=\mathrm{PSL}(2,8)$, the subdegrees of G are 1, 7^3 and 14, and G has only one conjugacy class of subgroups of index 36. Thus for any $B\in\mathcal{B}$, G_x is conjugate to G_B. Without loss of generality, let $G_x=G_{B_0}$ for some block B_0. The flag-transitivity of G forces G_{B_0} to act transitively on the points of B_0. Hence the points of B_0 form an orbit of G_x, which implies that a subdegree of G is $k=21$, a contradiction. Now assume $G=\mathrm{P\Gamma L}(2,8)$. Then the subdegrees of G are $1,14$ and 21, and G has only one conjugacy class of subgroups of index 36. So let $G_x=G_{B_0}$ for some block B_0 as above. Then B_0 is an orbit of size 21 of G_x. By using Magma, we obtain that $|\mathcal{B}|=|B^G|=36$, but $|B_i\cap B_j|=10$ or 15 for any two distinct blocks B_i and B_j. This is a contradiction since in our situation any two distinct blocks should have $\lambda=12$ common points.

Case 3 $X\cap G_x = D_{2(q+1)}$.

Here $v=\frac{1}{2}q(q-1)$, $|\mathrm{Out}(X)|=f$, $|G|=\frac{1}{2}eq(q^2-1)$ and $|G_x|=2e(q+1)$, where $e\,|\,f$.

Since $k\,|\,|G_x|$, there exists a positive integer m such that $k=\frac{2e(q+1)}{m}$. Thus Lemma 2.1 (i) yields $\frac{2e(q+1)}{m}(\frac{2e(q+1)}{m}-1)=\lambda(\frac{1}{2}q(q-1)-1)$, and so $(m^2\lambda-8e^2)q=8e^2-4em+2m^2\lambda=8(e-\frac{1}{2}m)^2+2(\lambda-1)m^2>0$. We then have

$$2^f=q=\frac{8e^2-4em+2m^2\lambda}{m^2\lambda-8e^2}=\frac{24e^2-4em}{m^2\lambda-2e^2}+2,$$

which implies $2^f<24e^2+2\leq 24f^2+2$. Hence $2\leq f\leq 11$. Since $k\,|\,2e(q+1)$ and $e\,|\,f$, we have $k\,|\,u=2f(q+1)$. For $2\leq f\leq 11$, the pairs (v,u) are as follows:

$(6,20)$, $(28,54)$, $(120,136)$, $(496,330)$, $(2016,780)$,
$(8128,1806)$, $(32640,4112)$, $(130816,9234)$, $(523776,20500)$, $(2096128,45078)$.

Applying Algorithm 2.8 to these pairs (v,u), we obtain $(v,k,\lambda)=(496,55,6)$ or $(2016,156,12)$.

If $(v,k,\lambda)=(496,55,6)$, then $G=\mathrm{PSL}(2,2^5)$ or $\mathrm{P\Gamma L}(2,2^5)$. Let $G=\mathrm{PSL}(2,2^5)$ (or $\mathrm{P\Gamma L}(2,2^5)$). Then the subdegrees of G are 1 and 33^{15} (or 1 and 165^3), and G has only one conjugacy class of subgroups of index 496. Thus there exists a block-stabilizer G_{B_0} such that $G_x=G_{B_0}$, which implies that B_0 should be an orbit of G_x. But this is impossible because $|B_0|=55$. Now suppose $(v,k,\lambda)=(2016,156,12)$. Then $G=\mathrm{PSL}(2,2^6)$, $\mathrm{PSL}(2,2^6):i$ $(i=2,3)$ or $\mathrm{P\Sigma L}(2,2^6)$. By the fact that G has only one conjugacy class of subgroups of index 2016, similar to the analysis above, there exists a block B_0 such that B_0 is an orbit of G_x. Thus G_x should have an orbit of size 156. The subdegrees of G, however, are as follows:

(i) 1, and 65^{31} when $G=\mathrm{PSL}(2,2^6)$;

(ii) 1, 65^7 and 130^{12} when $G=\mathrm{PSL}(2,2^6):2$;

(iii) 1, 65 and 195^{10} when $G=\mathrm{PSL}(2,2^6):3$;

(iv) 1, 65, 195^2 and 390^4 when $G=\mathrm{P\Sigma L}(2,2^6)$.

Case 4 $X\cap G_x=\mathrm{PSL}(2,q_0)=\mathrm{PGL}(2,q_0)$, where $q=q_0^r$ for some prime r and $q_0\neq 2$.

Here $v=\frac{q_0^{r-1}(q_0^{2r}-1)}{q_0^2-1}$, $|X_x|=|X\cap G_x|=q_0(q_0^2-1)$, $|\mathrm{Out}(X)|=f$, $|G|=\frac{1}{2}eq(q^2-1)$ and $|G_x|=eq_0(q_0^2-1)$, where $e\,|\,f$. Let $q_0=2^a$, so that $f=ra$.

From $|G_x|^3>|G|$, $q=q_0^r$ and $e\,|\,f$, we get

$$f^2\geq e^2>q_0^{r-3}\frac{q_0^{2r}-1}{q_0^6-3q_0^4+3q_0^2-1}.$$

If $r\geq 5$, then

$$f^2>q_0^{r-3}\frac{q_0^{2r}-1}{q_0^6-3q_0^4+3q_0^2-1}\geq q_0^{r-3}\frac{q_0^{10}-1}{q_0^6-3q_0^4+3q_0^2-1}>q_0^r=q=2^f.$$

But for $f\geq r\geq 5$ the inequality $f^2>2^f$ is not satisfied. Hence $r=2$ or 3.

Suppose first that $r=3$, so that $q=q_0^3=2^{3a}$, $v=q_0^2(q_0^4+q_0^2+1)$ and $f=3a$. The subdegrees of $\mathrm{PSL}(2,q_0^3)$ on the cosets of $\mathrm{PSL}(2,q_0)$ are as follows [13]:

$$1,\ (q_0^2-1)^{q_0+1},\ (q_0(q_0-1))^{\frac{q_0(q_0-1)}{2}},\ (q_0(q_0+1))^{\frac{q_0(q_0+1)}{2}},\ (q_0(q_0^2-1))^{q_0^3+q_0-1}.$$

By Lemma 2.5, we have

$$k\,|\,\lambda\gcd\big((q_0+1)^2(q_0-1),\frac{q_0^2(q_0-1)^2}{2},\frac{q_0^2(q_0+1)^2}{2},q_0(q_0^2-1)(q_0^3+q_0-1)\big).$$

So $k\,|\,2\lambda$. This forces $k=2\lambda$ since $k>\lambda$. Thus $v=4\lambda-1$ by equation $k(k-1)=\lambda(v-1)$. Then $\lambda=\frac{v+1}{4}=\frac{q_0^6+q_0^4+q_0^2+1}{4}$ and $k=2\lambda=\frac{q_0^6+q_0^4+q_0^2+1}{2}$. By $k\,|\,|G_x|=\frac{1}{2}eq_0(q_0^2-1)$ and $e\,|\,f=3a$, we get $\frac{2^{6a}+2^{4a}+2^{2a}+1}{2}\leq\frac{3a}{2}\cdot 2^a(2^{2a}-1)$, and so $2^{6a}\leq 3a\cdot 2^a\cdot 2^{2a}$, i.e., $2^{3a}\leq 3a$, which is impossible.

Now suppose $r=2$. Then $q=q_0^2=2^{2a}$, $v=q_0(q_0^2+1)$ and $f=2a$. The subdegrees of $\mathrm{PSL}(2,q_0^2)$ on the cosets of $\mathrm{PGL}(2,q_0)$ are as follows [13]:

$$1,\ q_0^2-1,\ (q_0(q_0-1))^{\frac{q_0-2}{2}},\ (q_0(q_0+1))^{\frac{q_0}{2}}.$$

By Lemma 2.5, we have

$$k\,|\,\lambda\gcd\big(q_0^2-1,\frac{q_0(q_0-1)(q_0-2)}{2},\frac{q_0^2(q_0+1)}{2}\big),$$

and so $k\,|\,3\lambda$. Now, $k>\lambda$ implies that $k=3\lambda$ or $\frac{3\lambda}{2}$.

If $k=3\lambda$, then $v=9\lambda-2$ by $k(k-1)=\lambda(v-1)$. So $\lambda=\frac{v+2}{9}=\frac{q_0^3+q_0+2}{9}$ and $k=3\lambda=\frac{q_0^3+q_0+2}{3}$. From $k\mid|G_x|=eq_0(q_0^2-1)$ and $e\mid f=2a$, we have $k\mid 2aq_0(q_0^2-1)$. By the facts that $\gcd(q_0^3+q_0+2,q_0)=2$ and $\gcd(q_0^3+q_0+2,q_0-1)=\gcd(4,q_0-1)=1$, we get $\frac{q_0^2-q_0+2}{3}\mid 4a$, and so $\frac{2^{2a}-2^a+2}{3}\le 4a$, which implies that $a=1$ or 2. Since $q_0\ne 2$, $a\ne 1$. Hence $a=2$ and $q_0=4$, but then $k=\frac{70}{3}$ is not an integer.

If $k=\frac{3\lambda}{2}$, then $v=\frac{9\lambda-2}{4}$. Thus $\lambda=\frac{4v+2}{9}=\frac{4q_0^3+4q_0+2}{9}$ and $k=\frac{2q_0^3+2q_0+1}{3}$. Since $k\mid|G_x|$ and $e\mid f=2a$, we have $\frac{2q_0^3+2q_0+1}{3}\mid 2aq_0(q_0^2-1)$. It follows that $2q_0^3+2q_0+1\mid 90a$, and hence $2^{3a+1}+2^{a+1}+1\le 90a$. It follows that $a=1$ or 2. If $a=1$, then $q_0=2$, a contradiction. If $a=2$, then $q_0=4$ which implies $k=\frac{137}{3}$ is not an integer.

This completes the proof of Theorem 1.1. $\Box$

Acknowledgements We thank the referees for their time and comments.

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