*Journal of Mathematical Research with Applications*
Mar., 2016, Vol. 36, No. 2, pp. 162–170
DOI:10.3770/j.issn:2095-2651.2016.02.004
Http://jmre.dlut.edu.cn

# On Jordan Biderivations of Triangular Matrix Rings

**Driss AIAT HADJ AHMED**
*Centre Régional des Métiers de l'éducation et de la formation, Tanger, Morocco*

**Abstract** Let $R$ and $S$ be rings with identity, $M$ be a unitary $(R,S)$-bimodule and $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the upper triangular matrix ring determined by $R$, $S$ and $M$. In this paper we prove that under certain conditions a Jordan biderivation of an upper triangular matrix ring $T$ is a biderivation of $T$.

**Keywords** triangular matrix ring; triangular matrix ring; derivation; biderivation; Jordan biderivation; prime ring

**MR(2010) Subject Classification** 16W20; 16S50; 16U60; 16W25; 16N60; 16U80

## 1. Introduction

Let $R$ be a ring and $Z(R)$ be the center of $R$. For each $x, y$ in $R$, denote the commutator of $x, y$ by $[x, y] = xy - yx$. An additive mapping $d$ from $R$ into $R$ is said to be a derivation of $R$ if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. If a derivation $d$ is of the form $d(x) = [x, a]$, where $a \in R$, then $d$ is said to be an inner derivation.

A biadditive mapping $\varphi$ from $R \times R$ into $R$ is called a biderivation if it is a derivation with respect to both components, meaning that

$$\varphi(xy, z) = \varphi(x, z)y + x\varphi(y, z) \quad \text{and} \quad \varphi(x, yz) = \varphi(x, y)z + y\varphi(x, z)$$

for all $x, y, z \in R$.

If $R$ is a non commutative ring, then the map $\phi(x, y) = \lambda[x, y]$ for all $x, y \in R$, where $\lambda \in Z(R)$, is a biderivation, which is called an inner biderivation.

We say that the mapping $\psi : R \times R \to R$ is an extremal biderivation if $\psi(x, y) = [x, [y, a]]$ for all $x, y \in R$, where $a \in R$ and $a \notin Z(R)$ such that $[[R, R], a] = 0$.

Let $\varphi$ be the biadditive mapping from $R \times R$ into $R$. $\varphi$ is called a Jordan biderivation if it is a Jordan biderivation with respect to both components, meaning that

$$\varphi(x^2, y) = \varphi(x, y)x + x\varphi(x, y) \quad \text{and} \quad \varphi(x, y^2) = \varphi(x, y)y + y\varphi(x, y)$$

for all $x, y \in R$.

Let $R$, $S$ be rings with identity and $M$ be a unitary $(R, S)$-bimodule. Let $f : M \times M \to M$ be a biadditive mapping. We say that $f$ is an $(R, S)$-bimodule homomorphism if it is a bimodule

Received February 15, 2015; Accepted October 21, 2015
E-mail address: ait_hadj@yahoo.com

homomorphism in each argument; namely,

$$f(rms, m') = rf(m, m')s \text{ and } f(m, r'm's') = r'f(m, m')s',$$

for all $r, r' \in R$, $s, s' \in S$, and $m, m' \in M$.

Let $R$, $S$ and $M$ be as above. In the sequel, we denote by $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ the upper triangular matrix ring determined by $R$, $S$ and $M$ with the usual addition and multiplication of matrices.

In 1993, Brešar et al. [1] proved that all biderivations of noncommutative prime rings are inner. Somewhat later, Brešar [2] investigated biderivations of semiprime rings. More details about biderivations and their generalizations can be found in [3, Section 3] where applications of biderivations to other fields are also described.

In 2009, Benkovič [4] obtained a description of biderivations for a certain class of triangular algebras, which in fact generalized some results on biderivations of nest algebras and upper triangular matrix algebras [5,6].

In 2013, Ghosseiri [7], obtained interesting results on biderivations of upper triangular matrix rings.

In 2013, Du and Wang [8], gave a description of biderivations for a certain class of generalized matrix algebras.

The aim of the paper is to give a description of Jordan biderivations for an upper triangular matrix ring. We prove that under certain conditions a Jordan biderivation of an upper triangular matrix ring is a biderivation.

## 2. Main results and proofs

This section is dedicated to the treatment of Jordan biderivations of the upper triangular matrix ring. The central question of this section is when Jordan biderivations of the upper triangular matrix ring are biderivations.

Let $1_R$ and $1_S$ be identities of the ring $R$ and $S$, respectively, and let 1 be the identity of the upper triangular matrix ring $T$. Throughout this paper we shall use following notation $1 = \begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix}$, $e = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$, and $f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_S \end{pmatrix}$.

We immediately notice that $e$ and $f$ are orthogonal idempotents of $T$ and so $T$ may be represented as $T = 1T1 = (e+f)T(e+f) = eTe + eTf + fTf$.

Here $eTe$ is a subring of $T$ isomorphic to $R$, $fTf$ is a subring of $T$ isomorphic to $S$ and $eTf$ is an $(eTe, fTf)$-bimodule isomorphic to the bimodule $M$. To simplify notation we will use the following convention: $r = ere \in R = eTe$, $s = fsf \in S = fTf$ and $m = emf \in M = eTf$ . Then each element $x \in T$ can be represented in the form

$$x = exe + exf + fxf = r + m + s,$$

where $r \in R$, $s \in S$, $m \in M$.

**Lemma 2.1** *Let $\varphi : R \times R \to R$ be a Jordan biderivation. Then, for all $a; b; c; x; y \in R$, the following statements hold:*

$$\varphi(ab + ba, x) = \varphi(a, x)b + a\varphi(b, x) + \varphi(b, x)a + b\varphi(a, x),$$

$$\varphi(a, xy + yx) = \varphi(a, x)y + x\varphi(a, y) + \varphi(a, y)x + y\varphi(a, x). \tag{1}$$

$$\varphi(aba, x) = \varphi(a, x)ba + a\varphi(b, x)a + ab\varphi(a, x),$$

$$\varphi(a, xyx) = \varphi(a, x)yx + x\varphi(a, y)x + xy\varphi(a, x). \tag{2}$$

$$\varphi(abc + cba, x) = \varphi(a, x)bc + a\varphi(b, x)c + ab\varphi(c, x) + \varphi(c, x)ba + c\varphi(b, x)a + cb\varphi(a, x),$$

$$\varphi(a, xyz + zyx) = \varphi(a, x)yz + x\varphi(a, y)z + xy\varphi(a, z) + \varphi(a, z)yx + z\varphi(a, y)x + zy\varphi(a, x). \tag{3}$$

**Proposition 2.2** *Let $\varphi : R \times R \to R$ be a Jordan biderivation. Then*

$$[\varphi(a, x), [y, b]] + [\varphi(a, y), [x, b]] + [[a, x], \varphi(b, y)] + [[a, y], \varphi(b, x)] = 0$$

*for all $x, y, a, b \in R$.*

**Proof** Consider $\varphi(ab + ba, xy + yx)$ for arbitrary $x, y, a, b \in R$. Since $\varphi$ is a Jordan derivation in the first argument, we have

$$\begin{aligned}\varphi(ab + ba, xy + yx) =& \varphi(a, xy + yx)b + a\varphi(b, xy + yx) + \varphi(b, xy + yx)a + b\varphi(a, xy + yx)\\ =& \left(\varphi(a, x)y + x\varphi(a, y) + \varphi(a, y)x + y\varphi(a, x)\right) b+\\ & a\left(\varphi(b, x)y + x\varphi(b, y) + \varphi(b, y)x + y\varphi(b, x)\right) +\\ & \left(\varphi(b, x)y + x\varphi(b, y) + \varphi(b, y)x + y\varphi(b, x)\right) a+\\ & b\left(\varphi(a, x)y + x\varphi(a, y) + \varphi(a, y)x + y\varphi(a, x)\right).\end{aligned}$$

Also, since $\varphi$ is a derivation in the second argument, we then have

$$\begin{aligned}\varphi(ab + ba, xy + yx) =& \varphi(ab + ba, x)y + x\varphi(ab + ba, y) + \varphi(ab + ba, y)x + y\varphi(ab + ba, x)\\ =& \left(\varphi(a, x)b + a\varphi(b, x) + \varphi(b, x)a + b\varphi(a, x)\right) y+\\ & x\left(\varphi(a, y)b + a\varphi(b, y) + \varphi(b, y)a + b\varphi(a, y)\right) +\\ & \left(\varphi(a, y)b + a\varphi(b, y) + \varphi(b, y)a + b\varphi(a, y)\right) x+\\ & y\left(\varphi(a, x)b + a\varphi(b, x) + \varphi(b, x)a + b\varphi(a, x)\right).\end{aligned}$$

Comparing both relations, we obtain

$$[\varphi(a, x), [y, b]] + [\varphi(a, y), [x, b]] + [[a, x], \varphi(b, y)] + [[a, y], \varphi(b, x)] = 0. \quad \square$$

**Lemma 2.3** *Let $\varphi : T \times T \to T$ be a Jordan biderivation. Then*

*(i)* $\varphi(1, x) = \varphi(x, 1) = 0$ *for all $x \in A$;*

*(ii)* $\varphi(x, 0) = 0 = \varphi(0, x)$ *for all $x \in A$;*

*(iii)* $\varphi(e, e) = -\varphi(e, f) = -\varphi(f, e) = \varphi(f, f)$.

**Proof** The identity $\varphi(1, x) = 0$ follows from

$$\varphi(1; x) = \varphi(1 \times 1 \times 1; x) = \varphi(1; x) + \varphi(1; x) + \varphi(1; x).$$

Similarly, $\varphi(x,1)=0$, while $\varphi(x,0)=0=\varphi(0,x)$ is obvious.

To prove (iii), we use (i) and the equality $e+f=1$. Then we may write

$$\varphi(e,e)=\varphi(e,1-f)=\varphi(e,1)-\varphi(e,f)=-\varphi(e,f).$$

Similarly, $\varphi(f,e)=-\varphi(e,e)$. It is also true that

$$\varphi(f,f)=\varphi(1-e,1-e)=\varphi(1,1)-\varphi(1,e)-\varphi(e,1)+\varphi(e,e)=\varphi(e,e).$$

Thus we conclude that $\varphi(e,e)=-\varphi(e,f)=-\varphi(f,e)=\varphi(f,f)$. □

**Lemma 2.4** *Let $T$ be the upper triangular matrix ring and let $\varphi : T\times T\to T$ be a Jordan biderivation. If $\varphi(e,e)\neq 0$, then $\varphi=\psi+\theta$, where $\psi(x,y)=[x,[y,\varphi(e,e)]]$ is an extremal biderivation and $\theta$ is a Jordan biderivation that satisfies $\theta(e,e)=0$.*

**Proof** For every $x,y,a,b\in T$, it follows from Lemma 2.1 that

$$[\varphi(a,x),[y,b]]+[\varphi(a,y),[x,b]]+[[a,x],\varphi(b,y)]+[[a,y],\varphi(b,x)]=0.$$

If we substitute $a=x=e$, then we obtain that

$$[\varphi(e,e),[y,b]]+[\varphi(e,y),[e,b]]+[[e,e],\varphi(b,y)]+[[e,y],\varphi(b,e)]=0.$$

Let $b,y\in T$. We have $[e,b]=eb-be=e\,[e,b]\,f$, and $\varphi(e,y)=e\varphi(e,y)e+e\varphi(e,y)f+f\varphi(e,y)f$.

It follows from the fact $e^2=e$ that $\varphi(e;y)=\varphi(e^2,y)=\varphi(e,y)e+e\varphi(e,y)$. This implies that $e\varphi(e,y)e=0=f\varphi(e,y)f$. Thus, $\varphi(e,y)=e\varphi(e,y)f$.

Consequently, $[\varphi(e,y),[e,b]]=0$. Similarly, $[[e,y],\varphi(b,e)]=0$. Hence

$$[\varphi(e,e),[y,b]]=0=\varphi(y,b)[e,e]=0.$$

By [4, Remark 4.4], if $x_0\in T$, $x_0\notin Z(T)$ and suppose that $[[x,y],x_0]=0$ for all $x,y\in A$. Then the map $\psi : T\times T\to T$ defined by $\psi(x,y)=[x,[y,x_0]]$ for all $x,y\in T$ is a biderivation. We see that $\varphi(e;e)=e\varphi(e,e)f\notin Z\,(T)$, and it satisfies $[\varphi(e,e),[y,b]]=0$.

Write $\theta=\varphi-\psi$. Clearly, $\theta$ is also a Jordan biderivation of $T$ satisfying $\theta(e,e)=0$. □

**Proposition 2.5** *Let $T$ be the upper triangular matrix ring given above and let $\varphi : T\times T\to T$ be a Jordan biderivation satisfying $\varphi(e,e)=0$. Then*

$$\varphi\left(\begin{pmatrix} r & m \\ & s\end{pmatrix},\begin{pmatrix} r' & m' \\ & s'\end{pmatrix}\right)=\begin{pmatrix}\delta\,(r,r') & rg(m')-g(m')s+r'h(m)-h(m)s'+\xi(m;m') \\ & \gamma\,(s,s')\end{pmatrix}$$

*for all $r,r\prime\in R$, $s,s'\in S$, and $m,m'\in M$ where*

*(i) $\delta$ is a Jordan biderivation of $R$ and $\gamma$ is a Jordan biderivation of $S$.*

*(ii) $g : M\to M$ an $(R,S)$-bimodule homomorphism such that*

$$\delta\,(r,r')\,m=[r,r']\,g(m);\quad m\gamma\,(s,s')=g(m)\,[s,s']$$

*for all $r,r'\in R, m\in M$, and $s,s'\in S$.*

*(iii) $h : M\to M$ an $(R,S)$-bimodule homomorphism such that*

$$\delta\,(r,r')\,m=[r',r]\,h(m);\quad m\gamma\,(s,s')=h(m)\,[s',s]$$

*for all $r, r' \in R, m \in M$, and $s, s' \in S$.*

*(iv) $\xi : M \times M \to M$ an $(R,S)$-bimodule homomorphism such that for all $m, m' \in M$, $r, r' \in R$, and $s, s' \in S$ we have*

$$[r, r']\,\xi(m, m') = \xi(m, m')\,[s, s'] = 0.$$

Before characterizing the Jordan biderivations of the triangular ring $T$, let us begin with the following lemma.

**Lemma 2.6** *Let $T$ be the upper triangular matrix ring given above and let $\varphi : T \times T \to T$ be a Jordan biderivation satisfying $\varphi(e, e) = 0$. For all $r, r' \in R$, $s, s' \in S$ we have*

$$\begin{aligned} \varphi(re, f) = 0;\ \ \varphi(re, e) = 0 \\ \varphi(f, r'e) = 0;\ \ \varphi(e, r'e) = 0 \\ \varphi(e, s'f) = 0;\ \ \varphi(f, s'f) = 0 \\ \varphi(sf, e) = 0;\ \ \varphi(sf, f) = 0 \end{aligned} \tag{4}$$

$$\varphi(re, s'f) = 0. \tag{5}$$

**Proof** Put $\varphi(re, f) = \begin{pmatrix} x_{11} & x_{12} \\ & x_{22} \end{pmatrix}$.

Applying $\varphi$ on both sides of $(re, f) = (re, f^2)$, one observes that $x_{12} = x_{22} = 0$.

Now applying, $\varphi$ on both sides of $(re, f) = (e\,(re)\,e, f)$, one observes that $x_{11} = 0$. Hence, we have $\varphi(re, e) = 0$.

In a similar manner we can prove that $\varphi(f, r'e) = 0$; $\varphi(e, r'e) = 0$, $\varphi(e, s'f) = 0$; $\varphi(f, s'f) = 0$; $\varphi(sf, e) = 0$; $\varphi(sf, f) = 0$.

To prove (2), put $\varphi(re, s'f) = \begin{pmatrix} y_{11} & y_{12} \\ & y_{22} \end{pmatrix}$.

Likewise, applying $\varphi$ on both sides of $(re, s'f) = (e\,(re)\,e, s'f)$ using the Lemma 2.1 (2), one observes that $y_{12} = y_{22} = 0$.

Moreover, applying $\varphi$ on both sides of $(re, s'f) = (re, f\,(sf)\,f)$ using the Lemma 2.1 (2), one observes that $y_{11} = 0$. $\square$

**Lemma 2.7** *Let $T$ be the upper triangular matrix ring given above and let $\varphi : T \times T \to T$ be a Jordan biderivation satisfying $\varphi(e, e) = 0$.*

*(i) There exists a Jordan biderivation $\delta : R \times R \to R$ such that for all $r, r' \in R$, $s, s' \in S$ we have $\varphi(re, r'e) = \delta\,(r, r')$.*

*(ii) There exists a Jordan biderivation $\gamma : S \times S \to S$ such that for all $s, s' \in S$ we have $\varphi\,(sf, s'f) = \gamma\,(s, s')\,f$.*

**Proof** Let $r, r'$ be in $R$. Using Lemma 2.1 (2), we have

$$\begin{aligned} \varphi(re, r'e) =& \varphi(e\,(re)\,e, r'e) = \varphi(e; r'e)re + e\varphi(re; r'e)e + re\varphi(e; r'e) \\ =& e\varphi(re; r'e)e = \delta\,(r, r')\,e, \end{aligned}$$

where $\delta : R \times R \to R$ is a mapping that satisfies $\varphi(re, r'e) = \delta(r, r')\, e$, since $\varphi$ is biadditive, so is $\delta$.

Let $r, r' \in R$. We have

$$\begin{aligned}\varphi(r^2e, r'e) =&\delta\left(r^2, r'\right) e = \varphi(\left(re\right)^2, r'e) = r\varphi(re, r'e)e + \varphi(re, r'e)re\\ =&r\delta(r, r')\, e + \delta(r, r')\, re.\end{aligned}$$

Hence $\delta\left(r^2, r'\right) = r\delta(r, r')\, e + \delta(r, r')\, r$, and $\delta$ is a Jordan derivation on the first component. Similarly, one can show that $\delta$ is also a Jordan derivation on the second component. Thus, $\delta$ is a Jordan biderivation of $R$. The proof of (ii) is similar, hence omitted. □

**Lemma 2.8** *Let $T$ be the upper triangular matrix ring given above and let $\varphi : T \times T \to T$ be a Jordan biderivation satisfying $\varphi(e, e) = 0$.*

*(i) There exists an $(R, S)$-bimodule homomorphism $g : M \to M$ such that*

$$\varphi(re, me_{12}) = rg(m)e_{12}; \quad \varphi(sf, me_{12}) = -g(m)se_{12},$$

$$\delta(r, r')\, m = [r, r']\, g(m); \quad m\gamma(s, s') = g(m)\,[s, s'],$$

*for all $r, r' \in R, m \in M$, and $s, s' \in S$.*

*(ii) There exists an $(R, S)$-bimodule homomorphism $h : M \to M$ such that*

$$\varphi(me_{12}, re) = rh(m)e_{12}; \quad \varphi(me_{12}, sf) = -h(m)se_{12},$$

$$\delta(r, r')\, m = [r', r]\, h(m); \quad m\gamma(s, s') = h(m)\,[s', s],$$

*for all $r, r' \in R, m \in M$, and $s, s' \in S$.*

**Proof** Let $r \in R$ and $m \in M$. First assume that $\varphi(e, me_{12}) = \begin{pmatrix} v_{11} & v_{12} \\ & v_{22} \end{pmatrix}$.

Applying $\varphi$ on both sides of $(e, me_{12}) = (e^2, me_{12})$, one observes that $v_{11} = v_{22} = 0$. Thus, there exists a mapping $g$ from $M$ to itself such that $\varphi(e, me_{12}) = g(m)e_{12}$.

Now, applying $\varphi$ on both sides of $(re, me_{12}) = (e\,(re)\,e, me_{12})$, and using Lemma 2.1 (2), we conclude that $\varphi(re, me_{12}) = e\varphi(re, me_{12})e + rg(m)e_{12}$.

Moreover, applying $\varphi$ on both sides of $(re, me_{12}) = (re, e\,(me_{12})\,f + f\,(me_{12})\,e)$, and using Lemma 2.1 (2), we find that $\varphi(re, me_{12}) = e\varphi(re, me_{12})f$, one observes that $\varphi(re, me_{12}) = rg(m)e_{12}$.

Let $s \in S$ and $m \in M$. First note that, by Lemma 2.3 and the preceding result, $\varphi(f, me_{12}) = -\varphi(e, me_{12}) = -\ g(m)e_{12}$. Now, applying $\varphi$ on both sides of $(sf, me_{12}) = (f\,(sf)\,f, me_{12})$, and using Lemma 2.1 (2), we conclude that $\varphi(sf, me_{12}) = f\varphi(sf, me_{12})f - g(m)se_{12}$. Applying $\varphi$ on both sides of $(sf, me_{12}) = (sf, e\,(me_{12})\,f + f\,(me_{12})\,e)$, and using Lemma 2.1 (3), we conclude that $\varphi(sf, me_{12}) = e\varphi(sf, me_{12})f$, we find that $\varphi(sf, me_{12}) = -g(m)se_{12}$.

Since $\varphi$ is additive on the second component, $g$ is additive. Moreover, from $\varphi(e, rme_{12}) = g(rm)e_{12}$ and applying $\varphi$ on both sides of $(e, rme_{12}) = (e, (re)\,(me_{12})\,f + f\,(me_{12})\,(re))$, and using Lemma 2.1 (3), we conclude that $g(rm) = rg(m)$. We infer that $g$ is a left $R$-homomorphism.

Likewise, applying $\varphi$ on both sides of $(f, mse_{12}) = (f, e\,(me_{12})\,(sf) + (sf)\,(me_{12})\,e)$, we see that $g$ is also a right $S$-homomorphism.

Next, we show that $\delta\,(r, r')\,m = [r, r']\,g(m)$.

Applying $\varphi$ on both sides of $(re, r'me_{12}) = (re, (r'e)\,(me_{12})\,f + f\,(me_{12})\,(r'e))$, and using Lemma 2.1 (3), we conclude that $\varphi(re, r'me_{12}) = \delta\,(r, r')\,me_{12} + r'rg(m)e_{12}$.

Now, applying $\varphi$ on both sides of $(re, me_{12}) = (e\,(re)\,e, r'me_{12})$, and using Lemma 2.1 (2), we conclude that $\varphi(re, me_{12}) = e\varphi(re, r'me_{12})e + rg(r'm)e_{12}$, one observes that $rg(r'm) = \delta\,(r, r')\,m + r'rg(m) = rr'g(m)$. Hence $\delta\,(r, r')\,m = [r, r']\,g(m)$. Similarly, one can show that $m\gamma\,(s, s') = g(m)\,[s, s']$.

The proof of (2) is similar to that of (1), except for the coordinates, so it is omitted. □

**Lemma 2.9** *Let $T$ be the upper triangular matrix ring given above and let $\varphi : T \times T \to T$ be a Jordan biderivation satisfying $\varphi(e, e) = 0$. There exists an $(R, S)$-bimodule homomorphism $\xi : M \times M \to M$ such that for all $m, m' \in M$, $r, r' \in R$, and $s, s' \in S$ we have*

$$\varphi\,(me_{12}, m'e_{12}) = \xi(m, m')e_{12}, \quad [r, r']\,\xi(m, m') = \xi(m, m')\,[s, s'] = 0. \tag{6}$$

**Proof** Applying $\varphi$ on both sides of $(me_{12}, m'e_{12}) = ((e)\,(me_{12})\,f + f\,(me_{12})\,(e)\,, m'e_{12})$, and using Lemma 2.1 (3), we conclude that $\varphi\,(me_{12}, m'e_{12}) = e\varphi(me_{12}, m'e_{12})f$, therefore, there exists a mapping $\xi : M \times M \to M$ such that $\varphi\,(me_{12}, m'e_{12}) = \xi(m, m')e_{12}$. Biadditivity of $\xi$ is inherited from $\varphi$.

To show that $\xi$ is a left $R$-homomorphism on the first component, let $r \in R$, and $m, m' \in M$. Applying $\varphi$ on both sides of $(rme_{12}, m'e_{12}) = ((re)\,(me_{12})\,f + f\,(me_{12})\,(re)\,, m'e_{12})$, and using Lemma 2.1 (3), we conclude that $\varphi(rme_{12}, m'e_{12}) = r\xi(m, m')e_{12}$, that is, $\xi(rm, m') = r\xi(m, m')$. Likewise, one can show that $\xi$ is also a right $S$-homomorphism on the first component, and an $(R, S)$-bimodule homomorphism on the second component.

Next, we show that $[r, r']\,\xi(m, m') = 0$.

Applying $\varphi$ on both sides of $(rme_{12}, r'm'e_{12}) = (rme_{12}, (r'e)\,(m'e_{12})\,f + f\,(m'e_{12})\,(r'e))$, and using Lemma 2.1 (3), we conclude that $\varphi(rme_{12}, r'm'e_{12}) = (r'e)\,\varphi(rme_{12}, (m'e_{12}))f = r'\xi(rm, m')e_{12}$, and $\xi(rm, r'm') = r'\xi(rm, m') = r'r\xi(m, m')$. One observes that $\xi(rm, r'm') = rr'\xi(m, m') = r'r\xi(m, m')$. Hence $[r, r']\,\xi(m, m') = 0$. Similarly, one can show that $\xi(m, m')\,[s, s'] = 0$. □

**Proof of Proposition 2.5** For all $m, m' \in M$, $r, r' \in R$, and $s, s' \in S$ we have

$$\begin{aligned}
\varphi\left(\begin{pmatrix} r & m \\ & s \end{pmatrix}, \begin{pmatrix} r' & m' \\ & s' \end{pmatrix}\right) &= \varphi\,(r + m + s, r' + m' + s') \\
&= \varphi(re, r'e) + \varphi(re, m'e_{12}) + \varphi(re, s'f) + \\
&\quad \varphi(sf, r'e) + \varphi(sf, m'e_{12}) + \varphi(sf, s'f) + \\
&\quad \varphi(me_{12}, r'e) + \varphi(me_{12}, m'e_{12}) + \varphi(me_{12}, s'f) + \\
&= \delta\,(r, r')\,e + \gamma\,(s, s')\,f + \xi(m, m')e_{12} +
\end{aligned}$$

$$rg(m')e_{12}-g(m')se_{12}+r'h(m)e_{12}-h(m)s'e_{12},$$

$$\varphi\left(\left(\begin{array}{cc} r & m \\ & s \end{array}\right),\left(\begin{array}{cc} r' & m' \\ & s' \end{array}\right)\right)=\left(\begin{array}{cc} \delta(r,r') & rg(m')-g(m')s+r'h(m)-h(m)s'+\xi(m,m') \\ & \gamma(s,s') \end{array}\right). \square$$

**Theorem 2.10** *Let $T$ be the upper triangular matrix ring given above and let $\varphi: T\times T\to T$ be a Jordan biderivation. Suppose that*

*(i) Every Jordan biderivation of $R$ is a biderivation;*

*(ii) Every Jordan biderivation of $S$ is a biderivation.*

*Then all the Jordan biderivation of $T$ is a biderivation.*

**Proof** For all $m_1,m_2,m'\in M$, $r_1,r_2,r'\in R$, and $s_1,s_2,s'\in S$, put $X=\left(\begin{array}{cc} r_1 & m_1 \\ & s_1 \end{array}\right)$, $Y=\left(\begin{array}{cc} r_2 & m_2 \\ & s_2 \end{array}\right)$ and $Z=\left(\begin{array}{cc} r' & m' \\ & s' \end{array}\right)$, we have

$$\begin{aligned}&\varphi(XY,Z)-\varphi(X,Z)Y-X\varphi(Y,Z)\\ &=\left(\begin{array}{cc} \delta(r_1r_2,r')-\delta(r_1,r')r_2-r_1\delta(r_2,r') & 0 \\ & \gamma(s_1s_2,s')-\gamma(s_1,s')s_2-s_1\gamma(s_2,s') \end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\varphi(X,YZ)-\varphi(X,Y)Z+Y\varphi(X,Z)\\ &=\left(\begin{array}{cc} \delta(r_1,r_2r')-\delta(r_1,r_2)r'+r_2\delta(r_1,r') & 0 \\ & \gamma(r_1,r_2r')-\gamma(r_1,r_2)r'+r_2\gamma(r_1,r') \end{array}\right).\end{aligned}$$

Since every Jordan biderivation of $R$ is a biderivation and every Jordan biderivation of $S$ is a biderivation. This completes the proof. $\square$

**Remark 2.11** ([9]) Let $R$ be a prime ring, and $\mathrm{char}(R)\neq 2$. Then every Jordan biderivation of $R$ is a biderivation.

**Corollary 2.12** ([10]) *Let $T$ be the upper triangular matrix ring given above and let $\varphi$ : $T\times T\to T$ be a Jordan biderivation. Suppose that $R$ and $S$ be a prime ring,* $\mathrm{char}(R)\neq 2$, *and* $\mathrm{char}(S)\neq 2$. *Then Jordan biderivation of $T$ is a biderivation.*

**Acknowledgements** The author would like to express his sincere thanks to the referees for carefully reading the manuscript and useful suggestions.

## References

[1] M. BREŠAR, W. S. MARTINDALE, C. R. MIERS. *Centralizing maps in prime rings with involution*. J. Algebra, 1993, **161**(2): 342–357.

[2] M. BREŠAR. *On certain pairs of functions of semiprime rings*. Proc. Amer. Math. Soc., 1994, **120**(3): 709–713.

[3] M. BREŠAR. *Commuting maps: a survey*. Taiwanese J. Math., 2004, **8**(3): 361–397.

[4] D. BENKOVIČ. *Biderivations of triangular algebras*. Linear Algebra Appl., 2009, **431**(9): 1587–1602.

[5] Jianhua ZHANG, Shan FENG, Hongxia LI, et al. *Generalized biderivations of nest algebras*. Linear Algebra Appl., 2006, **418**(1): 225–233.

[6] Yanxia ZHAO, Dengyin WANG, Ruiping YAO. *Biderivations of upper triangular matrix algebras over commutative rings.* Int. J. Math. Game Theory Algebra, 2009, **18**(6): 473–478.
[7] N. M. GHOSSEIRI. *On biderivations of upper triangular matrix rings.* Linear Algebra Appl., 2013, **438**(1): 250–260.
[8] Yiqiu DU, Yu WANG. *Biderivations of generalized matrix algebras.* Linear Algebra Appl., 2013, **438**(11): 4483–4499.
[9] M. BREŠAR, J. VUKMAN. *Jordan derivations on prime rings.* Bull. Austral. Math. Soc., 1988, **37**(3): 321–322.
[10] D. BENKOVIČ. *Generalized Lie derivations on triangular algebras.* Linear Algebra Appl., 2011, **434**(6): 1532–1544.
[11] D. BENKOVIČ, D. EREMITA. *Multiplicative Lie $n$-derivations of triangular rings.* Linear Algebra Appl., 2012, **436**(11): 4223–4240.
[12] D. BENKOVIČ, N. ŠIROVNIK. *Jordan derivations of unital algebras with idempotents.* Linear Algebra Appl., 2012, **437**(9): 2271–2284.
[13] M. BREŠAR. *On generalized biderivations and related maps.* J. Algebra, 1995, **172**(3): 764–786.
[14] W. S. CHEUNG. *Commuting maps of triangular algebras.* J. London Math. Soc. (2), 2001, **63**(1): 117–127.
[15] W. S. CHEUNG. *Lie derivations of triangular algebras.* Linear Multilinear Algebra, 2003, **51**(3): 299–310.
[16] Yiqiu DU, Yu WANG. *Lie derivations of generalized matrix algebras.* Linear Algebra Appl., 2012, **437**(11): 2719–2726.
[17] Yiqiu DU, Yu WANG. *$k$-commuting maps on triangular algebras.* Linear Algebra Appl., 2012, **436**(5): 1367–1375.
[18] Yanbo LI, Feng WEI. *Semi-centralizing maps of generalized matrix algebras.* Linear Algebra Appl., 2012, **436**(5): 1122–1153.
[19] Yanbo LI, Zhankui XIAO. *Additivity of maps on generalized matrix algebras.* Electron. J. Linear Algebra, 2011, **22**: 743–757.
[20] A. D. SANDS. *Radicals and Morita contexts.* J. Algebra, 1973, **24**: 335–345.
[21] Yao WANG, Yu WANG. *Multiplicative Lie $n$-derivations of generalized matrix algebras.* Linear Algebra Appl., 2013, **438**(5): 2599–2616.
[22] Zhankui XIAO, Feng WEI. *Commuting mappings of generalized matrix algebras.* Linear Algebra Appl., 2010, **433**(11-12): 2178–2197.