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# The Twin Domination Number of Cartesian Product of Directed Cycles

**Hongxia MA, Juan LIU***
*College of Mathematics Sciences, Xinjiang Normal University, Xinjiang* 830017*, P. R. China*

**Abstract** Let $\gamma^*(D)$ denote the twin domination number of digraph $D$ and let $C_m\Box C_n$ denote the Cartesian product of $C_m$ and $C_n$, the directed cycles of length $m, n \geq 2$. In this paper, we determine the exact values: $\gamma^*(C_2\Box C_n) = n$; $\gamma^*(C_3\Box C_n) = n$ if $n \equiv 0 \pmod 3$, otherwise, $\gamma^*(C_3\Box C_n) = n + 1$; $\gamma^*(C_4\Box C_n) = n + \lceil \frac{n}{2} \rceil$ if $n \equiv 0, 3, 5 \pmod 8$, otherwise, $\gamma^*(C_4\Box C_n) = n + \lceil \frac{n}{2} \rceil + 1$; $\gamma^*(C_5\Box C_n) = 2n$; $\gamma^*(C_6\Box C_n) = 2n$ if $n \equiv 0 \pmod 3$, otherwise, $\gamma^*(C_6\Box C_n) = 2n + 2$.

**Keywords** twin domination number; Cartesian product; directed cycles

**MR(2010) Subject Classification** 05C69; 05C76

## 1. Introduction

Let $D = (V, A)$ be a finite digraph without loops and multiple arcs where $V = V(D)$ is the vertex set and $A = A(D)$ is the arc set. For a vertex $v \in V(D)$, $N_D^+(v)$ and $N_D^-(v)$ denote the set of out-neighbors and in-neighbors of $v$, $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$ denote the out-degree and in-degree of $v$ in $D$, respectively. A digraph $D$ is $r$-regular if $d_D^+(v) = d_D^-(v) = r$ for any vertices $v$ in $D$. Given two vertices $u$ and $v$ in $D$, we say $u$ out-dominates $v$ if $u = v$ or $uv \in A(D)$, and we say $v$ in-dominates $u$ if $u = v$ or $uv \in A(D)$. Let $N_D^+[v] = N_D^+(v) \cup \{v\}$. A vertex $v$ dominates all vertices in $N_D^+[v]$. A set $S \subseteq V(D)$ is a dominating set of $D$ if $S$ dominates $V(D)$. The domination number of $D$, denoted by $\gamma(D)$, is the minimum cardinality of a dominating set of $D$. The notion of twin domination in digraphs has been studied in [1,7]. A set $S \subseteq V(D)$ is a twin dominating set of $D$ if for any vertex $v \in V - S$, there exist $u, w \in S$ (possibly $u = w$) such that arcs $uv, vw \in A(D)$. The twin domination number of $D$, denoted by $\gamma^*(D)$, is the minimum cardinality of a twin dominating set of $D$. Clearly, $\gamma(D) \leqslant \gamma^*(D)$.

Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs which have disjoint vertex sets $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ and $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$ and disjoint arc sets $A_1$ and $A_2$, respectively. The Cartesian product $D = D_1\Box D_2$ has vertex set $V = V_1 \times V_2$ and $(x_i, y_j)(x_{i'}, y_{j'}) \in A(D)$ if and only if one of the following holds:

(a) $x_i = x_{i'}$ and $y_j y_{j'} \in A_2$;

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* Corresponding author
E-mail address: 598254233@qq.com (Hongxia MA); liujuan1999@126.com (Juan LIU)

(b) $y_j = y_{j'}$ and $x_i x_{i'} \in A_1$.

The subdigraph $D_1^{y_i}$ of $D$ has vertex set $V_1^{y_i} = \{(x_j, y_i)$: for any $x_j \in V_1$, fixed $y_i \in V_2\} \cong V_1$, and arc set $A_1^{y_i} = \{(x_j, y_i)(x_{j'}, y_i)$: $x_j x_{j'} \in A_1\} \cong A_1$. It is clear that $D_1^{y_i} \cong D_1$. Similarly, the subdigraph $D_2^{x_i}$ of $D$ has vertex set $V_2^{x_i} = \{(x_i, y_j)$: for any $y_j \in V_2$, fixed $x_i \in V_1\} \cong V_2$, and arc set $A_2^{x_i} = \{(x_i, y_j)(x_i, y_{j'})$: $y_j y_{j'} \in A_2\} \cong A_2$. It is clear that $D_2^{x_i} \cong D_2$. In recent years, the domination number of the Cartesian product of directed cycles and paths has been discussed in [2–6,8]. However, to date no research about the twin domination number has been done for the Cartesian product of directed cycles. In this paper, we study the twin domination number of $C_m \Box C_n$. We mainly determine the exact values: $\gamma^*(C_2 \Box C_n) = n$; $\gamma^*(C_3 \Box C_n) = n$ if $n \equiv 0 \pmod 3$, otherwise, $\gamma^*(C_3 \Box C_n) = n + 1$; $\gamma^*(C_4 \Box C_n) = n + \lceil \frac{n}{2} \rceil$ if $n \equiv 0, 3, 5 \pmod 8$, otherwise, $\gamma^*(C_4 \Box C_n) = n + \lceil \frac{n}{2} \rceil + 1$; $\gamma^*(C_5 \Box C_n) = 2n$; $\gamma^*(C_6 \Box C_n) = 2n$ if $n \equiv 0 \pmod 3$, otherwise, $\gamma^*(C_6 \Box C_n) = 2n + 2$.

## 2. Main results

We denote the vertices of a directed cycle $C_n$ by the integers $\{0, 1, \ldots, n-1\}$ considering modulo $n$. There is an arc $xy$ from $x$ to $y$ in $C_n$ if and only if $y = x + 1 \pmod n$.

Observe that in $C_m \Box C_n$ the vertices of $C_m^i$ are out-dominated by vertices of $C_m^{i-1}$ or $C_m^i$ and in-dominated by vertices of $C_m^{i+1}$ or $C_m^i$ for $i \in \{0, 1, \ldots, n-1\}$. Especially, the vertices of $C_m^0$ are out-dominated by vertices of $C_m^{n-1}$ or $C_m^0$ and in-dominated by vertices of $C_m^1$ or $C_m^0$. First we investigate the twin domination number of $C_2 \Box C_n$.

**Lemma 2.1** ([2]) *Let $n \geq 2$. Then $\gamma(C_2 \Box C_n) = n$.*

By Lemma 2.1, $\gamma^*(C_2 \Box C_n) \geq \gamma(C_2 \Box C_n) = n$. Let $S_0 = \{(0, i) | i \in \{0, 1, \ldots, n-1\}\}$. Then $S_0$ is a twin dominating set of $C_2 \Box C_n$. According to the discussion, we can get the following theorem.

**Theorem 2.2** *Let $n \geq 2$. Then $\gamma^*(C_2 \Box C_n) = n$.*

**Lemma 2.3** ([2]) *Let $n \geq 2$. Then* $\gamma(C_3 \Box C_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 3; \\ n+1, & \text{otherwise.} \end{cases}$

Now we consider the twin domination number of Cartesian product $C_3 \Box C_n$, and define a set $S_1$ (see Figure 1) as follows: $S_1$ consists of vertices $(0, i)$, $i \equiv 0 \pmod 3$; $(1, i)$, $i \equiv 1 \pmod 3$; $(2, i)$, $i \equiv 2 \pmod 3$. Note that $|S_1| = n$.

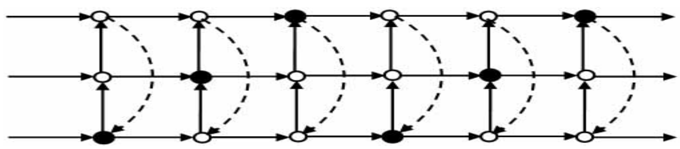

Figure 1 The set $S_1$

**Theorem 2.4** *Let $n \geq 2$. Then* $\gamma^*(C_3 \Box C_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 3; \\ n+1, & \text{otherwise.} \end{cases}$

**Proof** By Lemma 2.3, if $n \equiv 0 \pmod 3$, then $\gamma^*(C_3\square C_n) \geq \gamma(C_3\square C_n) = n$. $S_1$ is a twin dominating set of $C_3\square C_n$, when $n \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $\gamma^*(C_3\square C_n) \geq \gamma(C_3\square C_n) = n+1$. Note that $S_1 \cup \{(1,0)\}$ is a twin dominating set of $C_3\square C_n$, when $n \equiv 1 \pmod 3$. Similarly, $S_1 \cup \{(2,0)\}$ is a twin dominating set of $C_3\square C_n$, when $n \equiv 2 \pmod 3$. Thus $\gamma^*(C_3\square C_n) = n+1$, when $n \equiv 1, 2 \pmod 3$. $\square$

**Lemma 2.5** ([4]) *Let $S$ be a dominating set of $C_m\square C_n$. Then $|S \cap C_m^{i-1}| + 2|S \cap C_m^i| \geq m$ for all $i$ in $\{0, 1, \ldots, n-1\}$ considered modulo $n$.*

By Lemma 2.5, we have $|S \cap C_m^{i-1}| + 2|S \cap C_m^i| \geq m$ for $0 \leqslant i \leqslant n-1$, where $S$ is a twin dominating set of $C_m\square C_n$.

**Lemma 2.6** ([4]) *Let $n \geq 2$. Then* $\gamma(C_4\square C_n) = \begin{cases} \frac{3n}{2}, & \text{if } n \equiv 0 \pmod 8; \\ n + \lceil \frac{n+1}{2} \rceil, & \text{otherwise.} \end{cases}$

Next we consider the twin domination number of Cartesian product $C_4\square C_n$, and define a set $S_2$ (see Figure 2) as follows: $S_2$ consists of vertices $(0,i)$, $(3,i)$, $i \equiv 0 \pmod 8$; $(1,i)$, $i \equiv 1 \pmod 8$; $(2,i)$, $(3,i)$, $i \equiv 2 \pmod 8$; $(0,i)$, $i \equiv 3 \pmod 8$; $(1,i)$, $(2,i)$, $i \equiv 4 \pmod 8$; $(3,i)$, $i \equiv 5 \pmod 8$; $(0,i)$, $(1,i)$, $i \equiv 6 \pmod 8$; $(2,i)$, $i \equiv 7 \pmod 8$. Note that $|S_2| = n + \lceil \frac{n}{2} \rceil$.

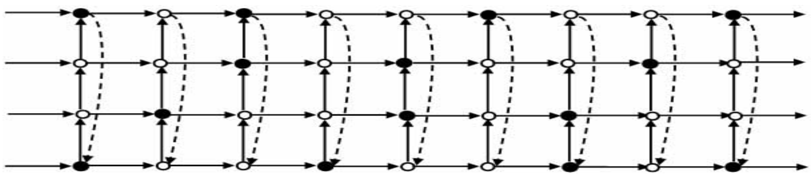

Figure 2 The set $S_2$

**Theorem 2.7** *Let $n \geq 2$. Then* $\gamma^*(C_4\square C_n) = \begin{cases} n + \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0, 3, 5 \pmod 8; \\ n + \lceil \frac{n}{2} \rceil + 1, & \text{otherwise.} \end{cases}$

**Proof** We consider three cases.

**Case 1** $n \equiv 0, 3, 5 \pmod 8$.

By Lemma 2.6, $\gamma^*(C_4\square C_n) \geq \gamma(C_4\square C_n) \geq n + \lceil \frac{n}{2} \rceil$. The set $S_2$ defined above is a twin dominating set. Thus $\gamma^*(C_4\square C_n) = n + \lceil \frac{n}{2} \rceil$, when $n \equiv 0, 3, 5 \pmod 8$.

**Case 2** $n \equiv 2, 4, 6 \pmod 8$.

By Lemma 2.6, $\gamma^*(C_4\square C_n) \geq \gamma(C_4\square C_n) \geq n + \lceil \frac{n+1}{2} \rceil$. If $n \equiv 2, 4, 6 \pmod 8$, then $n + \lceil \frac{n+1}{2} \rceil = n + \lceil \frac{n}{2} \rceil + 1$, so $\gamma^*(C_4\square C_n) \geq n + \lceil \frac{n}{2} \rceil + 1$. It is easy to see that $S_2 \cup \{(2, n-1)\}$ is a minimum twin dominating set of $C_4\square C_n$ and $|S_2 \cup \{(2, n-1)\}| = n + \lceil \frac{n}{2} \rceil + 1$. Thus $\gamma^*(C_4\square C_n) = n + \lceil \frac{n}{2} \rceil + 1$, when $n \equiv 2, 4, 6 \pmod 8$.

**Case 3** $n \equiv 1,\ 7 \pmod 8$.

By Lemma 2.6, $\gamma^*(C_4\square C_n) \geq \gamma(C_4\square C_n) \geq n + \lceil \frac{n+1}{2} \rceil$. Suppose that $\gamma^*(C_4\square C_n) = n + \lceil \frac{n}{2} \rceil = n + \frac{n+1}{2}$. Let $S$ be a minimum twin dominating set of $C_4\square C_n$ and $a_i = |S \cap C_4^i|$ for all $i \in \{0, 1, \ldots, n-1\}$, $|S| = n + \frac{n+1}{2}$.

Let $J$ be the set of $i \in \{0, 1, \ldots, n-1\}$ such that $a_i \leq 1$ ($|J| \leq \frac{n-1}{2}$). Let $J' = \{i | i +$

$1 \pmod n) \in J\}$. If $i \in J$, by Lemma 2.5, $a_{i-1}+2a_i \geq 4$, and thus $a_{i-1}+a_i \geq 3$. Then $J \cap J' = \emptyset$ and $\Sigma_{i\in J\cup J'} a_i \geq 3|J|$. Therefore, $\Sigma_{i\in\{0,1,\ldots,n-1\}} a_i \geq 3|J| + 2(n-2|J|)$. Next we consider the following two subcases.

**Subcase 3.1** $a_i = 0$.

In order to in-dominate and out-dominate the vertices of $C_4^i$, let $a_{i-1} = 4$ and $a_{i+1} = 4$. So $\Sigma_{i\in\{0,1,\ldots,n-1\}} a_i \geq 3(|J|-1)+4+0+4+2(n-1-2|J|) = 2n-|J|+3 = \frac{3n+7}{2} > \frac{3n+1}{2}$, a contradiction.

**Subcase 3.2** $a_i = 4$.

If $i-1 \in J$, then $a_{i-1} = 0$ or 1. If $a_{i-1} = 0$, we can obtain $\Sigma_{i\in\{0,1,\ldots,n-1\}} a_i \geq \frac{3n+7}{2}$ by the same argument as that of Subcase 3.1. If $a_{i-1} = 1$, then $\Sigma_{i\in\{0,1,\ldots,n-1\}} a_i \geq 3(|J|-1)+2+1+4+2(n-1-2|J|) = 2n-|J|+2 = \frac{3n+5}{2} > \frac{3n+1}{2}$, a contradiction.

If $i-1 \notin J$, $\Sigma_{i\in\{0,1,\ldots,n-1\}} a_i \geq 3|J|+4+2(n-1-2|J|) = 2n-|J|+2 = \frac{3n+5}{2} > \frac{3n+1}{2}$, a contradiction.

From above, we can obtain $1 \leq a_i \leq 3$, $i \in \{0,1,\ldots,n-1\}$. Since $J \cap J' = \emptyset$, there do not exist two consecutive integers $t$ and $t+1$, modulo $n$, such that $a_t = a_{t+1} = 1$. Therefore, $S$ can only be of the form $(a_0, a_1, \ldots, a_{n-1}) = (2,1,2,\ldots,2)$.

Next we will prove that $(|S\cap C_4^0|, \ldots, |S\cap C_4^{n-1}|)$ does not exist in the form $(a_0, a_1, \ldots, a_{n-1}) = (2,1,2,\ldots,2)$ when $|S| = n + \frac{n+1}{2}$. Without loss of generality, suppose that $|C_4^0 \cap S| = 2$ and $(0,0) \in S$. We claim that $(1,0) \in S$ or $(3,0) \in S$, otherwise, if $(2,0) \in S$, in order to in-dominate (1,1) and (3,1), $|C_4^1 \cap S| = 2$, a contradiction. In fact, the proofs of cases that $(1,0) \in S$ or $(3,0) \in S$ are exactly the same, considered modulo $n$. Hence we consider only the case that $(3,0) \in S$. We have $(1,1) \in S$ and consequently $(2,2),(3,2) \in S$, otherwise, $|C_4^3 \cap S| = 2$. Thus $(0,3) \in S$ and $(1,4),(2,4) \in S$; $(3,5) \in S$; and $(0,6),(1,6) \in S$; $\cdots$. We conclude that $S \supseteq S_2$. Note that $S_2$ is not a twin dominating set of $C_4\Box C_n$, when $n \equiv 1,7 \pmod 8$. Thus $\gamma^*(C_4\Box C_n) > n+\lceil\frac{n}{2}\rceil$, a contradiction. So $\gamma^*(C_4\Box C_n) \geq n+\lceil\frac{n}{2}\rceil+1$. While $S_3\cup\{(2,n-1)\}$ is a twin dominating set of $C_4\Box C_n$ and $|S_3\cup\{(2,n-1)\}| = n+\lceil\frac{n}{2}\rceil+1$. Thus $\gamma^*(C_4\Box C_n) = n+\lceil\frac{n}{2}\rceil+1$, when $n \equiv 1,7 \pmod 8$. $\Box$

**Lemma 2.8** ([8]) *Let $n \geq 2$. Then $\gamma(C_5\Box C_n) = 2n$.*

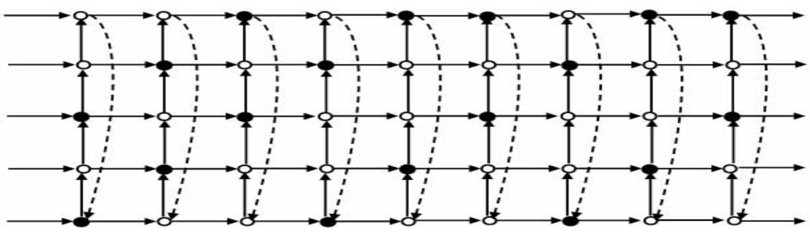

Figure 3 The set $S_3$

We consider the twin domination number of Cartesian product $C_5\Box C_n$, and define two sets $S_3$ (see Figure 3) and $S_4$ (see Figure 4) as follows: $S_3$ consists of vertices $(0,0),(2,0),(1,1),(3,1),(2,2),(4,2)$; $(0,i),(3,i)$, $i \equiv 0 \pmod 3$; $(1,i),(4,i)$, $i \equiv 1 \pmod 3$; $(2,i),(4,i)$, $i \equiv 2 \pmod 3$, when $i \geq 3$. By Theorem 2.8, $\gamma^*(C_5 \Box C_n) \geq \gamma(C_5 \Box C_n) = 2n$. If $n \equiv 0, 2 \pmod 3$, it is clear that $S_3$ is a twin dominating set of $C_5 \Box C_n$ and $|S_3| = 2n$. $S_4$ consists of vertices $(0,0), (2,0), (0,1), (3,1), (1,2), (3,2)$; $(1,i), (4,i)$, $i \equiv 0 \pmod 3$; $(0,i), (2,i)$, $i \equiv 1 \pmod 3$; $(0,i), (3,i)$, $i \equiv 2 \pmod 3$, when $i \geq 3$.

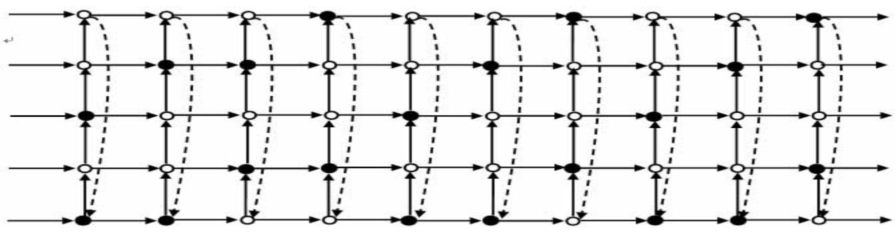

Figure 4 The set $S_4$

Similarly, $S_4$ is a twin dominating set of $C_5 \Box C_n$, when $n \equiv 1 \pmod 3$, and $|S_4| = 2n$. Therefore, we can obtain the following Theorem.

**Theorem 2.9** *Let $n \geq 2$. Then $\gamma^*(C_5 \Box C_n) = 2n$.*

**Lemma 2.10** ([8]) *Let $n \geq 3$. Then* $\gamma(C_6 \Box C_n) = \begin{cases} 2n, & \text{if } n \equiv 0 \pmod 3; \\ 2n+2, & \text{otherwise.} \end{cases}$

Finally we consider the twin domination number of Cartesian product $C_6 \Box C_n$, and define a set $S_5$ (see Figure 5) as follows: $S_5$ consists of vertices $(0,i), (3,i)$, $i \equiv 0 \pmod 3$; $(1,i), (4,i)$, $i \equiv 1 \pmod 3$; $(2,i), (5,i)$, $i \equiv 2 \pmod 3$. Note that $|S_5| = 2n$.

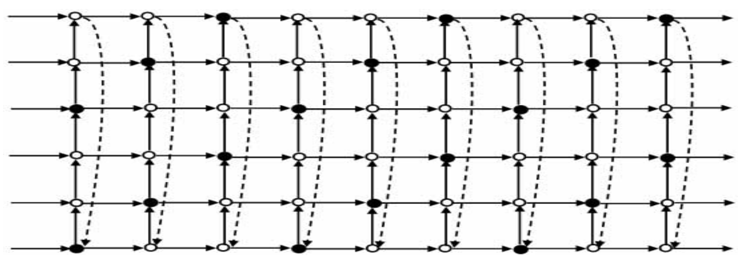

Figure 5 The set $S_5$

**Lemma 2.11** *Let $n \geq 3$. Then* $\gamma^*(C_6 \Box C_n) = \begin{cases} 2n, & \text{if } n \equiv 0 \pmod 3; \\ 2n+2, & \text{otherwise.} \end{cases}$

**Proof** By Lemma 2.10, if $n \equiv 0 \pmod 3$, then $\gamma^*(C_6 \Box C_n) \geq \gamma(C_6 \Box C_n) = 2n$. $S_5$ is a twin dominating set of $C_6 \Box C_n$, when $n \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $\gamma^*(C_3 \Box C_n) \geq \gamma(C_3 \Box C_n) = 2n+2$. Note that $S_5 \cup \{(1,0), (4,0)\}$ is a twin dominating set of $C_6 \Box C_n$, when $n \equiv 1 \pmod 3$. Similarly, $S_5 \cup \{(2,0), (5,0)\}$ is a twin dominating set of $C_6 \Box C_n$, when $n \equiv 2 \pmod 3$. Thus $\gamma^*(C_3 \Box C_n) = 2n+2$, when $n \equiv 1, 2 \pmod 3$. □

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