

Trace Formulae for the Nonlinearization of Periodic Finite-Bands Dirac Spectral Problem

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Abstract This paper deals with a Dirac operator with periodic and finite-bands potentials. Taking advantage of the commutativity of the monodromy operator and the Dirac operator, we define the Bloch functions and multiplier curve, which leads to the formula of Dubrovin-Novikov's type. Further, by calculation of residues on the complex sphere and via gauge transformation, we get the trace formulae of eigenfunctions corresponding to the left end-points and right end-points of the spectral bands, respectively. As an application, we obtain a completely integrable Hamiltonian system in Liouville sense through nonlinearization of the Dirac spectral problem.

Keywords trace formulae; periodic N -bands Dirac operator; nonlinearization; integrable Hamiltonian system

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1. Introduction

The nonlinearization of linear spectral problems is a powerful weapon to get completely integrable Hamiltonian systems in Liouville sense [1,2]. Further, the compatible condition of the spectral problems always gives a soliton equation, i.e., the compatible spectral problems are an integrable decomposition of the corresponding soliton equation, whose quasi-periodic solutions may be found out subsequently [3–5]. However, the constraints that are the foundations of the nonlinearization technique are usually given formally from the Lenard gradients of the spectral problems. Fortunately, trace formulae of the eigenfunctions of the spectral problem may provide sound constraints that are necessary in nonlinearization [6,7].

Hill's equation is famous for its application in the research of moon performed by a U.S. mathematician Hill. Hill's operator possesses bands spectra where all end-points are eigenvalues under periodic boundary conditions. In case that all spectral bands tending to infinity merge into one spectrum while the remaining N spectral bands still separate with each other, the corresponding potential is called periodic N -bands potential [8]. The eigenfunctions corresponding to the left end-points and the right end-points of spectral bands satisfy the famous McKean-Trubowitz's and Cao's identities [7], respectively. Both of them play dramatic roles in theory

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of integrable systems, yield the Neumann constraint of the Neumann system and the Bargmann constraint of the restricted KdV system [6,7,9], respectively.

Several trace formulae of Dirac operator have already been known [10–13], though almost all of them are trace formulae for eigenvalues, helpless for the nonlinearization of the spectral problem. Due to the known facts, Dirac operator and Schrödinger operator share many common properties [14,15]. It hints that the periodic finite-bands Dirac operator must possess parallel results with Hill's operator.

In this paper, we deal with the periodic Dirac operator $L = B\partial_x + Q(x)$, where

$$\partial_x = \frac{d}{dx}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} u(x) & v(x) \\ v(x) & -u(x) \end{pmatrix},$$

with $u(x)$ and $v(x)$ being real smooth periodic functions for period T . It possesses spectral bands $\bigcup_{j=-\infty}^{\infty} [\lambda_{2j}, \lambda_{2j+1}]$, where the band-ends λ_j 's are eigenfunctions under periodic conditions of period $2T$. Suppose that the right spectral bands merge into one band $(\lambda_{2N_+}, +\infty)$ as $j > N_+$, and the left spectral bands change into one band $(-\infty, \lambda_{-2N_- - 1})$ as $j < N_-$, while the remaining $N = N_+ + N_-$ spectral bands still separate each other as $-N_- < j < N_+$. For this case, we say $Q(x)$ is a periodic N -bands potential [14,15]. In Section 2, we define the monodromy operator M , commutative with the Dirac operator L , whose common eigenfunctions are called Bloch functions, and the multiplicative curve \mathcal{C} . In Section 3, we show the formula of Duborovin-Novikov's type which illustrates the inherent relation between the Bloch functions and periodic N -bands potential. In Section 4, we derive trace formulae of eigenfunctions by calculation of residues on the complex sphere. Finally, we gain an integrable Hamiltonian system through nonlinearization of the periodic finite-bands Dirac spectral problem.

2. The monodromy operator and Bloch functions

Define a translation (monodromy) operator \hat{T} as

$$\begin{aligned} \hat{T} : C^\infty(\mathbb{R}, \mathbb{C}^2) &\rightarrow C^\infty(\mathbb{R}, \mathbb{C}^2) \\ (f_1(x), f_2(x))^T &\mapsto (f_1(x+T), f_2(x+T))^T, \end{aligned}$$

where $C^\infty(\mathbb{R}, \mathbb{C}^2)$ is the class of 2-dimensional complex vector-valued smooth functions on real axis. It is obvious that \hat{T} and L are commutative. Hence, the annihilator subspace of $L - \lambda$, denoted as $D_\lambda = \ker(L - \lambda)$, is invariant under the action of \hat{T} for any complex λ .

Choose a real number x_0 as the reference point arbitrarily. Let

$$\theta(x, x_0, \lambda) = (\theta_1(x, x_0, \lambda), \theta_2(x, x_0, \lambda))^T \text{ and } \varphi(x, x_0, \lambda) = (\varphi_1(x, x_0, \lambda), \varphi_2(x, x_0, \lambda))^T$$

be the solutions of the following two initial value problems, respectively:

$$\begin{aligned} L\theta &= \lambda\theta, & \theta_1(x_0) &= 1, & \theta_2(x_0) &= 0; \\ L\varphi &= \lambda\varphi, & \varphi_1(x_0) &= 1, & \varphi_2(x_0) &= 0. \end{aligned}$$

The matrix of \hat{T} on D_λ with bases θ, φ reads

$$\begin{pmatrix} \theta_1 & \varphi_1 \\ \theta_2 & \varphi_2 \end{pmatrix}_{x_0+T} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

The eigenpolynomial $\det(\mu - \hat{T})$ of \hat{T} is a constant independent of the choice of bases. Hence, it is also independent of the choice of the reference point x_0 . Noting that

$$\det \hat{T} = \begin{vmatrix} \theta_1 & \varphi_1 \\ \theta_2 & \varphi_2 \end{vmatrix}_{x_0+T} = \begin{vmatrix} \theta_1 & \varphi_1 \\ \theta_2 & \varphi_2 \end{vmatrix}_{x_0} = 1,$$

$$\text{tr} \hat{T} = \theta_1(x_0 + T) + \varphi_2(x_0 + T) = F(\lambda),$$

we have

$$\det(\mu - \hat{T}) = \mu^2 - F(\lambda)\mu + 1 = 0. \quad (2.1)$$

The roots μ_+, μ_- of Eq.(2.1) are the eigenvalues of M , known as Floquet multiplier. For $F^2(\lambda) \neq 4$, $\mu_+ \neq \mu_-$. The corresponding eigenfunctions $\psi_+(x, \lambda), \psi_-(x, \lambda)$, called Block-Floquet solutions, are common eigenfunctions of L, \hat{T} :

$$L\psi_\pm = \lambda\psi_\pm, \quad \hat{T}\psi_\pm = \psi_\pm(x + T) = \mu_\pm\psi_\pm(x). \quad (2.2)$$

Pairs $(\mu_+, \lambda), (\mu_-, \lambda)$ that satisfy Eq.(2.2) determine a complex curve on $(\mu, \lambda) \in \mathbb{C}^2$. Its analytic continuation \mathcal{C} in $P^2\mathbb{C}$ is an analytic variety, called multiplier curve, a double leaves Riemann surface.

Set $\psi_1(x_0) = 1$, thus Bloch-Floquet solutions are determined uniquely. It can be expressed as

$$\psi_\pm(x, x_0, \lambda) = \theta(x, x_0, \lambda) + m_\pm\varphi(x, x_0, \lambda),$$

where m_\pm satisfy the following equation:

$$\begin{pmatrix} \mu_\pm - \alpha_{11} & -\alpha_{12} \\ -\alpha_{21} & \mu_\pm - \alpha_{22} \end{pmatrix} \begin{pmatrix} 1 \\ m_\pm \end{pmatrix} = 0.$$

Thus,

$$m_\pm(x_0, \lambda) = \frac{\mu_\pm - \alpha_{11}}{\alpha_{12}},$$

where $\alpha_{12} = \varphi_1(x_0 + T, x_0, \lambda)$, whose zeros $\lambda = r_j(x_0)$, $j = 0, \pm 1, \pm 2, \dots$, are eigenvalues of an ordinary Dirac boundary value problem that constitutes an auxiliary spectral problem:

$$Ly = \lambda y, \quad y_1(x_0) = y_1(x_0 + T) = 0. \quad (2.3)$$

These zero points are obviously dependent on the choice of the reference point x_0 .

Define meromorphic functions on the multiplier curve \mathcal{C} (remove the infinite point):

$$m(x_0, \mathbf{p}) = \frac{\mu - \theta_1(x_0 + T, x_0, \lambda)}{\varphi_1(x_0 + T, x_0, \lambda)},$$

$$\psi(x, x_0, \mathbf{p}) = \theta(x, x_0, \lambda) + m(x_0, \mathbf{p})\varphi(x, x_0, \lambda),$$

with $\mathbf{p} = (\mu, \lambda)$. The values of ψ at points $\mathbf{p}_+ = (\mu_+, \lambda), \mathbf{p}_- = (\mu_-, \lambda)$ are ψ_+, ψ_- , respectively.

The zeros of $F^2(\lambda) - 4$ are real, independent of x_0 , and can be numbered in increasing order as

$$\cdots \leq \lambda_{-2} < \lambda_{-1} \leq \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \cdots .$$

For $j = 0, \pm 1, \pm 2, \dots$, intervals $(\lambda_{2j-1}, \lambda_{2j})$ are unstable bands where $|F(\lambda)| > 2$; intervals $(\lambda_{2j}, \lambda_{2j+1})$ are stable bands where $|F(\lambda)| < 2$; μ_+ and μ_- are complex conjugate with each other; $F(\lambda) = 2, \mu = 1$ as $\lambda_{4j}, \lambda_{4j-1}$, which are the eigenvalues of periodic boundary condition $y(x+T) = y(x)$; $F(\lambda) = -2, \mu = -1$ as $\lambda_{4j+1}, \lambda_{4j+2}$, which are the eigenvalues of semi-periodic boundary condition $y(x+T) = -y(x)$ (see [14,15]).

3. Formula of Dubrovin-Novikov's type

Taking advantage of the Bloch functions, we define a new meromorphic function χ on the multiplier curve \mathcal{C} . The values of χ at $\mathfrak{p}_\pm = (\mu_\pm, \lambda)^T$ reads

$$i\chi_\pm = \frac{\psi_{2\pm}}{\psi_{1\pm}},$$

which are independent of x_0 , since the role of x_0 is only to adjust the coefficients of ψ_+ and ψ_- . And, χ_\pm satisfy Riccati equations:

$$i\chi'_\pm = \lambda - u - (\lambda + \mu)\chi_\pm^2 - 2iv\chi_\pm. \tag{3.1}$$

Proposition 3.1 *Let λ be a real number belonging to the stable intervals and $\chi_+ = \chi_R + i\chi_I$. Then the following equalities hold:*

$$\bar{\psi}_+(x, x_0, \lambda) = \psi_-(x, x_0, \lambda), \tag{3.2}$$

$$\chi_- = -\chi_R + i\chi_I, \tag{3.3}$$

$$\chi'_R = -2(\lambda + \mu)\chi_R\chi_I - 2v\chi_R, \tag{3.4}$$

$$\chi'_I = u - \lambda + (\lambda + \mu)(\chi_R^2 - \chi_I^2) - 2v\chi_I. \tag{3.5}$$

Proof Noting that $\mu_- = \bar{\mu}_+$, take complex conjugate of $L\psi_+ = \lambda\psi_+, \psi_+(x+T) = \mu_+\psi_+$, we get Eq. (3.2) by the uniqueness of the solution. It is easy to get Eq. (3.3) from the definition and Eq. (3.2). Substituting Eq. (3.3) into the Riccati equations Eq. (3.1) and separating the real and imaginary parts gives Eqs. (3.4) and (3.5). \square

Corollary 3.2 *Let λ be a real number belonging to the stable intervals. Then there holds that*

$$\chi_R(x_0, \lambda) = \frac{\sqrt{4 - F^2(\lambda)}}{2\alpha_{12}}, \quad \chi_I(x_0, \lambda) = \frac{\alpha_{11} - \alpha_{12}}{2\alpha_{12}}.$$

Proof For $|F(\lambda)| < 2$, we have

$$\mu_+ = \frac{\alpha_{11} + \alpha_{22} + i\sqrt{4 - F^2(\lambda)}}{2}. \tag{3.6}$$

Noting that $\psi_{1+}(x_0) = 1$, we get

$$i\chi_+(x_0) = \psi_{2+}(x_0) = m_+ = \frac{\mu_+ - \alpha_{11}}{\alpha_{12}}. \tag{3.7}$$

Substituting Eq. (3.6) into Eq. (3.7) and separating the real and imaginary parts, we get what was supposed immediately.

Proposition 3.3 *Let λ be a real number belonging to the stable intervals. Then we have*

$$\psi_{1+}(x, x_0, \lambda)\psi_{1-}(x, x_0, \lambda) = \frac{\chi_R(x_0)}{\chi_R(x)} = \frac{\varphi_1(x+T, x, \lambda)}{\varphi_1(x_0+T, x_0, \lambda)}.$$

Proof Define

$$W(x, x_0, \lambda) = \begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix} = \frac{1}{m_- - m_+} \begin{pmatrix} \psi_{1+}\psi_{1-} & \frac{\psi_{1+}\psi_{2-} + \psi_{2+}\psi_{1-}}{2} \\ \frac{\psi_{1+}\psi_{2-} + \psi_{2+}\psi_{1-}}{2} & \psi_{2+}\psi_{2-} \end{pmatrix}.$$

Noting that $\det(\psi_+, \psi_-) = m_- - m_+$, which is independent of x , we have

$$i\chi_+ = \frac{\psi_{2+}}{\psi_{1+}} = \frac{2\psi_{1-}\psi_{2+}}{2\psi_{1-}\psi_{1+}} = \frac{2w_3 + \frac{\psi_{1-}\psi_{2+} - \psi_{1+}\psi_{2-}}{m_- - m_+}}{2w_1} = \frac{2w_3 - 1}{2w_1}, \quad (3.8)$$

$$i\chi_- = \frac{\psi_{2-}}{\psi_{1-}} = \frac{2\psi_{1+}\psi_{2-}}{2\psi_{1-}\psi_{1+}} = \frac{2w_3 - \frac{\psi_{1-}\psi_{2+} - \psi_{1+}\psi_{2-}}{m_- - m_+}}{2w_1} = \frac{2w_3 + 1}{2w_1}. \quad (3.9)$$

Subtracting Eq. (3.9) from Eq. (3.8), we get $2\chi_R w_1 = i$, i.e., $w_1 = \frac{i}{2\chi_R}$. Noting that $\psi_{1+}(x_0)\psi_{1-}(x_0) = 1$, we get

$$\psi_{1+}\psi_{1-} = \frac{w_1(x)}{w_1(x_0)} = \frac{\chi_R(x_0)}{\chi_R(x)}.$$

By Corollary 3.2, we get what was supposed immediately. \square

Let $\chi_R = \sum_{n=0}^{\infty} a_n \lambda^{-n}$, $\chi_I = \sum_{n=0}^{\infty} b_n \lambda^{-n}$, as $|\lambda| \rightarrow \infty$. Substituting them into Eq. (3.4, 3.5), and comparing the coefficients of λ with the same power, we get

$$\chi_R = 1 - u\lambda^{-1} + O(\lambda^{-2}), \quad \chi_I = -v\lambda^{-1} + O(\lambda^{-2}). \quad (3.10)$$

Since φ_1 is analytic on λ , the infinite product expansion with respect to its zeros reads [14,15]

$$\varphi_1(x_0 + T, x_0, \lambda) = \text{const.} \prod_{j=-\infty}^{+\infty} \left(1 - \frac{\lambda}{r_j(x_0)}\right),$$

where r_j 's fall in the unstable bands $[\lambda_{2j-1}, \lambda_{2j}]$. Suppose that $Q(x)$ is the periodic N -bands potential, i.e., $\lambda_{2j-1} = \lambda_{2j}$ as $j \geq N_+ + 1$ or $j \leq N_- - 1$, the unstable bands disappear. Thus, we have

$$r_j(x_0) = \lambda_{2j-1} = \lambda_{2j}, \quad \text{for } j \geq N_+ + 1 \text{ or } j \leq N_- - 1,$$

independent of the choice of x_0 . Hence, $r_j(x) = r_j(x_0)$ for $j \geq N_+ + 1$ or $j \leq N_- - 1$. Further, we obtain

$$\chi_R(x, \lambda) = \frac{\sqrt{\prod_{j=-2N_- - 1}^{2N_+} (\lambda - \lambda_j)}}{\prod_{k=-N_-}^{N_+} (\lambda - r_k(x))}, \quad (3.11)$$

$$\frac{\varphi_1(x+T, x, \lambda)}{\varphi_1(x_0+T, x_0, \lambda)} = \text{const.} \frac{\prod_{j=-N_-}^{N_+} (\lambda - r_j(x))}{\prod_{k=-N_-}^{N_+} (\lambda - r_k(x_0))}. \quad (3.12)$$

Comparing the coefficients of λ with the same power of χ_R in Eqs. (3.10) and (3.11), we get

$$u(x) = \frac{1}{2} \sum_{j=-2N_- - 1}^{2N_+} \lambda_j - \sum_{k=-N_-}^{N_+} r_k(x). \tag{3.13}$$

It is easy to see that $\text{const.} = 1$ in Eq. (3.12) from the asymptotic expansion of φ_1 as $|\lambda| \rightarrow \infty$. Since the stable bands are sets of accumulation points, it is easy to see that Eq. (3.12) is valid for all $\lambda \in \mathbb{C}$ by analytic continuation. Hence, we obtain

Proposition 3.4 (Formula of Dubrovin-Novikov’s type) *Let $Q(x)$ be the periodic N -bands potential of the Dirac equation. Then there holds*

$$\psi_{1+}(x, x_0, \lambda)\psi_{1-}(x, x_0, \lambda) = \frac{P_N(x, \lambda)}{P_N(x_0, \lambda)},$$

with $P_N(x, \lambda) = \prod_{k=-N_-}^{N_+} (\lambda - r_k(x))$.

4. Trace formulae

Lemma 4.1 *Let ω be a meromorphic differential on the complex sphere $\mathcal{S} = \mathbb{C} \cup \{\infty\}$:*

$$\omega = \frac{\prod_{j=1}^{\beta} (\lambda - a_j)}{\prod_{j=1}^{\beta} (\lambda - b_j)} d\lambda,$$

where β is a nonnegative integer, a_j, b_j are complex constants. Then the residue of ω at infinite point reads

$$\text{Res}_{\lambda=\infty} \omega = \sum_{j=1}^{\beta} (a_j - b_j).$$

Proof Take local coordinate $\zeta = \lambda^{-1}$ in the neighborhood of infinite point ∞ , we may verify the lemma by direct calculation. \square

Since ψ_+, ψ_- are values of Bloch function ψ on the multiplier \mathcal{C} according to $\mathbf{p}_+ = (\mu_+, \lambda)$, $\mathbf{p}_- = (\mu_-, \lambda)$, respectively; and \mathbf{p}_+ and \mathbf{p}_- may coincide with each other as λ tends to the main spectral point λ_k ; it results that $\psi_+ = \psi_-$ when $\mathbf{p}_+ = \mathbf{p}_-$. According to Dubrovin-Novikov’s formula, it reads

$$y_{1,k}(x) = \psi_{1+}(x, x_0, \lambda_k) = \psi_{1-}(x, x_0, \lambda_k) = \pm \sqrt{\frac{P_N(x, \lambda_k)}{P_N(x_0, \lambda_k)}},$$

where $y_k(x)$ ’s are the eigenfunction with respect to the eigenvalue λ_k ’s of the Dirac operator with periodic or semi-periodic conditions $y_k(x + T) = \pm y_k(x)$, taking “+” for $k = 4j - 1, 4j$, while taking “-” for $k = 4j + 1, 4j + 2$.

Proposition 4.2 *The eigenfunctions $(\psi_{1,2k}, \psi_{2,2k})^T$ corresponding to the left end-points λ_{2k} of the periodic N -bands Dirac equation, by some gauge, satisfy that*

$$\sum_{k=-N_-}^{N_+} \psi_{1,2k}^2 - \frac{1}{2}\sigma = u(x),$$

where σ is the summation of the band gaps:

$$\sigma = - \sum_{j=-N_-}^{N_+} (\lambda_{2j-1} - \lambda_{2j}). \quad (4.2)$$

Proof Consider the meromorphic differential ω_1 on complex sphere \mathcal{S} :

$$\omega_1 = \frac{\prod_{j=-N_-}^{N_+} (\lambda - r_j(x))}{\prod_{k=-N_-}^{N_+} (\lambda - \lambda_{2k}(x))} d\lambda = \frac{P_N(x_0, \lambda)}{\prod_{k=-N_-}^{N_+} (\lambda - \lambda_{2k}(x))} \psi_{1+}(x, x_0, \lambda) \psi_{1-}(x, x_0, \lambda) d\lambda.$$

Since the Riemann surface \mathcal{S} is compact, by the residue theorem we have

$$\sum_{k=-N_-}^{N_+} \operatorname{Res}_{\lambda=\lambda_{2k}} \omega_1 + \operatorname{Res}_{\lambda=\infty} \omega_1 = 0,$$

i.e.,

$$\sum_{k=-N_-}^{N_+} \rho_{2k} y_{1,2k}^2(x) + \operatorname{Res}_{\lambda=\infty} \omega_1 = 0, \quad (4.2)$$

with

$$\rho_{2k} = \frac{P_N(x_0, \lambda_{2k})}{\prod_{j \neq k, j=-N_-}^{N_+} (\lambda_{2k} - \lambda_{2j})}.$$

For $r_j \in (\lambda_{2j-1}, \lambda_{2j})$, the symbol of the numerator is $(-1)^{N_+-k}$, while the symbol of the denominator is also $(-1)^{N_+-k}$. Hence, ρ_{2k} is always positive. Thus the real eigenfunctions can be gauged as $\psi_{2k}(x) = \sqrt{\rho_{2k}} y_{2k}(x)$. Further, by Lemma 4.1 we have

$$\begin{aligned} \operatorname{Res}_{\infty} \omega_1 &= \sum_{j=-N_-}^{N_+} (r_j(x) - \lambda_{2j}) \\ &= - \left(\frac{1}{2} \sum_{j=-2N_- - 1}^{2N_+} \lambda_j - \sum_{j=-N_-}^{N_+} r_j(x) \right) + \frac{1}{2} \sum_{j=-N_-}^{N_+} (\lambda_{2j-1} - \lambda_{2j}) \\ &= -u(x) - \frac{1}{2} \sigma. \end{aligned} \quad (4.3)$$

Substituting Eq. (4.3) into Eq. (4.2), we get what is expected immediately. \square

Proposition 4.3 *The eigenfunctions $(\psi_{1,2k-1}, \psi_{2,2k-1})^T$ corresponding to the right end-points λ_{2k-1} of the periodic N -bands Dirac equation, by some gauge, satisfy that*

$$\sum_{k=-N_-}^{N_+} \psi_{1,2k-1}^2 + \frac{1}{2} \sigma = u(x),$$

where σ is the summation of the band gaps given by Eq. (4.1).

Proof Consider the meromorphic differential ω_2 on complex sphere \mathcal{S} :

$$\omega_2 = \frac{\prod_{j=-N_-}^{N_+} (\lambda - r_j(x))}{\prod_{k=-N_-}^{N_+} (\lambda - \lambda_{2k-1}(x))} d\lambda = \frac{P_N(x_0, \lambda)}{\prod_{k=-N_-}^{N_+} (\lambda - \lambda_{2k-1}(x))} \psi_{1+}(x, x_0, \lambda) \psi_{1-}(x, x_0, \lambda) d\lambda.$$

Since the Riemann surface \mathcal{S} is compact, by the residue theorem we have

$$\sum_{k=-N_-}^{N_+} \operatorname{Res}_{\lambda=\lambda_{2k-1}} \omega_2 + \operatorname{Res}_{\lambda=\infty} \omega_2 = 0,$$

i.e.,

$$\sum_{k=-N_-}^{N_+} \rho_{2k-1} y_{1,2k-1}^2(x) + \operatorname{Res}_{\lambda=\infty} \omega_2 = 0, \quad (4.4)$$

with

$$\rho_{2k-1} = \frac{P_N(x_0, \lambda_{2k-1})}{\prod_{j \neq k, j=-N_-}^{N_+} (\lambda_{2k-1} - \lambda_{2j-1})}.$$

For $r_j \in (\lambda_{2j-1}, \lambda_{2j})$, the symbol of the numerator is $(-1)^{N_+ - k + 1}$, while the symbol of the denominator is also $(-1)^{N_+ - k + 1}$. Hence, ρ_{2k-1} is always positive. Thus the real eigenfunctions can be gauged as $\psi_{2k-1}(x) = \sqrt{\rho_{2k-1}} y_{2k-1}(x)$. Further, by Lemma 4.1 we have

$$\begin{aligned} \operatorname{Res}_{\infty} \omega_2 &= \sum_{j=-N_-}^{N_+} (r_j(x) - \lambda_{2j-1}) \\ &= -\left(\frac{1}{2} \sum_{j=-2N_- - 1}^{2N_+} \lambda_j - \sum_{j=-N_-}^{N_+} r_j(x)\right) - \frac{1}{2} \sum_{j=-N_-}^{N_+} (\lambda_{2j-1} - \lambda_{2j}) \\ &= -u(x) + \frac{1}{2} \sigma. \end{aligned} \quad (4.5)$$

Substituting Eq. (4.5) into Eq. (4.4), we get what is expected immediately. \square

Proposition 4.4 *The eigenfunctions $(\psi_{1,2k}, \psi_{2,2k})^T$ corresponding to the left end-points λ_{2k} of the periodic N -bands Dirac equation, by some gauge, satisfy the following identities:*

$$-\frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k} - \psi_{2,2k})^2 + \frac{1}{2} \sigma = v(x), \quad (4.6)$$

$$\frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k} + \psi_{2,2k})^2 - \frac{1}{2} \sigma = v(x), \quad (4.7)$$

$$-\sum_{k=-N_-}^{N_+} \psi_{2,2k}^2 + \frac{1}{2} \sigma = u(x), \quad (4.8)$$

where σ is given by Eq. (4.1).

Proof Define a linear transformation

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (4.9)$$

For $\phi = -\frac{\pi}{4}$, the Dirac operator $L = B\partial_x + Q(x)$ is transformed to the following form:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -v(x) & u(x) \\ u(x) & v(x) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

From Proposition 4.2, we get Eq.(4.6) immediately. Similarly, taking $\phi = \frac{\pi}{4}, \frac{\pi}{2}$ in Eq.(4.9) respectively, we see that Eqs. (4.7) and (4.8) hold due to Proposition 4.2. \square

Proposition 4.5 *The eigenfunctions $(\psi_{1,2k-1}, \psi_{2,2k-1})^T$ corresponding to the right end-points λ_{2k-1} of the periodic N -bands Dirac equation, by some gauge, satisfy the following identities:*

$$-\frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k-1} - \psi_{2,2k-1})^2 - \frac{1}{2}\sigma = v(x), \quad (4.10)$$

$$\frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k-1} + \psi_{2,2k-1})^2 + \frac{1}{2}\sigma = v(x), \quad (4.11)$$

$$-\sum_{k=-N_-}^{N_+} \psi_{2,2k}^2 - \frac{1}{2}\sigma = u(x), \quad (4.12)$$

where σ is given by Eq. (4.1).

Proof Taking $\phi = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ in Eq. (4.9), respectively, we see that Eqs. (4.10)–(4.12) hold due to Proposition 4.3. \square

5. Integrable Hamiltonian systems

Due to the trace formulae gained in Propositions 4.2–4.5, we have expressions for potentials by eigenfunctions (i.e., constraints) as follows:

$$u(x) = \frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k}^2 - \psi_{2,2k}^2) = \frac{1}{2} \sum_{k=-N_-}^{N_+} (\psi_{1,2k-1}^2 - \psi_{2,2k-1}^2),$$

$$v(x) = \sum_{k=-N_-}^{N_+} \psi_{1,2k} \psi_{2,2k} = \sum_{k=-N_-}^{N_+} \psi_{1,2k-1} \psi_{2,2k-1}.$$

This is not similar to the case of Hill's equation whose left and right end-points of the spectral bands lead to two formally extremely different constraints. Introduce canonical variables

$$p = (p_1, p_2, \dots, p_N)^T = (\psi_{1,-2N_-}, \psi_{1,-2N_-+2}, \dots, \psi_{1,2N_+})^T,$$

$$q = (q_1, q_2, \dots, q_N)^T = (\psi_{2,-2N_-}, \psi_{2,-2N_-+2}, \dots, \psi_{2,2N_+})^T,$$

and denote that $\Lambda = \text{diag}(\lambda_{-2N_-}, \lambda_{-2N_-+2}, \dots, \lambda_{2N_+})$. Then the Dirac equations that the eigenfunctions satisfy corresponding to the left end-points of the spectral bands lead to a Hamiltonian system through nonlinearization:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial H}{\partial p}, \quad (5.1)$$

where

$$H = -\frac{1}{8}(\langle p, p \rangle + \langle q, q \rangle)^2 + \frac{1}{2}(\langle p, p \rangle \langle q, q \rangle - \langle p, q \rangle^2) + \frac{1}{2}(\langle \Lambda p, p \rangle) + \langle \Lambda q, q \rangle,$$

with $\langle \cdot, \cdot \rangle$ denoting the usual inner product in \mathbb{R}^N . The Hamiltonian system (5.1) is completely integrable in Liouville sense, i.e., possesses conserved integrals $\{F_1, F_2, \dots, F_N\}$ (see [16]), which

are functionally independent and involute with each other:

$$F_k(p, q) = p_k^2 + q_k^2 + \sum_{j=1, j \neq k}^N \frac{(p_x q_j - p_j q_k)^2}{\lambda_j - \lambda_k}, \quad k = 1, 2, \dots, N.$$

The expressions for the Hamiltonian system (5.1) by components read

$$\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = -\frac{\partial H}{\partial p_k}, \quad k = 1, 2, \dots, N, \\ H(p, q) &= -\frac{1}{2} \sum_{k=1}^N \lambda_k F_k(p, q) - \frac{1}{8} \left(\sum_{k=1}^N F_k(p, q) \right)^2. \end{aligned}$$

If we construct canonical variables by eigenfunctions corresponding to the right end-points of the spectral bands:

$$\begin{aligned} p &= (p_1, p_2, \dots, p_N)^T = (\psi_{1, -2N_- - 1}, \psi_{1, -2N_- + 1}, \dots, \psi_{1, 2N_+ - 1})^T, \\ q &= (q_1, q_2, \dots, q_N)^T = (\psi_{2, -2N_- - 1}, \psi_{2, -2N_- + 1}, \dots, \psi_{2, 2N_+ - 1})^T, \end{aligned}$$

and set $\Lambda = \text{diag}(\lambda_{-2N_- - 1}, \lambda_{-2N_- + 1}, \dots, \lambda_{2N_+ - 1})$; it may lead to another completely integrable Hamiltonian system formally the same as system (5.1), which is unlike the case of Hill's equation whose left and right end-points of the spectral bands lead to two extremely different systems.

Theorem 5.1 *The periodic N -bands Dirac equation may lead to a completely integrable Hamiltonian system through nonlinearization under constraints deduced from the trace formulae of eigenfunctions corresponding to the left (or right) end-points of the spectral bands.*

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