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n-Clean Rings with Involutions

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Abstract A $*$-ring is called $*$-clean if every element of the ring can be written as the sum of a projection and a unit. For an integer $n \geq 1$, we call a $*$-ring R n-$*$-clean if for any $a \in R$, $a = p + u_1 + \cdots + u_n$ where p is a projection and u_i are units for all i. Basic properties of n-$*$-clean rings are considered, and a number of illustrative examples of 2-$*$-clean rings which are not $*$-clean are provided. In addition, extension properties of n-$*$-clean rings are discussed.

Keywords $*$-clean ring; n-$*$-clean ring; clean ring; n-clean ring

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1. Introduction

Throughout this article, rings are associative with unity. Following Nicholson [1], an element of a ring R is called clean if it is the sum of an idempotent and a unit, and R is called clean if every element of R is clean. Unit regular rings and semiperfect rings are well known examples of clean rings [2]. For a positive integer n, Xiao and Tong [3] introduced the concept of n-clean rings. Recall that $a \in R$ is n-clean if it can be written as the sum of an idempotent and n units, and R is called n-clean if all of its elements are n-clean. Clearly, clean rings coincide with 1-clean rings. Various examples of 2-clean rings but not clean rings were provided in [4,5].

An involution of a ring R is an operation $* : R \to R$ satisfying

$$(x+y)^* = x^* + y^*,\ (xy)^* = y^*x^* \text{ and } (x^*)^* = x \text{ for all } x, y \in R.$$

A ring R with involution $*$ is called a $*$-ring. An element p of a $*$-ring is called a projection if $p^2 = p$ and $p^* = p$ (i.e., p is a self-adjoint idempotent). So 0 and 1 are projections of any $*$-ring. Following Vaš [6], a $*$-ring R is called $*$-clean if every element of R is the sum of a projection and a unit, and R is strongly $*$-clean if every element of R is the sum of a projection and a unit that commute. Clearly, $*$-clean rings are clean and strongly $*$-clean rings are strongly clean (i.e., each element of the ring is the sum of an idempotent and a unit that commute [7]). Strongly $*$-clean rings were studied further in [8,9].

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In this article, we introduce the notion of n-$*$-clean rings which can be regarded as both generalization of $*$-clean rings and n-clean rings. Several examples of n-$*$-clean rings are given, and the relationship among $*$-clean rings, n-$*$-clean rings, clean rings and n-clean rings are discussed. In addition, extension properties of n-$*$-clean rings are studied. For a ring R, the set of all idempotents, all projections and all units of R are denoted by $\mathrm{Id}(R)$, $P(R)$ and $U(R)$, respectively. We write $M_n(R)$ for the ring of all $n\times n$ matrices over R. Let $\mathbb{Z}_{(p)}$ be the localization of the ring of integers $\mathbb{Z}$ at the prime ideal (p) and C_n be a cyclic group of order n. The ring of integers modulo n is denoted by $\mathbb{Z}_n$. For a $*$-ring R, the matrix ring $M_k(R)$ has a natural involution inherited from R : if $A=(a_{ij})\in M_k(R)$, $A^*=(a_{ij}^*)^t$ is the transpose of (a_{ij}^*), and in this way, $M_k(R)$ becomes a $*$-ring.

2. Main results

We first introduce the following concept.

Definition 2.1 *Let n be a positive integer. An element a of a $*$-ring R is called n-$*$-clean if $a=p+u_1+\cdots+u_n$ where $p\in P(R)$ and $u_1,\ldots,u_n\in U(R)$. The $*$-ring R is called n-$*$-clean if every element of R is n-$*$-clean.*

It is clear that $*$-clean rings coincide with 1-$*$-clean rings, and n-$*$-clean rings are n-clean.

Lemma 2.2 *If R is n-$*$-clean, then R is m-$*$-clean for any $m>n$.*

Proof By assumption, it is enough to prove that R is $n+1$-$*$-clean. Let $a=p+u_1+\cdots+u_n$ with $p\in P(R)$ and $u_1,\ldots,u_n\in U(R)$. Take $q=1-p$ and $u_{n+1}=2p-1$. Then $q\in P(R)$ and $u_{n+1}^2=1$. Thus, $a=q+u_1+\cdots+u_n+u_{n+1}$ is $(n+1)$-$*$-clean in R. □

Recall that R is called an (S,n)-ring if every element of R is a sum of no more than n units of R (see [10]).

Corollary 2.3 *(i) Every (S,n)-ring with involution $*$ is n-$*$-clean.*

(ii) If R is an n-$*$-clean ring with the only projections 0 and 1, then R is an $(S,n+1)$-ring.

For a commutative ring R and a group G, the standard involution $*$ of the group ring RG is defined by $(\sum r_g g)^*=\sum r_g g^{-1}$. According to [5, Proposition 2.7] and Proposition 2.4, the $*$-ring $\mathbb{Z}_{(2)}C_3$ is 2-$*$-clean but not an $(S,2)$-ring.

Proposition 2.4 *The group ring $\mathbb{Z}_{(p)}C_3$ is 2-$*$-clean for any prime p.*

Proof Let $C_3=\{1,\ b,\ b^2\}$ with $b^3=1$. If $p\neq 2$, then $\mathbb{Z}_{(p)}C_3$ is an $(S,2)$-ring by [5, Proposition 2.5], and thus 2-$*$-clean by Corollary 2.3. Next we assume that $p=2$.

Firstly, we claim that all idempotents in $\mathbb{Z}_{(2)}C_3$ are projections. Let $e=e_0+e_1b+e_2b^2\in\mathbb{Z}_{(2)}C_3$ and $e^2=e$. Then we have

$$e_1=e_2^2+2e_0e_1 \tag{2.1}$$

$$e_2=e_1^2+2e_0e_2. \tag{2.2}$$

Now, performing (2.1)–(2.2) yields

$$(e_2 - e_1)(e_1 + e_2 - 2e_0 + 1) = 0. \tag{2.3}$$

Since $2 \in J(\mathbb{Z}_{(2)})$, by Eqs. (2.1) and (2.2) one easily gets that either $e_1,\ e_2 \in U(\mathbb{Z}_{(2)})$ or $e_1,\ e_2 \in J(\mathbb{Z}_{(2)})$. Note that $\mathbb{Z}_{(2)}/J(\mathbb{Z}_{(2)}) \cong \mathbb{Z}_2$. Thus $\overline{e_1} = \overline{e_2}$, that is $e_2 - e_1 \in J(\mathbb{Z}_{(2)})$. It follows that $e_1 + e_2 - 2e_0 + 1 = 1 + (e_2 - e_1) + 2(e_1 - e_0) \in U(\mathbb{Z}_{(2)})$. By Eq. (2.3), we obtain that $e_2 = e_1$. So $e^2 = e = e^*$, and the claim follows. In view of [3, Theorem 3.2], $\mathbb{Z}_{(2)}C_3$ is clean. Therefore, $\mathbb{Z}_{(2)}C_3$ is $*$-clean (and hence 2-$*$-clean). □

A $*$-ring R is called $*$-regular [11] if for any $x \in R$, there exists $p \in P(R)$ such that $xR = pR$. Due to [8], a $*$-ring R is $*$-unit regular if it is unit regular and $*$-regular.

Theorem 2.5 *Every clean $*$-regular ring is 2-$*$-clean.*

Proof Let R be a clean $*$-regular ring. Given any $a \in R$. Then $a = e + u$ for some $e \in \mathrm{Id}(R)$ and $u \in U(R)$. We next show that e is $*$-clean in R. Since R is $*$-regular, there exists a projection p such that $(1-e)R = pR$. So we have $1 - e = p(1-e)$ and $p = (1-e)p$, and thus $ep = 0$. Note that

$$(e-p)(e-p) = e - ep - pe + p = e + p(1-e) = e + (1-e) = 1.$$

So $e - p \in U(R)$ and $e = p + (e-p)$ is a $*$-clean expression of e in R. So $a = p + (e-p) + u$ is 2-$*$-clean. Hence, R is 2-$*$-clean. □

According to [12, Theorem 1], unit regular rings are clean. So we have the following result.

Corollary 2.6 *Every $*$-unit regular ring is 2-$*$-clean.*

Example 2.7 (i) Let $R = \mathbb{Z}_{(7)}C_3$. In view of [13, Example 1], R is not clean, and thus not $*$-clean. However, R is 2-$*$-clean by Proposition 2.4 (and thus 2-clean).

(ii) Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Clearly, R is a commutative clean ring (and thus 2-clean). Define a map $* : R \to R$ by $(a,b)^* = (b,a)$. It is easy to check that $*$ is an involution of R. Note that $P(R) = \{(0,0),(1,1)\}$ and $U(R) = \{(1,1)\}$. Take $a = (1,0)$. We conclude that a is not n-$*$-clean. Indeed, if $a = p + u_1 + \cdots + u_n$ with $p \in P(R)$ and $u_i \in U(R)$ for all i, then $a = p^* + u_1^* + \cdots + u_n^* = a^*$. This is a contradiction.

(iii) Let $R = T_2(\mathbb{Z}_2)$ be the 2×2 upper triangular matrix ring, and an involution $*$ of R given by $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} c & b \\ 0 & a \end{smallmatrix}\right)$. By [9, Example 2.6], R is clean. However, R is not n-$*$-clean. Note that $u^* = u$ for every $u \in U(R)$. Given any $a \in R$ with $a \neq a^*$. If $a = p + u_1 + \cdots + u_n$ for some $p \in P(R)$ and $u_i \in U(R)$ for each i, then $a^* = a$. This contradicts $a \neq a^*$.

Vaš [6] asked whether there is an example of a $*$-ring that is clean but not $*$-clean. Example 2.7(ii) gives an affirmative answer [8,9].

Remark 2.8 By virtue of Example 2.7, we have the following implications (for the class of

$*$-rings):

$$\begin{array}{ccc} *\text{-clean ring} & \Longrightarrow & n\text{-}*\text{-clean ring} \\ \Downarrow & & \Downarrow \\ \text{clean ring} & \Longrightarrow & n\text{-clean ring.} \end{array}$$

In this diagram, each of the implications is irreversible, and there are no other implications between these rings.

Proposition 2.9 *Let R be a $*$-ring with $2 \in U(R)$. The following are equivalent:*

(i) R is n-clean and every unit of R is self-adjoint (i.e., $u^* = u$ for every $u \in U(R)$).

(ii) R is n-$*$-clean and $* = 1_R$ is the identity endomorphism of R.

Proof (ii)$\Rightarrow$(i) is trivial.

(i)$\Rightarrow$(ii) Let $a \in R$. Then $a = e + u_1 + \cdots + u_n$ for some $e^2 = e$ and $u_i \in U(R)$ for all i. Since $(1-2e)^2 = 1$, we have $1 - 2e$ is a unit. The hypothesis implies that $(1-2e)^* = 1 - 2e$, and so $2(e^* - e) = 0$. Since the condition $2 \in U(R)$, $e^* = e$. Thus $a \in R$ is n-$*$-clean and $a^* = a$, whence $* = 1_R$. $\square$

Example 2.7(ii) reveals that "$2 \in U(R)$" in Proposition 2.9 can not be removed. Recall that an element t of a $*$-ring R is self-adjoint square root of 1 if $t^2 = 1$ and $t^* = t$.

Proposition 2.10 *Let R be a $*$-ring with $2 \in U(R)$. Then R is n-$*$-clean if and only if every element of R is a sum of n units and a self-adjoint square root of* 1.

Proof Assume that R is n-$*$-clean and $a \in R$. Then $\frac{1+a}{2} = p + u_1 + \cdots + u_n$ for some $p \in P(R)$ and $u_1, \ldots, u_n \in U(R)$. It follows that $a = (2p-1) + 2u_1 + \cdots + 2u_n$ with $(2p-1)^* = 2p-1$, $(2p-1)^2 = 1$ and $2u_i \in U(R)$ for $i = 1, \ldots, n$.

Conversely, given any $a \in R$. Then there exist $y \in R$ and $v_1, \ldots, v_n \in U(R)$ such that $2a - 1 = y + v_1 + \cdots + v_n$ with $y^* = y$, $y^2 = 1$. Thus, $a = \frac{y+1}{2} + \frac{v_1}{2} + \cdots + \frac{v_n}{2}$ is an n-$*$-clean expression since $(\frac{y+1}{2})^* = \frac{y+1}{2}, (\frac{y+1}{2})^2 = \frac{y+1}{2}$ and $\frac{v_i}{2} \in U(R)$ for each i. $\square$

By Proposition 2.10, any n-$*$-clean ring R with $2 \in U(R)$ is an $(S, n+1)$-ring. However, we do not know whether $2 \in U(R)$ is still necessary in Proposition 2.10 when $n \geq 2$.

Let I be an ideal of a $*$-ring R. We call I is $*$-invariant if $I^* \subseteq I$. In this case, the involution $*$ of R can be extended to the factor ring R/I, which is still denoted by $*$.

Lemma 2.11 *Let R be n-$*$-clean. If I is a $*$-invariant ideal of R, then R/I is n-$*$-clean.*

Proof Since the homomorphism image of a projection (resp., unit) is also a projection (resp., unit), the result follows. $\square$

For a ring R, the set of all nilpotent elements of R is denoted by $N(R)$.

Corollary 2.12 *Let R be n-$*$-clean. We have*

(i) $R/J(R)$ is n-$*$-clean.

(ii) if $N(R)$ is an ideal, then $R/N(R)$ is n-$*$-clean.

Proof (i) In view of Lemma 2.11, it suffices to show that $J(R)$ is $*$-invariant. For any $a^* \in (J(R))^*$, we show that $a^* \in J(R)$. Note that $a \in J(R)$. Take any $x \in R$. Then $1 - x^*a \in U(R)$. Thus $1 - a^*x = (1 - x^*a)^*$ is a unit of R, as desired.

(ii) By assumption, one easily checks that $N(R)$ is a $*$-invariant ideal. The rest follows from Lemma 2.11. □

Proposition 2.13 *Let R be a $*$-ring and I a $*$-invariant ideal of R with $I \subseteq J(R)$. If R/I is n-$*$-clean and projections can be lifted modulo I, then R is n-$*$-clean.*

Proof Given any $x \in R$. Write $\bar{x} = x + I \in R/I$. Then $\bar{x} = \bar{p} + \bar{u_1} + \bar{u_2} + \cdots + \bar{u_n}$, where $p^2 - p \in I,\ p^* - p \in I$ and $\bar{u_i} \in U(R/I)$ for $i = 1, \dots, n$. Since projections can be lifted modulo I, we may assume that $p^2 = p \in R$ and $p = p^*$. Also, units can be lifted modulo I (since for any $\bar{u},\ \bar{v} \in R/I$ with $\bar{u}\bar{v} = \bar{v}\bar{u} = \bar{1}$, one has $uv - 1 \in I \subseteq J(R)$ and $vu - 1 \in I \subseteq J(R)$. Thus, both u and v are invertible in R). So we can assume that u_i are all units of R for $i = 1, \dots, n$. Therefore,

$$x = p + u_1 + u_2 + \cdots + u_n.$$

This proves that x is n-$*$-clean, and hence R is an n-$*$-clean ring. □

For a $*$-ring R. Then $*$ induces an involution of the polynomial ring $R[x]$ (resp., power series ring $R[[x]]$), denoted by $*$, where $(\sum_{i=0}^{m} a_i x^i)^* = \sum_{i=0}^{m} a_i^* x^i$ (resp., $(\sum_{i=0}^{\infty} a_i x^i)^* = \sum_{i=0}^{\infty} a_i^* x^i$).

Proposition 2.14 *Let R be a $*$-ring. Then $R[[x]]$ is n-$*$-clean if and only if R is so.*

Proof Suppose that $R[[x]]$ is n-$*$-clean. Note that $R \cong R[[x]]/(x)$ and (x) is a $*$-invariant ideal of $R[[x]]$. By Lemma 2.11, R is n-$*$-clean. Conversely, assume that R is n-$*$-clean. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Write $a_0 = p + u_1 + u_2 + \cdots + u_n$ with $p \in P(R)$ and $u_i \in U(R)$ for each i. Then $f(x) = p + (u_1 + \sum_{i=1}^{\infty} a_i x^i) + u_2 + \cdots + u_n$, where $p \in P(R) \subseteq P(R[[x]])$, $u_1 + \sum_{i=1}^{\infty} a_i x^i \in U(R[[x]])$ and $u_i \in U(R[[x]])$. Hence $f(x)$ is n-$*$-clean in $R[[x]]$. □

Recall that a ring R is *semicommutative* if $ab = 0$ implies that $aRb = 0$ for any $a, b \in R$. Semicommutative rings are abelian (i.e., all idempotents of the ring are central).

Remark 2.15 (i) In view of [5, Theorem 3.8], if R is a semicommutative $*$-ring, then $R[x]$ is not n-$*$-clean (as it is not n-clean).

(ii) For any $*$-ring R and $k > 1$, by [10, Theorem 3] the matrix ring $M_k(R)$ is an $(S, 3)$-ring. So $M_k(R)$ is n-$*$-clean by Corollary 2.3 where $n \geq 3$.

The following example reveals that there exists a non-semicommutative $*$-ring R such that $R[x]$ is n-$*$-clean where $n \geq 2$.

Example 2.16 Let F be a field and $R = M_k(F)$ with $k \geq 2$. Then $R[x] = M_k(F)[x] \cong M_k(F[x])$. Since $F[x]$ is an elementary divisor ring, by [10, Theorem 11] $R[x]$ is an $(S, 2)$-ring. In view of Corollary 2.3, $R[x]$ is n-$*$-clean for any $n \geq 2$. Notice that R is not semicommutative (since it is not abelian).

It was proved in [3] that for any $e \in \mathrm{Id}(R)$, R is n-clean whenever both eRe and $(1-e)R(1-e)$

are n-clean. Let R be a $*$-ring and $p \in P(R)$. The involution $*$ of R is inherited naturally to the corner ring pRp. We have an analogous result.

Theorem 2.17 *Let R be a $*$-ring and $p \in P(R)$. If pRp and $(1-p)R(1-p)$ are n-$*$-clean, then R is n-$*$-clean.*

Proof Denote $q = 1 - p$. We apply the Pierce decomposition for the ring R. Then

$$R = \begin{pmatrix} pRp & pRq \\ qRp & qRq \end{pmatrix}$$

Let $x \in R$. Then we can write

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a \in pRp,\ b \in pRq,\ c \in qRp$ and $d \in qRq$.

By hypothesis, $a = e + u_1 + u_2 + \cdots + u_n$ with $e \in P(pRp)$ and $u_i \in U(pRp)$ with inverses u_i^{-1} $(1 \le i \le n)$. Note that $d - cu_1^{-1}b \in qRq$. From the n-$*$-cleanness of qRq, there exist $g,\ v_1, \ldots, v_n \in R$ such that

$$d - cu_1^{-1}b = g + v_1 + v_2 + \cdots + v_n,$$

where $g \in P(qRq)$ and v_i are units of qRq $(1 \le i \le n)$. Thus, we have

$$\begin{aligned} x &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix} + \begin{pmatrix} u_1 & b \\ c & v_1 + cu_1^{-1}b \end{pmatrix} + \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix}. \end{aligned}$$

Note that $eg = ge = 0$. Thus $\left(\begin{smallmatrix} e & 0 \\ 0 & g \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} e & 0 \\ 0 & g \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e & 0 \\ 0 & g \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} e & 0 \\ 0 & g \end{smallmatrix}\right)$, which implies that $\left(\begin{smallmatrix} e & 0 \\ 0 & g \end{smallmatrix}\right) \in P(R)$. In addition,

$$\begin{aligned} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} &= \begin{pmatrix} u_1 & b \\ c & v_1 + cu_1^{-1}b \end{pmatrix} \begin{pmatrix} u_1^{-1} + u_1^{-1}bv_1^{-1}cu_1^{-1} & -u_1^{-1}bv_1^{-1} \\ -v_1^{-1}cu_1^{-1} & v_1^{-1} \end{pmatrix} \\ &= \begin{pmatrix} u_1^{-1} + u_1^{-1}bv_1^{-1}cu_1^{-1} & -u_1^{-1}bv_1^{-1} \\ -v_1^{-1}cu_1^{-1} & v_1^{-1} \end{pmatrix} \begin{pmatrix} u_1 & b \\ c & v_1 + cu_1^{-1}b \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix} \begin{pmatrix} u_i^{-1} & 0 \\ 0 & v_i^{-1} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} u_i^{-1} & 0 \\ 0 & v_i^{-1} \end{pmatrix} \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix}.$$

This proves that $x \in R$ is n-$*$-clean. Hence, R is an n-$*$-clean ring. □

By Theorem 2.17, an inductive argument gives immediately the following.

Corollary 2.18 *Let R be a $*$-ring. If $1 = p_1 + p_2 + \cdots + p_k$ in R where p_i are orthogonal projections and each p_iRp_i is n-$*$-clean, then R is n-$*$-clean.*

Corollary 2.19 *If R is n-$*$-clean, then $M_k(R)$ is n-$*$-clean for any $k \ge 1$.*

The converse of Theorem 2.17 is not true in general.

Example 2.20 Let $R = F[x]$ with F a field, and define $* = 1_R$. In view of [10, Theorem 11],

$M_2(R)$ is 2-$*$-clean (being an $(S,2)$-ring). Take $E = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \in M_2(R)$. Then $E \in P(M_2(R))$ and $EM_2(R)E \cong R$. However, $R = F[x]$ is not 2-$*$-clean by Remark 2.15 (i).

Proposition 2.21 *Let R be a $*$-ring and p a central projection of R. If R is n-$*$-clean, then so is pRp.*

Proof Let $x \in pRp \subseteq R$. By hypothesis, $x = q + u_1 + \cdots + u_n$ with $q \in P(R)$ and $u_1, \ldots, u_n$ are units of R. Since p is central, $x = pxp = pqp + pu_1p + \cdots + pu_np$ is an n-$*$-clean expression in pRp. $\square$

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