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Persistence and Almost Periodic Solutions in a Model of Plankton Allelopathy with Impulsive Effects

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Abstract This paper is concerned with an almost periodic model of plankton allelopathy with impulsive effects. By using the comparison theorem and the Lyapunov method of the impulsive differential equations, sufficient conditions which guarantee the permanence and existence of a unique uniformly asymptotically stable positive almost periodic solution of the model are obtained. The main results in this paper improve the work in recent years. And the method used in this paper provides a new method to study the permanence, uniform asymptotical stability and almost periodic solution of the models with impulsive perturbations in biological populations. An example and numerical simulations are provided to illustrate the main results of this paper. Finally, a conclusion is also given to discuss how the impulsive effects influence the permanence, almost periodic solutions and uniform asymptotical stability of the model.

Keywords almost periodic solution; permanence; uniformly asymptotically stable; impulse; plankton allelopathy

MR(2010) Subject Classification 34K14; 34K20; 34K45; 92D25

1. Introduction

Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. Related to a continuous function f, we use the following notations:

$$f^- = \inf_{s\in\mathbb{R}} f(s), \quad f^+ = \sup_{s\in\mathbb{R}} f(s).$$

In the last twenty years, the permanence and positive almost periodic solutions of the biological models have been studied extensively by many authors [1–13]. However, there are few papers that consider impulsive models. The ecological system is often deeply perturbed by human exploitation activities such as planting and harvesting and so on, which makes it unsuitable to be considered continually. To obtain a more accurate description of such systems, one needs to consider the impulsive differential equations. Then He et al. in [6] studied the

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following impulsive model of plankton allelopathy:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1(t)x_2(t)], \\ \dot{x}_2(t) = x_2(t)[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_2(t)x_1(t)], \quad t \neq \tau_k, \\ \Delta x_1(\tau_k) = h_{1k}x_1(\tau_k), \\ \Delta x_2(\tau_k) = h_{2k}x_2(\tau_k), \quad k \in \{0, 1, \dots\} = \mathbb{Z}^+, \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ are population densities of species x_1, x_2 at time t, respectively; $r_i(t)$, $a_{ij}(t)$, $b_i(t)$; $i, j = 1, 2$, are all continuous almost periodic functions which are bounded above and below by positive constants; $h_{1k}, h_{2k} > -1$ are constants, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots$, are impulse points with $\lim_{k\to+\infty} \tau_k = +\infty$, and the set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}^+, j \in \mathbb{Z}$ is uniformly almost periodic (see Definition 2.1 in Section 2).

By the relation between the solutions of impulsive system and the corresponding non-impulsive system, the authors [6] obtained the following results ensuring the permanence and existence of a unique positive almost periodic solution of system (1.1).

Theorem 1.1 ([6]) *Assume that*

(F_1) There exist positive constants ϱ_i and ρ_i such that $\varrho_i \leq \prod_{0<\tau_k<t}(1 + h_{ik}) \leq \rho_i$ for $t > 0$, $i = 1, 2$;

(F_2) $r_1^- - a_{12}^+\rho_2\bar{M}_2 > 0$ and $r_2^- - a_{21}^+\rho_1\bar{M}_1 > 0$, where $\bar{M}_i := \frac{r_i^+}{a_{ii}^-\varrho_i}$, $i = 1, 2$.

Then system (1.1) is permanent.

Theorem 1.2 ([6]) *Assume that (F_1)–(F_2) hold, suppose further that*

(F_3) $\prod_{0<\tau_k<t}(1 + h_{ik})$ is almost periodic;

(F_4) There exist three constants λ_1, λ_2 and μ such that

$$\lambda_1 a_{11}^-\varrho_1 + \lambda_1 b_1^-\varrho_1\varrho_2\bar{N}_2 - \lambda_2 a_{21}^+\rho_1 - \lambda_2 b_2^+\rho_1\rho_2\bar{M}_2 > \mu,$$

$$\lambda_2 a_{22}^-\varrho_2 + \lambda_2 b_2^-\varrho_1\varrho_2\bar{N}_1 - \lambda_1 a_{12}^+\rho_2 - \lambda_1 b_1^+\rho_1\rho_2\bar{M}_1 > \mu,$$

where $\bar{N}_i := \frac{r_i^- - a_{ij}^+\rho_j\bar{M}_j}{b_i^+\rho_1\rho_2\bar{M}_j + a_{ii}^+\rho_i}$, $i \neq j$, $i, j = 1, 2$.

Then system (1.1) admits a unique almost periodic solution, which is uniformly asymptotically stable.

Remark 1.3 Obviously, (F_1) in Theorem 1.1 and (F_3) in Theorem 1.2 are harsh. For example, if the impulse coefficient $h_{ik} \equiv 0.5$ or $h_{ik} \equiv -0.5$ in system (1.1), then conditions (F_1) and (F_3) are both invalid. Therefore, the application of Theorems 1.1 and 1.2 are narrow.

In order to remedy the above shortcoming, the main purpose of this paper is to establish some new sufficient conditions which guarantee the permanence and existence of a unique uniformly asymptotically stable positive almost periodic solution of system (1.1) without conditions (F_1) and (F_3) by using the comparison theorem and the Lyapunov method of the impulsive differential equations [14,15] (see Theorems 3.1 and 4.1 in Sections 3–4).

The organization of this paper is as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, by using the comparison theorem of the impulsive differential equations [14], we give the permanence of system (1.1). In Section 4, we study the existence of a unique uniformly asymptotically stable positive almost periodic solution of system (1.1) by applying the Lyapunov method of the impulsive differential equations [15]. An example and numerical simulations are also given to illustrate our main results.

2. Preliminaries

Now, let us state the following definitions and lemmas, which will be useful in proving our main result.

By $\mathbb{I}$, $\mathbb{I} = \{\{\tau_k\} \in \mathbb{R} : \tau_k < \tau_{k+1},\ k \in \mathbb{Z},\ \lim_{k\to\pm\infty} \tau_k = \pm\infty\}$, we denote the set of all sequences that are unbounded and strictly increasing. Introduce the following notations:

For $J \subset \mathbb{R}^2$, $PC(J, \mathbb{R}^2)$ is the space of all piecewise continuous functions from J to $\mathbb{R}^2$ with points of discontinuity of the first kind τ_k, at which it is left continuous. By the basic theories of impulsive differential equations in [14,16], system (1.1) has a unique solution $X(t) = X(t, X_0) \in PC([0, +\infty), \mathbb{R}^2)$.

Since the solution of system (1.1) is a piecewise continuous function with points of discontinuity of the first kind τ_k, $k \in \mathbb{Z}$ we adopt the following definitions for almost periodicity.

Definition 2.1 ([17]) *The set of sequences* $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{I}$ *is said to be uniformly almost periodic if for arbitrary* $\epsilon > 0$ *there exists a relatively dense set of* ϵ*-almost periods common for any sequences.*

Definition 2.2 ([17]) *The function* $\varphi \in PC(\mathbb{R}, \mathbb{R})$ *is said to be almost periodic, if the following hold:*

(1) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k \in \mathbb{Z}, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{I}$ *is uniformly almost periodic.*

(2) For any $\epsilon > 0$ *there exists a real number* $\delta > 0$ *such that if the points* t' *and* t'' *belong to one and the same interval of continuity of* $\varphi(t)$ *and satisfy the inequality* $|t' - t''| < \delta$, *then* $|\varphi(t') - \varphi(t'')| < \epsilon$.

(3) For any $\epsilon > 0$ *there exists a relatively dense set* T *such that if* $\eta \in T$, *then* $|\varphi(t + \eta) - \varphi(t)| < \epsilon$ *for all* $t \in \mathbb{R}$ *satisfying the condition* $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$. *The elements of* T *are called* ϵ*-almost periods.*

Lemma 2.3 ([17]) *Let* $\{\tau_k\} \in \mathbb{I}$. *Then there exists a positive integer* A *such that on each interval of length* 1, *we have no more than* A *elements of the sequence* $\{\tau_k\}$, *i.e.,*

$$i(s, t) \leq A(t - s) + A,$$

where $i(s, t)$ *is the number of the points* τ_k *in the interval* (s, t).

Theoretically, one can investigate the existence, uniqueness and stability of almost periodic solution for functional differential equations by using Lyapunov functional as follows [15, p. 109]:

Consider the system of impulsive differential equations:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq \tau_k, \\ \Delta x(\tau_k) = I_k x(\tau_k), \end{cases} \tag{2.1}$$

where $t \in \mathbb{R}$, $\{\tau_k\} \in \mathbb{I}$, $f : \mathbb{R} \times D \to \mathbb{R}^n$, $I_k : D \to \mathbb{R}^n$, $k \in \mathbb{Z}$, D is an open set in $\mathbb{R}^n$.

Introduce the following conditions:

(C_1) The function $f(t, x)$ is almost periodic in t uniformly with respect to $x \in D$.

(C_2) The sequence $\{I_k(x)\}$, $k \in \mathbb{Z}$ is almost periodic uniformly with respect to $x \in D$.

Lemma 2.4 ([15, p. 109]) *Suppose that there exists a Lyapunov functional $V(t, x, y)$ defined on $\mathbb{R}^+ \times D \times D$ satisfying the following conditions:*

(1) $u(\|x - y\|) \leq V(t, x, y) \leq v(\|x - y\|)$, where $u, v \in \mathcal{P}$ with $\mathcal{P} = \{u : \mathbb{R}^+ \to \mathbb{R}^+ | u$ is continuous increasing function and $u(s) \to 0$ as $s \to 0\}$.

(2) $|V(t, \bar{x}, \bar{y}) - V(t, \hat{x}, \hat{y})| \leq K(\|\bar{x} - \hat{x}\| + \|\bar{y} - \hat{y}\|)$, where $K > 0$ is a constant.

(3) For $t = \tau_k$, $V(t^+, x + I_k(x), y + I_k(y)) \leq V(t, x, y)$; For $t \neq \tau_k$, $\dot{V}_{(2.2)}(t, x, y) \leq -\gamma V(t, x, y)$, $\forall k \in \mathbb{Z}$, where $\gamma > 0$ is a constant.

Moreover, one assumes that system (2.1) has a solution that remains in a compact set $S \subset D$. Then system (2.1) has a unique almost periodic solution which is uniformly asymptotically stable.

3. Permanence

In this section, we establish a permanence result for system (1.1).

Lemma 3.1 ([14]) *Assume that $x \in PC(\mathbb{R})$ with points of discontinuity at $t = \tau_k$ and is left continuous at $t = \tau_k$ for $k \in \mathbb{Z}^+$, and*

$$\begin{cases} \dot{x}(t) \leq f(t, x(t)), & t \neq \tau_k, \\ x(\tau_k^+) \leq I_k(x(\tau_k)), & k \in \mathbb{Z}^+, \end{cases} \tag{3.1}$$

where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$ and $I_k(x)$ is nondecreasing in x for $k \in \mathbb{Z}^+$. Let $u^*(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} \dot{u}(t) = f(t, u(t)), & t \neq \tau_k, \\ u(\tau_k^+) = I_k(u(\tau_k)) \geq 0, & k \in \mathbb{Z}^+, \\ u(t_0^+) = u_0 \end{cases} \tag{3.2}$$

existing on $[t_0, \infty)$. Then $x(t_0^+) \leq u_0$ implies $x(t) \leq u^*(t)$ for $t \geq t_0$.

Remark 3.2 If the inequalities (3.1) in Lemma 3.1 is reversed and $u_*(t)$ is the minimal solution of system (3.2) existing on $[t_0, \infty)$, then $x(t_0^+) \geq u_0$ implies $x(t) \geq u_*(t)$ for $t \geq t_0$.

Lemma 3.3 *Assume that $a, b > 0$, then the following impulsive logistic equation*

$$\begin{cases} \dot{x}(t) = x(t)[a - bx(t)], & t \neq \tau_k, \\ \Delta x(\tau_k^+) = h_k x(\tau_k), & k \in \mathbb{Z}^+ \end{cases} \tag{3.3}$$

has a unique globally asymptotically stable positive almost periodic solution x^* which can be *expressed as follows:*

$$\frac{\alpha}{e^{\xi A} b} \le x^*(t) = \Big[b \int_{-\infty}^{t} W(t,s)\mathrm{d}s\Big]^{-1} \le \frac{a}{\eta b(1-e^{-a\theta})}, \tag{3.4}$$

where A is defined as that in Lemma 2.3, $\xi := \ln \sup_{k\in\mathbb{Z}} \frac{1}{1+h_k}$, $\alpha := a - \xi A$, $\theta := \inf_{k\in\mathbb{Z}} \tau_k^1$, $\eta := \inf_{k\in\mathbb{Z}} \prod_{j=1}^{2} \frac{1}{1+h_{j+k}}$ and

$$W(t,s) = \begin{cases} e^{-a(t-s)}, & \tau_{k-1} < s < t < \tau_k; \\ \prod_{j=m}^{k+1} \frac{1}{1+h_j} e^{-a(t-s)}, & \tau_{m-1} < s \le \tau_m < \tau_k < t \le \tau_{k+1}. \end{cases}$$

Proof Let $u = \frac{1}{x}$. Then system (3.3) changes to

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = -au(t) + b, & t \ne \tau_k, \\ \Delta u(\tau_k^+) = -\frac{h_k}{1+h_k} u(\tau_k), & k \in \mathbb{Z}^+. \end{cases} \tag{3.5}$$

Similar to the proof as that in [16], we can easily obtain

$$u^*(t) = b \int_{-\infty}^{t} W(t,s)\mathrm{d}s,$$

which is an almost periodic solution of system (3.5). Then system (3.3) has a unique almost periodic solution x^* which can be expressed by (3.4). By Lemma 2.3, we have

$$x^*(t) \ge \Big[b \int_{-\infty}^{t} e^{\xi A} e^{-\alpha(t-s)}\mathrm{d}s\Big]^{-1} = \frac{\alpha}{e^{\xi A} b}.$$

On the other hand,

$$x^*(t) \le \Big[b \int_{t-\theta}^{t} \eta e^{-a(t-s)}\mathrm{d}s\Big]^{-1} = \frac{a}{\eta b(1-e^{-a\theta})}.$$

Suppose that $x(t)$ is another positive solution of system (3.3). Define

$$V(t) = |\ln x^*(t) - \ln x(t)|, \quad \forall t \in \mathbb{R}.$$

For $t \ne \tau_k$, $k \in \mathbb{Z}^+$, calculating the upper right derivative of $V(t)$ along the solution of system (3.3), we have

$$D^+V(t) = -b|x^*(t) - x(t)|. \tag{3.6}$$

For $t = \tau_k$, $k \in \mathbb{Z}^+$, we have

$$V(\tau_k^+) = |\ln x^*(\tau_k^+) - \ln x(\tau_k^+)| = \big|\ln \frac{(1+h_k)x^*(\tau_k)}{(1+h_k)x(\tau_k)}\big| = |\ln x^*(\tau_k) - \ln x(\tau_k)| = V(\tau_k).$$

Therefore, V is non-increasing. Integrating (3.6) from 0 to t leads to

$$V(t) + b\int_0^t |x(s) - x^*(s)|\mathrm{d}s \le V(0) < +\infty, \quad \forall t \ge 0,$$

that is,

$$\int_0^{+\infty} |x(s) - x^*(s)|\mathrm{d}s < +\infty,$$

which implies that

$$\lim_{s\to+\infty} |x(s) - x^*(s)| = 0.$$

Thus, the almost periodic solution of system (3.3) is globally asymptotically stable. This completes the proof. □

Proposition 3.4 *Every solution $(x_1, x_2)^T$ of system (1.1) satisfies*

$$\limsup_{t\to\infty} x_i(t) \le M_i := \frac{r_i^+}{\eta_i a_{ii}^-(1-e^{-r_i^+\theta})}, \quad \eta_i := \inf_{k\in\mathbb{Z}} \prod_{j=1}^{2} \frac{1}{1+h_{i(j+k)}}, \quad i=1,2.$$

Proof From the first equation of system (1.1), we have

$$\begin{cases} \dot{x}_1(t) \le x_1(t)[r_1^+ - a_{11}^- x_1(t)], & t \ne \tau_k, \\ x_1(\tau_k^+) = (1+h_{1k})x_1(\tau_k), & k \in \mathbb{Z}^+. \end{cases}$$

Consider the auxiliary system

$$\begin{cases} \dot{z}_1(t) = z_1(t)[r_1^+ - a_{11}^- z_1(t)], & t \ne \tau_k, \\ z_1(\tau_k^+) = (1+h_{1k})z_1(\tau_k), & k \in \mathbb{Z}^+. \end{cases} \tag{3.7}$$

By Lemma 3.1, $x_1(t) \le z_1(t)$, where $z_1(t)$ is the solution of system (3.7) with $z_1(0^+) = x_1(0^+)$. By Lemma 3.3, system (3.7) has a unique globally asymptotically stable positive almost periodic solution z_1^* which can be expressed as follows:

$$z_1^*(t) = \Big[a_{11}^- \int_{-\infty}^{t} W_1(t,s)\mathrm{d}s\Big]^{-1} \le \frac{r_1^+}{\eta_1 a_{11}^-(1-e^{-r_1^+\theta})} := M_1,$$

where

$$W_1(t,s) = \begin{cases} e^{-r_1^+(t-s)}, & \tau_{k-1} < s < t < \tau_k; \\ \prod_{j=m}^{k+1} \frac{1}{1+h_{1j}} e^{-r_1^+(t-s)}, & \tau_{m-1} < s \le \tau_m < \tau_k < t \le \tau_{k+1}. \end{cases}$$

Then for any constant $\epsilon > 0$, there exists $T_1 > 0$ such that $x_1(t) \le z_1(t) < z_1^*(t) + \epsilon \le M_1 + \epsilon$ for $t > T_1$. So $\limsup_{t\to\infty} x_1(t) \le M_1$. Similarly, ones have $\limsup_{t\to\infty} x_2(t) \le M_2$. This completes the proof. □

Proposition 3.5 *Assume that*

(H$_1$) $r_i^- - a_{i(3-i)}^+ M_{3-i} - \xi_i A > 0$, *where* $\xi_i := \ln\sup_{k\in\mathbb{Z}} \frac{1}{1+h_{ik}}$, $i=1,2$.

Then every solution $(x_1, x_2)^T$ of system (1.1) satisfies

$$\liminf_{t\to\infty} x_i(t) \ge N_i := \frac{r_i^- - a_{i(3-i)}^+ M_{3-i} - \xi_i A}{e^{\xi_i A}(a_{ii}^+ + b_i^+ M_{3-i})}, \quad i=1,2.$$

Proof For any $\epsilon > 0$, according to Proposition 3.4, there exists $T_2 > 0$ such that

$$x_i(t) \le M_i + \epsilon \text{ for } t \ge T_2, \quad i=1,2.$$

From the first equation of system (1.1), we have

$$\begin{cases} \dot{x}_1(t) \ge x_1(t)[r_1^- - a_{11}^+ x_1(t) - a_{12}^+(M_2+\epsilon) - b_1^+(M_2+\epsilon)x_1(t)], & t \ne \tau_k,\ t > T_2, \\ x_1(\tau_k^+) = (1+h_{1k})x_1(\tau_k), & k \in \mathbb{Z}^+. \end{cases}$$

Consider the auxiliary system

$$\begin{cases} \dot{p}_1(t) = p_1(t)[R_1(\epsilon) - (a_{11}^+ + b_1^+(M_2+\epsilon))p_1(t)], & t \ne \tau_k,\ t > T_2, \\ p_1(\tau_k^+) = (1+h_{1k})p_1(\tau_k), & k \in \mathbb{Z}^+. \end{cases} \tag{3.8}$$

where $R_1(\epsilon) := r_1^- - a_{12}^+(M_2 + \epsilon)$. By Remark 3.2, $x_1(t) \geq p_1(t)$ for $t > T_2$, where $p_1(t)$ is the solution of system (3.8) with $p_1(T_2^+) = x_1(T_2^+)$. By Lemma 3.3, system (3.8) has a unique globally asymptotically stable positive almost periodic solution p_1^* which can be expressed as follows:

$$p_1^*(t) = \Big[(a_{11}^+ + b_1^+(M_2+\epsilon)) \int_{-\infty}^t W_2(t,s)\mathrm{d}s\Big]^{-1} \geq \frac{r_1^- - a_{12}^+(M_2+\epsilon) - \xi_1 A}{e^{\xi_1 A}(a_{11}^+ + b_1^+(M_2+\epsilon))} := N_1(\epsilon),$$

where

$$W_2(t,s) = \begin{cases} e^{-R_1(\epsilon)(t-s)}, & \tau_{k-1} < s < t < \tau_k; \\ \prod_{j=m}^{k+1} \frac{1}{1+h_{1j}} e^{-R_1(\epsilon)(t-s)}, & \tau_{m-1} < s \leq \tau_m < \tau_k < t \leq \tau_{k+1}. \end{cases}$$

Similar to the above argument as that in Proposition 3.4, we have $\liminf_{t\to\infty} x_1(t) \geq N_1(\epsilon)$. By the arbitrariness of ϵ, it leads to $\liminf_{t\to\infty} x_1(t) \geq N_1$. Similarly, $\liminf_{t\to\infty} x_2(t) \geq N_2$. This completes the proof. □

Remark 3.6 When h_{ik} $(i = 1, 2, k \in \mathbb{Z}) \equiv 0$ in system (1.1), then Propositions 3.1 and 3.2 improve the corresponding result in [6]. So Propositions 3.1 and 3.2 improve the corresponding result in [6].

Remark 3.7 In view of Proposition 3.2, the distance θ between impulse points, the values of impulse coefficients h_{ik} $(i = 1, 2, k \in \mathbb{Z})$ and the number A of the impulse points in each interval of length 1 have negative effect on the uniform persistence of system (1.1).

By Propositions 3.4 and 3.5, we have

Theorem 3.8 *Assume that* (H_1) *holds. Then system (1.1) is permanent.*

Remark 3.9 Without condition (F_1) in Theorem 1.1, system (1.1) is also permanent.

Remark 3.10 Theorem 3.8 gives the sufficient conditions for the permanence of system (1.1). Therefore, Theorem 3.8 provides a new method to study the permanence of the models with almost periodic impulsive perturbations in biological populations.

Remark 3.11 From the proof of Propositions 3.4 and 3.5, we know that under the conditions of Theorem 3.8, the set $S = \{(x_1, x_2)^T \in \mathbb{R}^2 : N_i \leq x_i \leq M_i, i = 1, 2\}$ is an invariant set of system (1.1).

4. Almost periodic solution

The main result of this paper concerns the existence of a unique uniformly asymptotically stable almost periodic solution for system (1.1).

Theorem 4.1 *Assume that* (H_1) *holds. Suppose further that*

(H_2) *there exist three constants* λ_1, λ_2 *and* μ *such that*

$$\lambda_1 a_{11}^- + \lambda_1 b_1^- N_2 - \lambda_2 a_{21}^+ - \lambda_2 b_2^+ M_2 > \mu, \ \lambda_2 a_{22}^- + \lambda_2 b_2^- N_1 - \lambda_1 a_{12}^+ - \lambda_1 b_1^+ M_1 > \mu,$$

where M_1, M_2, N_1 *and* N_2 *are defined as that in Section 3. Then system (1.1) admits a unique* *positive almost periodic solution, which is uniformly asymptotically stable.*

Proof Suppose that $Z(t)=(z_1(t),z_2(t))^T$ and $Z^*(t)=(z_1^*(t),z_2^*(t))^T$ are any two solutions of system (1.1). Consider the product system of system (1.1)

$$\begin{cases} \dot{z}_1(t)=z_1(t)[r_1(t)-a_{11}(t)z_1(t)-a_{12}(t)z_2(t)-b_1(t)z_1(t)z_2(t)],\\ \dot{z}_2(t)=z_2(t)[r_2(t)-a_{21}(t)z_1(t)-a_{22}(t)z_2(t)-b_2(t)z_2(t)z_1(t)],\\ \dot{z}_1^*(t)=z_1^*(t)[r_1(t)-a_{11}(t)z_1^*(t)-a_{12}(t)z_2^*(t)-b_1(t)z_1^*(t)z_2^*(t)],\\ \dot{z}_2^*(t)=z_2^*(t)[r_2(t)-a_{21}(t)z_1^*(t)-a_{22}(t)z_2^*(t)-b_2(t)z_2^*(t)z_1^*(t)],\quad t\neq\tau_k,\\ \Delta z_1(\tau_k)=h_{1k}z_1(\tau_k),\\ \Delta z_2(\tau_k)=h_{2k}z_2(\tau_k),\\ \Delta z_1^*(\tau_k)=h_{1k}z_1^*(\tau_k),\\ \Delta z_2^*(\tau_k)=h_{2k}z_2^*(\tau_k),\quad k\in\mathbb{Z}. \end{cases}\tag{4.1}$$

Set $S_1=\{(z_1,z_2)^T\in\mathbb{R}^2: N_i\le z_i\le M_i, i=1,2\}$, which is an invariant set of system (4.1) directly from Remark 3.11.

Construct a Lyapunov functional $V(t,Z,Z^*)=V(t,(z_1,z_2)^T,(z_1^*,z_2^*)^T)$ defined on $\mathbb{R}^+\times S_1\times S_1$ as follows:

$$V(t,Z,Z^*)=\lambda_1|\ln z_1(t)-\ln z_1^*(t)|+\lambda_2|\ln z_2(t)-\ln z_2^*(t)|.$$

It is obvious that

$$\begin{aligned} V(t,Z,Z^*)&\ge\min\{\lambda_1,\lambda_2\}\sum_{i=1}^2|\ln z_i(t)-\ln z_i^*(t)|\\ &\ge\min\{\lambda_1,\lambda_2\}\sum_{i=1}^2\frac{1}{M_i}|z_i(t)-z_i^*(t)|\ge\underline{\lambda}\|Z-Z^*\|, \end{aligned}$$

where $\underline{\lambda}:=\min\{\lambda_1,\lambda_2\}\min\{M_1^{-1},M_2^{-1}\}$. Further, we have

$$\begin{aligned} V(t,Z,Z^*)&\le\max\{\lambda_1,\lambda_2\}\sum_{i=1}^2|\ln z_i(t)-\ln z_i^*(t)|\\ &\le\max\{\lambda_1,\lambda_2\}\sum_{i=1}^2\frac{1}{N_i}|z_i(t)-z_i^*(t)|\le\overline{\lambda}\|Z-Z^*\|, \end{aligned}$$

where $\overline{\lambda}:=\max\{\lambda_1,\lambda_2\}\max\{N_1^{-1},N_2^{-1}\}$, thus condition (1) in Lemma 2.4 is satisfied.

Since

$$\begin{aligned} |V(t,Z,Z^*)-V(t,\bar{Z},\bar{Z}^*)|&=\sum_{i=1}^2\lambda_i|\ln z_i(t)-\ln z_i^*(t)|-\sum_{i=1}^2\lambda_i|\ln\bar{z}_i(t)-\ln\bar{z}_i^*(t)|\\ &\le\overline{\lambda}\sum_{i=1}^2[|z_i(t)-z_i^*(t)|+|\bar{z}_i(t)-\bar{z}_i^*(t)|]\\ &=\overline{\lambda}[|Z(t)-Z^*(t)|+|\bar{Z}(t)-\bar{Z}^*(t)|], \end{aligned}$$

condition (2) in Lemma 2.4 holds.

For $t \neq \tau_k$, $k \in \mathbb{Z}^+$, calculating the upper right derivative of $V(t)$ along the solution of system (4.1), we have

$$\begin{aligned} D^+V(t) =&\lambda_1(\dot{z}_1(t)-\dot{z}_1^*(t))\text{sgn}(z_1(t)-z_1^*(t))+\lambda_2(\dot{z}_2(t)-\dot{z}_2^*(t))\text{sgn}(z_2(t)-z_2^*(t)) \\ =&\lambda_1\text{sgn}(z_1(t)-z_1^*(t)\big\{-a_{11}(t)[z_1(t)-z_1^*(t)]- \\ &a_{12}(t)[z_2(t)-z_2^*(t)]-b_1(t)[z_1(t)z_2(t)-z_1^*(t)z_2^*(t)]\big\}+ \\ &\lambda_2\text{sgn}(z_2(t)-z_2^*(t)\big\{-a_{21}(t)[z_1(t)-z_1^*(t)]- \\ &a_{22}(t)[z_2(t)-z_2^*(t)]-b_2(t)[z_1(t)z_2(t)-z_1^*(t)z_2^*(t)]\big\} \\ \leq&-\left[\lambda_1a_{11}^-+\lambda_1b_1^-N_2-\lambda_2a_{21}^+-\lambda_2b_2^+M_2\right]|z_1(t)-z_1^*(t)|- \\ &[\lambda_2a_{22}^-+\lambda_2b_2^-N_1-\lambda_1a_{12}^+-\lambda_1b_1^+M_1]|z_2(t)-z_2^*(t)| \\ \leq&-\frac{\mu}{\lambda_1N_1}\lambda_1|\ln z_1(t)-\ln z_1^*(t)|-\frac{\mu}{\lambda_2N_2}\lambda_2|\ln z_2(t)-\ln z_2^*(t)| \\ \leq&-\min\{\frac{\mu}{\lambda_1N_1},\frac{\mu}{\lambda_2N_2}\}V(t,Z,Z^*). \end{aligned} \tag{4.2}$$

For $t=\tau_k$, $k\in\mathbb{Z}^+$, we have

$$\begin{aligned} V(\tau_k^+,Z(\tau_k^+),Z^*(\tau_k^+)) &= \sum_{i=1}^2\lambda_i|\ln z_i(\tau_k^+)-\ln z_i^*(\tau_k^+)| = \sum_{i=1}^2\lambda_i\Big|\ln\frac{(1+h_{ik})z_i(\tau_k)}{(1+h_{ik})z_i^*(\tau_k)}\Big| \\ &= \sum_{i=1}^2\lambda_i|\ln z_i(\tau_k)-\ln z_i^*(\tau_k)| = V(\tau_k,Z(\tau_k),Z^*(\tau_k)). \end{aligned} \tag{4.3}$$

In view of (4.2) and (4.3), condition (3) in Lemma 2.4 is satisfied.

By Lemma 2.4, system (1.1) admits a unique uniformly asymptotically stable positive almost periodic solution $(z_1(t),z_2(t))^T$. This completes the proof. □

Remark 4.2 Without conditions (F_1) and (F_3) in Theorem 1.2, system (1.1) also admits a unique uniformly asymptotically stable positive almost periodic solution.

Remark 4.3 Theorem 4.1 gives sufficient condition for the global asymptotical stability of a unique positive almost periodic solution of system (1.1). From Remark 4.2, Theorem 4.1 improves the corresponding result in [6] and provides a new method to study the existence, uniqueness and stability of positive almost periodic solution of the models with impulsive perturbations in biological populations.

5. An example and numerical simulations

Example 5.1 Consider the following impulsive model of plankton allelopathy:

$$\begin{cases} \dot{x}_1(t)=x_1(t)[1-0.1x_1(t)-0.02\cos(\sqrt{2}t)x_2(t)-0.001\sin(\sqrt{3}t)x_1(t)x_2(t)], \\ \dot{x}_2(t)=x_2(t)[1-0.002x_1(t)-0.2x_2(t)-0.01\sin(\sqrt{5}t)x_2(t)x_1(t)], \quad t\neq\tau_k, \\ \Delta x_1(\tau_k)=0.5x_1(\tau_k), \\ \Delta x_2(\tau_k)=-0.5x_2(\tau_k), \quad \{\tau_k:k\in\mathbb{Z}\}\subset\{10k:k\in\mathbb{Z}\},\ \theta=\inf_{k\in\mathbb{Z}}\tau_k^1=10, \end{cases} \tag{5.1}$$

then system (5.1) is permanent and admits a unique uniformly asymptotically stable positive almost periodic solution.

Proof Corresponding to system (1.1), $r_1 = r_2 \equiv 1$, $a_{11} \equiv 0.1$, $a_{21} \equiv 0.002$, $a_{22} \equiv 0.2$, $a_{12} = 0.02\cos(\sqrt{2}t)$, $b_1 = 0.001\sin(\sqrt{3}t)$, $b_2 = 0.01\sin(\sqrt{5}t)$, $h_{1k} \equiv 0.5$, $h_{2k} \equiv -0.5$, $k \in \mathbb{Z}^+$.

By easy calculation, $M_1 \approx 35.7$, $M_2 \approx 8$, $\xi_1 \approx -0.36$, $\xi_2 \approx 0.69$, $A = 1$,

$$r_1^- - a_{12}^+ M_2 - \xi_1 A \approx 1.32, \quad r_2^- - a_{21}^+ M_1 - \xi_2 A \approx 0.3,$$

which implies that (H_1) in Theorem 3.8 holds. By Theorem 3.8, system (5.1) is permanent (see Figures 1 and 2).

Further, $N_1 \approx 6.7$, $N_2 \approx 0.13$. Taking $\lambda_1 = \lambda_2 = 1$ in (H_2) of Theorem 4.1, then

$$a_{11}^- + b_1^- N_2 - a_{21}^+ - b_2^+ M_2 \approx 0.06, \quad a_{22}^- + b_2^- N_1 - a_{12}^+ - b_1^+ M_1 \approx 0.1443.$$

So (H_2) of Theorem 4.1 is satisfied. By Theorem 4.1, system (5.1) admits a unique uniformly asymptotically stable positive almost periodic solution (see Figures 1–4). This completes the proof. □

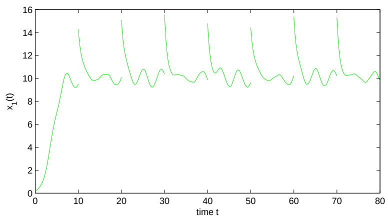

Figure 1 Almost periodic oscillation of state variable x_1 system (5.1)

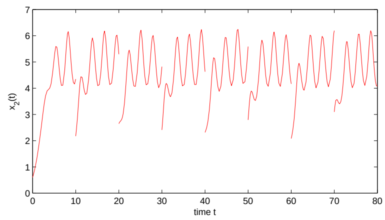

Figure 2 Almost periodic oscillation of state variable x_2 system (5.1)

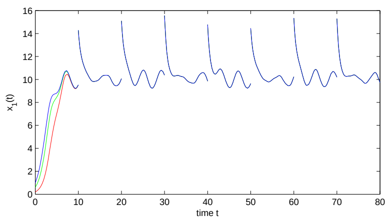

Figure 3 Uniform asymptotical stability of state variable x_1 of system (5.1)

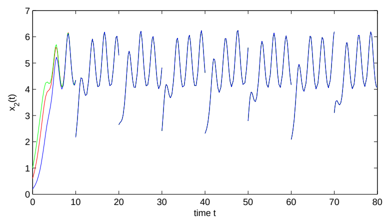

Figure 4 Uniform asymptotical stability of state variable x_2 of system (5.1)

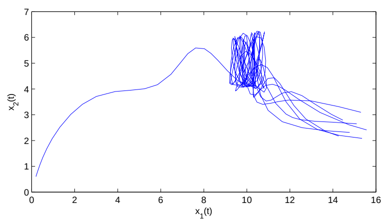

Figure 5 Phase response of state variables x_1, x_2 of system (5.1)

Remark 5.2 In Example 5.1, the impulse coefficients ($h_{1k} = 0.5$ and $h_{2k} = -0.5$) do not satisfy (F_1) in Theorem 1.1 and (F_3) in Theorem 1.2. So it is impossible to obtain the permanence and existence of a unique uniformly asymptotically stable positive almost periodic solution of system (5.1) by Theorems 1.1 and 1.2. Therefore, the work in this paper extends the results in paper [6].

6. Conclusion

This paper considers an impulsive differential equation model of plankton allelopathy. By using the comparison theorem and the Lyapunov method of the impulsive differential equations, sufficient conditions which guarantee the permanence and existence of a unique uniformly asymptotically stable positive almost periodic solution of the model are obtained. Proposition 3.2 indicates that the distance θ between impulse points, the values of impulse coefficients h_{ik} $(i = 1, 2, k \in \mathbb{Z})$ and the number A of the impulse points in each interval of length 1 are harm for the permanence of the model. Theorem 3.1 indicates that (F_1) in Theorem 1.1 has no effect on the permanence of the model. Theorem 4.1 indicates that (F_1) and (F_3) in Theorem 1.2 have no effect on the existence of a unique uniformly asymptotically stable positive almost periodic solution of the model. From the proof of Theorem 4.1, we can see that the impulsive effects have no effect on the uniform asymptotical stability of the model. The main results obtained in this paper are completely new and the method used in this paper provides a new method to study the permanence and existence of a unique globally asymptotically stable positive almost periodic solution of the models with impulsive perturbations in biological populations.

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References

[1] Xuezhong HE. *Almost periodic solutions of a competition system with dominated infinite delays.* Tohoku Math. J. (2), 1998, **50**(1): 71–89.

[2] Yonghui XIA, Jinde CAO, Huiying ZHANG, et al. *Almost periodic solutions of n-species competitive system with feedback controls.* J. Math. Anal. Appl., 2004, **294**(2): 503–522.

[3] Xinzhu MENG, Lansun CHEN. *Almost periodic solution of non-autonomous Lotka-Volterra predator-prey dispersal system with delays.* J. Theoret. Biol., 2006, **243**(4): 562–574.

[4] Fengde CHEN, Chunling SHI. *Global attractivity in an almost periodic multi-species nonlinear ecological model.* Appl. Math. Comput., 2006, **180**(1): 376–392.

[5] Xiao LIN, Fengde CHEN. *Almost periodic solution for a Volterra model with mutual interference and Beddington-DeAngelis functional response.* Appl. Math. Comput., 2009, **214**(2): 548–556.

[6] Mengxin HE, Fengde CHEN, Zhong LI. *Almost periodic solution of an impulsive differential equation model of plankton allelopathy.* Nonlinear Anal. Real World Appl., 2010, **11**(4): 2296–2301.

[7] Yongkun LI, Tianwei ZHANG. *Permanence and almost periodic sequence solution for a discrete delay logistic equation with feedback control.* Nonlinear Anal. Real World Appl., 2011, **12**(3): 1850–1864.

[8] Tianwei ZHANG, Yongkun LI, Yuan YE. *Persistence and almost periodic solutions for a discrete fishing model with feedback control.* Commun. Nonlinear Sci. Numer. Simul., 2011, **16**(3): 1564–1573.

[9] Tianwei ZHANG, Yongkun LI, Yuan YE. *On the existence and stability of a unique almost periodic solution of Schoener's competition model with pure-delays and impulsive effects.* Commun. Nonlinear Sci. Numer. Simul., 2012, **17**(3): 1408–1422.

[10] Tianwei ZHANG, Xiaorong GAN. *Existence and permanence of almost periodic solutions for Leslie-Gower predator-prey model with variable delays.* Electron. J. Differential Equations, 2013, **105**: 1–21.

[11] Tianwei ZHANG, Xiaorong GAN. *Almost periodic solutions for a discrete fishing model with feedback control and time delays*. Commun. Nonlinear Sci. Numer. Simul., 2014, **19**(1): 150–163.

[12] S. ABBAS, M. SEN, M. BANERJEE. *Almost periodic solution of a non-autonomous model of phytoplankton allelopathy*. Nonlinear Dynam., 2012, **67**(1): 203–214.

[13] Zengji DU, Yansen LV. *Permanence and almost periodic solution of a Lotka-Volterra model with mutual interference and time delays*. Appl. Math. Model., 2013, **37**(3): 1054–1068.

[14] V. LAKSHMIKANTHAM, D. D. BAINOV, P. S. SIMEONOV. *Theory of Impulsive Differential Equations*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[15] G. T. STAMOV. *Almost Periodic Solutions of Impulsive Differential Equations*. Springer, Heidelberg, 2012.

[16] D. D. BAINOV, P. S. SIMEONOV. *Impulsive Differential Equations: Periodic Solutions and Applications*. John Wiley & Sons, Inc., New York, 1993.

[17] A. M. SAMOILENKO, N. A. PERESTYUK. *Impulsive Differential Equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1995.