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The Construction of I-Bornological Vector Spaces

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Abstract In this paper, the concept of I-bornological vector spaces and two examples of the spaces are given. Two methods on constructing new I-bornological vector spaces are discussed, one is using a (crisp) bornological vector space to induce an I-bornological vector space, the other is utilizing I-bornological linear maps to generate an I-bornological vector space.

Keywords I-vector bornology; I-bornological vector space; I-bornological linear map

MR(2010) Subject Classification 46S40; 46A17

1. Introduction

The theory of bornological spaces was first introduced by Hogbe-Nlend [1] (It should be noticed that, the notion of bornological vector spaces is different from that in [2,3], the former [1] is a space without topology, however, the latter [2] is a space with topology). Since then, many authors have carried out various work on bornological spaces. They investigated lattice-valued bornological spaces [4], dicussed topologies of strong uniform convergence on bornologies [5–8], studied bornologies for metrically generated theories [9], established Ekeland-type variational principle and Caristi fixed point theorem in bornological vector spaces [10]. Nevertheless, the work on the construction of fuzzy bornological vector spaces was rarely discussed. In this paper, we introduce the notion of fuzzy bornological vector spaces. Based on that, we investigate the method of constructing new fuzzy bornological vector spaces.

It should be mentioned here that, according to the standardized terminology in [11], fuzzy set [12] should be called an $I(=[0,1])$-valued map. For convenience, we rename the fuzzy bornological vector space as $IBVS$ in the following paper. Our purpose is three-fold. First, we introduce the concept of $IBVS$. For specifically illustrating the definition, we show two examples of $IBVS$. By using Q-neighborhood [13], we prove the family consisting of all I-bounded subset [14,15] is an I-vector bornology; By employing the notion of I-semi-norm [16], we give the concept of I-λ-bounded subset for the I-semi-norm, and obtain that the fuzzy subsets which are I-λ-bounded subset for the I-semi-norm form an I-vector bornology. Thus we obtain the concrete

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$IBVSs$. Next, in consideration of the relation between r-cut sets and I-valued maps, we give special I-valued maps, which eventually induce an I-vector bornology. Further by employing an r-cut vector bornological space, we prove a characterization theorem of the induced $IBVS$. Finally, by using the fuzzy linear maps [17], we consider the I-bornological linear maps and prove that the primary image under inverse projective map about a family of $IBVS$ is still an $IBVS$, which provides another method of constructing $IBVS$.

2. Preliminaries

Throughout this paper, let X be a vector space over $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$ and θ denote the zero element of X. Let $I=[0,1]$ and I^X denote a family of all fuzzy subsets of X. A fuzzy subset which takes the constant value r on X $(0\le r\le 1)$ is denoted by r^*. A fuzzy subset of X is called a fuzzy point [13], denoted by x_λ, if it takes value 0 at $y\in X\backslash\{x\}$ and its value at x is λ. The set of all fuzzy points on X is denoted by $Pt(I^X)$. A fuzzy point x_λ is said to be quasi-coincident with a fuzzy subset A, denoted by $x_\lambda\widetilde{\in}A$, if $A(x)>1-\lambda$. $\mathscr{A}$ is a non-empty set.

Definition 2.1 ([18,19]) *A stratified I-topology τ on X is said to be an I-vector topology, if the following two mappings are continuous:*

$$f:\quad X\times X\to\quad X\qquad (x,y)\mapsto\quad x+y,$$

$$g:\quad \mathbb{K}\times X\to\quad X\qquad (k,x)\mapsto\quad kx,$$

where $\mathbb{K}$ is equipped with the I-topology induced by the usual topology, $X\times X$ and $\mathbb{K}\times X$ are equipped with the corresponding product I-topologies. A vector space X with an I-vector topology τ, denoted by (X,τ), is called an I-topological vector space.

Definition 2.2 ([18,19]) *Let $A,B\in I^X$ and $k\in\mathbb{K}$. Then $A+B$ and kA are defined respectively by*

$$(A+B)(x)=\bigvee\{A(s)\wedge B(t): s+t=x\};$$

$$(kA)(x)=A(x/k)\quad whenever\quad k\neq 0;$$

$$(0A)(x)=\begin{cases}\bigvee_{t\in X}A(t), & x=\theta,\\ 0, & x\neq\theta.\end{cases}$$

In particular, for $x_\lambda,y_\mu\in Pt(I^X)$, we have

$$x_\lambda+y_\mu=(x+y)_{\lambda\wedge\mu},\quad kx_\lambda=(kx)_\lambda.$$

Definition 2.3 ([1]) *A vector bornology on X is a collection $\mathfrak{B}$ of subsets of X which satisfies the following conditions:*

(B1) $X=\bigcup_{B\in\mathfrak{B}}B$;

(B2) $B_1\subseteq B_2$ *and* $B_2\in\mathfrak{B}$ *implies that* $B_1\in\mathfrak{B}$;

(B3) $B_1,B_2\in\mathfrak{B}$ *implies that* $B_1\cup B_2\in\mathfrak{B}$;

(B4) $B_1, B_2 \in \mathfrak{B}$ *implies that* $B_1 + B_2 \in \mathfrak{B}$, *where*

$$B_1 + B_2 = \{x_1 + x_2 : x_1 \in B_1, x_2 \in B_2\};$$

(B5) *For any* $\lambda \in \mathbb{K}$, $B \in \mathfrak{B}$ *implies that* $\lambda B \in \mathfrak{B}$ *and* $\bigcup_{|\lambda|\le 1} \lambda B \in \mathfrak{B}$.

The ordered pair $(X, \mathfrak{B})$ *is called a bornological vector space (briefly* BVS*) and every candidate of* $\mathfrak{B}$ *is called a vector bornological subset.*

It is easy to prove the following conclusion:

Remark 2.4 For any $r \in [0, 1)$ and $A \in I^X$, $\sigma_r(A) = \{x : A(x) > r\}$.

(1) If $A_\alpha \in I^X, \alpha \in \mathscr{A}$, where $\mathscr{A}$ denotes indicator set, then $\sigma_r(\bigcup_{\alpha\in\mathscr{A}} A_\alpha) = \bigcup_{\alpha\in\mathscr{A}} \sigma_r(A_\alpha)$;

(2) If $A, B \in I^X$, then $\sigma_r(A + B) = \sigma_r(A) + \sigma_r(B)$;

(3) For $r \in (0, 1]$, $\sigma_r(A) = r\sigma_r(\frac{1}{r}A)$;

(4) For any $\lambda \in \mathbb{K}$ and $A \in I^X$, $\lambda\sigma_r(A) = \sigma_r(\lambda A)$.

We just prove (4). It is not difficult to prove that when $\lambda \ne 0$, $\lambda\sigma_r(A) = \sigma_r(\lambda A)$. So, we just need to prove the conditions when $\lambda = 0$. For each $x \in \sigma_r(A)$, we have $0A(0x) = 0A(\theta) = \sup_{t\in X} A(t) \ge A(x) > r$, which means $0x \in \sigma_r(0A)$, thus $0\sigma_r(A) \subset \sigma_r(0A)$. Conversely, for each $y \in \sigma_r(0A)$, we have $0A(y) > r$, so $y = \theta = 0\sigma_r(A)$, which shows that $\sigma_r(0A) \subset 0\sigma_r(A)$. □

3. IBVS and examples

In this section, we introduce the notion of an $IBVS$ and give two examples of $IBVS$.

Definition 3.1 *An I-vector bornology on* X *is a family* $\mathfrak{IB}$ *of fuzzy subsets of* X ($\mathfrak{IB} \subset I^X$) *satisfying the following conditions:*

(IB1) $X = \bigcup_{B^*\in\mathfrak{IB}} B^*$, *i.e.*, $\sup_{B^*\in\mathfrak{IB}} B^*(x) = 1, \forall\, x \in X$;

(IB2) $B_1^* \subset B_2^*$ *and* $B_2^* \in \mathfrak{IB}$ *implies that* $B_1^* \in \mathfrak{IB}$;

(IB3) $B_1^*, B_2^* \in \mathfrak{IB}$ *implies that* $B_1^* \cup B_2^* \in \mathfrak{IB}$;

(IB4) $B_1^*, B_2^* \in \mathfrak{IB}$ *implies that* $B_1^* + B_2^* \in \mathfrak{IB}$, *where*

$$(B_1^* + B_2^*)(x) = \sup_{s+t=x} \min\{B_1^*(s), B_2^*(t)\}, \quad \forall x \in X;$$

(IB5) *For any* $\lambda \in \mathbb{K}$, $B^* \in \mathfrak{IB}$ *implies that* $\lambda B^* \in \mathfrak{IB}$ *and* $\bigcup_{|\lambda|\le 1} \lambda B^* \in \mathfrak{IB}$, *where*

$$(\lambda B^*)(x) = \begin{cases} B^*(x/\lambda), & \lambda \ne 0 \\ (0B^*)(x) = \begin{cases} \bigvee_{t\in X} B^*(t), & x = \theta \\ 0, & x \ne \theta \end{cases} \end{cases}$$

for each $x \in X$. *Every candidate of* $\mathfrak{IB}$ *is called an I-vector bornological subset. If* $\mathfrak{IB} \subset I^X$ *is an I-vector bornology on* X, *then the ordered pair* $(X, \mathfrak{IB})$ *is called an I-bornological vector space (briefly* $IBVS$*).*

Let X be a (crisp) topological vector space. A bounded subset of X is a subset that is absorbed by every neighborhood of zero. The collection of bounded subsets of X forms a vector bornology on X called the Vov Neumann Bornology of X (see [1]). Corresponding to the bounded subsets in (crisp) topological vector space, there is an I-bounded subset defined by Q-neighborhood in I-topological vector space (X,τ) (A fuzzy subset U of X is called Q-neighborhood of x_λ iff there exists $G\in\tau$ such that $x_\lambda\widetilde{\in}G\subset U$ (see [13]). Now, as a reference we mention the definition of the I-bounded subset. Then we prove the I-bounded subset defined by Q-neighborhood is an I-vector bornological subset. We refer to [14,19,20] for other symbols or definitions which are not mentioned here.

Definition 3.2 ([14,15]) *A fuzzy set B in (X,τ) is said to be λ-bounded ($\lambda\in(0,1]$), if for each Q-neighborhood U of θ_λ in X, there exist $t>0$ and $r\in(1-\lambda,1]$ such that $B\cap r^*\subset tU$. B is said to be I-bounded, if it is λ-bounded for each $\lambda\in(0,1]$. The family consisting of all I-bounded subsets is denoted by $\mathscr{U}$.*

Example 3.3 Let (X,τ) be an I-topological vector space. Then $\mathscr{U}$ is an I-vector bornology.

Proof We need to prove that $\mathscr{U}$ satisfies the conditions (IB1)–(IB5) in Definition 3.1.

(IB1) For any $x\in X$, we have $\{x\}\in\mathscr{U}$, which means $\{x\}$ is an I-bounded subset in the sense of Definition 3.2. In fact, for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, by [21], there exists a balanced and $Q-\lambda$ absorbing subset V such that $V\subset U$, where V is a $Q-\lambda$ absorbing subset, that is, there exists $t>0$ such that $V(tx)>1-\lambda$. Obviously, there exists $\varepsilon>0$ such that $V(tx)>1-\lambda+\varepsilon$, therefore, $tx_{1-\lambda+\varepsilon}\in V$. Notice that $\{x\}\cap(1-\lambda+\varepsilon)^*=x_{1-\lambda+\varepsilon}\in 1/tV\subset 1/tU$, so $\{x\}\in\mathscr{U}$. At the same time, it is easy to see that $\{x\}(x)=1$ for each $x\in X$. Combining it with $\{x\}\in\mathscr{U}$, we obtain that $\sup_{B\in\mathscr{U}}B(x)=1$. Thus $X=\bigcup_{B\in\mathscr{U}}B$.

(IB2) Let $B_1\subset B_2$ and $B_2\in\mathscr{U}$. Then for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, there exist $t>0$ and $r\in(1-\lambda,1]$ such that $B_2\cap r^*\subset tU$. Since $B_1\subset B_2$, $B_1\cap r^*\subset tU$, which means $B_1\in\mathscr{U}$.

(IB3) Let $B_1,B_2\in\mathscr{U}$. Then for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, there exists a balanced $Q-\lambda$ subset V such that $V\subset U$. Since $B_1,B_2\in\mathscr{U}$, there exist $t_1,t_2>0$ and $r_1,r_2\in(1-\lambda,1]$ such that $B_1\cap {r_1}^*\subset t_1V$, $B_2\cap {r_2}^*\subset t_2V$. Taking $r=\min\{r_1,r_2\}$, $t=\max\{t_1,t_2\}$, then we have $(B_1\cup B_2)\cap r^*=(B_1\cap r^*)\cup(B_2\cap r^*)\subset(B_1\cap r_1^*)\cup(B_1\cap r_2^*)\subset t_1V\cup t_2V\subset tV\subset tU$, thus $B_1\cup B_2\in\mathscr{U}$.

(IB4) Let $B_1,B_2\in\mathscr{U}$. Then for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, there exists a balanced $Q-\lambda$ subset V such that $V+V\subset U$ (see [19]). Considering that $B_1,B_2\in\mathscr{U}$, there exist $t_1,t_2>0$ and $r_1,r_2\in(1-\lambda,1]$ such that $B_1\cap {r_1}^*\subset t_1V$, $B_2\cap {r_2}^*\subset t_2V$. Let $r=\min\{r_1,r_2\}$, $t=\max\{t_1,t_2\}$. We have $(B_1+B_2)\cap r^*=(B_1\cap r^*)+(B_2\cap r^*)\subset(B_1\cap r_1^*)+(B_1\cap r_2^*)\subset t_1V+t_2V\subset tV+tV\subset tU$, thus $B_1+B_2\in\mathscr{U}$.

(IB5) Let $B\in\mathscr{U}$. For any $\alpha\in\mathbb{K}$, firstly, we need to prove $\alpha B\in\mathscr{U}$. In fact, for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, there exists a balanced $Q-\lambda$ subset V such that $V\subset U$. Note that $B\in\mathscr{U}$, so there exist $t>0$ and $r\in(1-\lambda,1]$ such that $B\cap r\subset tV$. Thus, we have $(\alpha B)\cap r^*=\alpha(B\cap r^*)\subset\alpha tV\subset(|\alpha t|+1)V\subset(|\alpha t|+1)U$. Next, we prove for each $B\in\mathscr{U}$, $\bigcup_{|\alpha|\le1}\alpha B\in\mathscr{U}$. In fact, for each $\lambda\in(0,1]$ and Q-neighborhood U of θ_λ, there exists a balanced $Q-\lambda$ subset V such that $V\subset U$. Note that $B\in\mathscr{U}$, so there exist $t>0$ and $r\in(1-\lambda,1]$ such that $B\cap r\subset tV$. Thus, we have $(\bigcup_{|\alpha|\le1}\alpha B)\cap r^*=\bigcup_{|\alpha|\le1}\alpha(B\cap r^*)\subset\bigcup_{|\alpha|\le1}\alpha tV\subset tV\subset tU$.

□

In [22], Wu and Fang introduced the definition of fuzzy norms. Then Wu and Ma gave the notion of fuzzy semi-norms [20], and indicate that the fuzzy norm is a special fuzzy semi-norm. Furthermore, in [16], Wu and Ma gave several examples of fuzzy norm on concrete vector spaces such as the vector space $L^w[0,1] = \cap_{p\geq 1} L_p[0,1]$ (where $L_p[0,1] = \{f(t) : f(t)$ is measurable function on $[0,1]$, and satisfies $\int_0^1 |f(t)|^p \mathrm{d}t < \infty\}$). In which, letting $p(f_\lambda) = (\int_0^1 |f(t)|^p \mathrm{d}t)^\lambda$, for each $\lambda \in (0,1]$ and for all $f \in L^w[0,1]$, they proved $p(f_\lambda)$ is a fuzzy norm on $L^w[0,1]$, so it is a fuzzy semi-norm naturally. The examples show that the definition of fuzzy semi-norm in the sense of Wu and Ma is practical. Now we use the definition of I-semi-norm in [16] to introduce another example of I-vector bornology.

Definition 3.4 ([16]) *A mapping $p : Pt(I^X) \to [0,+\infty)$ is called an I-semi-norm on X, if it satisfies the following conditions:*

(1) $p(kx_\lambda) = |k|p(x_\lambda)$ for all $x_\lambda \in Pt(I^X)$ and all $k \in \mathbb{K}$;

(2) $p(x_\lambda + y_\lambda) \leq p(x_\lambda) + p(y_\lambda)$;

(3) $p(x_\lambda)$ is nonincreasing and left continuous for λ.

Definition 3.5 *Let X be a vector space over $\mathbb{K}$, p be an I-semi-norm on X. A fuzzy subset A of X is said to be an I-λ-bounded subset for the I-semi norm p if $p(A) = \sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A\} < \infty$, $\lambda \in (0,1]$. (Appoint $\sup\emptyset = 0$).*

Example 3.6 The fuzzy subsets of X which are I-λ-bounded subsets for the I-semi-norm p form an I-vector bornology on X, $\lambda \in (0,1]$.

Proof Let $\mathscr{U}^* = \{A| \ \ p(A) = \sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A\} < \infty\}$. We need to prove it satisfies the conditions (IB1)–(IB5) in Definition 3.1.

(IB1) For any $x \in X$, it is easy to know that $\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} \{x\}\} = p(x_\lambda) < \infty$, which means that $\{x\} \in \mathscr{U}^*$, so $X = \bigcup_{A\in\mathscr{U}^*} A$.

(IB2) Let $A_1 \subset A_2$ and $A_2 \in \mathscr{U}^*$. Consider the fact that $\{p(x_\lambda) | x_\lambda \widetilde{\in} A_1\} \subset \{p(x_\lambda) | x_\lambda \widetilde{\in} A_2\}$, hence $\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A_1\} \leq \sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A_2\} < \infty$, which shows $A_1 \in \mathscr{U}^*$.

(IB3) Let $A_1, A_2 \in \mathscr{U}^*$. Notice that $\{p(x_\lambda) | x_\lambda \widetilde{\in} (A_1 \cup A_2)\} = \{p(x_\lambda) | x_\lambda \widetilde{\in} A_1\} \cup \{p(x_\lambda) | x_\lambda \widetilde{\in} A_2\}$, hence $\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} (A_1 \cup A_2)\} = \max\{\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A_1\}, \sup\{p(x_\lambda) | x_\lambda \widetilde{\in} A_2\}\} < \infty$, which implies that $A_1 \cup A_2 \in \mathscr{U}^*$.

(IB4) Let $A_1, A_2 \in \mathscr{U}^*$. Notice that

$$\{p(x_\lambda) | x_\lambda \widetilde{\in} (A_1 + A_2)\} \subset \{p(s_\lambda) | s_\lambda \widetilde{\in} A_1\} + \{p(t_\lambda) | t_\lambda \widetilde{\in} A_2\},$$

where $s + t = x$. So

$$\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} (A_1 + A_2)\} \leq \sup\{p(s_\lambda) | s_\lambda \widetilde{\in} A_1\} + \sup\{p(t_\lambda) | t_\lambda \widetilde{\in} A_2\} < \infty,$$

thus $A_1 + A_2 \in \mathscr{U}^*$.

(IB5) Let $A \in \mathscr{U}^*$. For any $\alpha \in \mathbb{K}$, if $\alpha = 0$, then $\sup\{p(x_\lambda) | x_\lambda \widetilde{\in} \alpha A\} = \sup\{p(\theta_\lambda)\} < \infty$; if $\alpha \neq 0$, then $\{p(x_\lambda) | x_\lambda \widetilde{\in} \alpha A\} = \{p(x_\lambda) | (1/\alpha x)_\lambda \widetilde{\in} \alpha A\} = \{p(\alpha \cdot 1/\alpha x)_\lambda) | (1/\alpha x)_\lambda \widetilde{\in} \alpha A\} =$

$|\alpha|\{p(y_\lambda)|y_\lambda\widetilde{\in}A\}$, obviously, $\alpha A\in\mathscr{U}^*$.

Meanwhile, since

$$\sup\{p(x_\lambda)|x_\lambda\widetilde{\in}\bigcup_{|\alpha|\le 1}\alpha A\}=\sup\{\bigcup_{|\alpha|\le 1}\{p(x_\lambda)|x_\lambda\widetilde{\in}\alpha A\}\}=\sup\{\bigcup_{|\alpha|\le 1}|\alpha|\{p(y_\lambda)|y_\lambda\widetilde{\in}A\}\}<\infty,$$

which means $\bigcup_{|\alpha|\le 1}\alpha A\in\mathscr{U}^*$. □

Remark 3.7 From Examples 3.3 and 3.6, we know easily that $(X,\mathscr{U})$ and $(X,\mathscr{U}^*)$ both are $IBVSs$.

4. Induced IBVS

In this section, we introduce one method of constructing an $IBVS$ and prove a characterization theorem of the induced $IBVS$.

At the beginning, we construct a family $\omega(\mathfrak{B})$ of fuzzy subsets on X, which satisfies that a fuzzy subset $A\in\omega(\mathfrak{B})$ iff $\sigma_r(A)\in\mathfrak{B}$ for each $r\in[0,1)$. Then, we prove the following theorem:

Theorem 4.1 *Let $(X,\mathfrak{B})$ be a BVS. Then $(X,\omega(\mathfrak{B}))$ is an $IBVS$.*

Proof We just need to prove that $\omega(\mathfrak{B})$ satisfies the conditions (IB1)–(IB5) in Definition 3.1.

(IB1) By (B1), we have $X=\bigcup_{A\in\mathfrak{B}}A$, which implies $\sup_{A\in\mathfrak{B}}A(x)=1,\forall x\in X$. From $A=\sigma_r(A)\in\mathfrak{B}$ for any $r\in[0,1)$, we know that $A\in\omega(\mathfrak{B})$. Thus $\sup_{A\in\omega(\mathfrak{B})}A(x)=1$, hence $X=\bigcup_{A\in\omega(\mathfrak{B})}A$.

(IB2) Let $A_1\subset A_2$ and $A_2\in\omega(\mathfrak{B})$. Thus $\sigma_r(A_2)\in\mathfrak{B}$ for each $r\in[0,1)$. Meanwhile, note that $\sigma_r(A_1)\subset\sigma_r(A_2)$, we have $\sigma_r(A_1)\in\mathfrak{B}$ for every $r\in[0,1)$, which means that $A_1\in\omega(\mathfrak{B})$.

(IB3) Let $A_1,A_2\in\omega(\mathfrak{B})$. Then $\sigma_r(A_1),\sigma_r(A_2)\in\mathfrak{B}$ for each $r\in[0,1)$. From (B3) we obtain that $\sigma_r(A_1)\cup\sigma_r(A_2)\in\mathfrak{B}$. Note that $\sigma_r(A_1\cup A_2)=\sigma_r(A_1)\cup\sigma_r(A_2)$. Thus $\sigma_r(A_1\cup A_2)\in\mathfrak{B}$ for each $r\in[0,1)$, which implies that $A_1\cup A_2\in\omega(\mathfrak{B})$.

(IB4) Let $A_1,A_2\in\omega(\mathfrak{B})$. Then $\sigma_r(A_1),\ \ \sigma_r(A_2)\in\mathfrak{B}$ for each $r\in[0,1)$. From (B4) we have $\sigma_r(A_1)+\sigma_r(A_2)\in\mathfrak{B}$. Note that $\sigma_r(A_1+A_2)=\sigma_r(A_1)+\sigma_r(A_2)$. Thus $\sigma_r(A_1+A_2)\in\mathfrak{B}$ for each $r\in[0,1)$, which implies that $A_1+A_2\in\omega(\mathfrak{B})$.

(IB5) Let $A\in\omega(\mathfrak{B})$. Then $\sigma_r(A)\in\mathfrak{B}$ for each $r\in[0,1)$. For any $\lambda\in\mathbb{K}$, by Remark 2.4 and (B5), we have $\lambda\sigma_r(A)=\sigma_r(\lambda A)$ and $\sigma_r(\lambda A)\in\mathfrak{B}$ for any $r\in[0,1)$, which implies that $\lambda A\in\omega(\mathfrak{B})$. Simultaneously, by Remark 2.4 and (B5), we have

$$\sigma_r(\bigcup_{|\lambda|\le 1}\lambda A)=\bigcup_{|\lambda|\le 1}\sigma_r(\lambda A)=\bigcup_{|\lambda|\le 1}\lambda\sigma_r(A),$$

and $\sigma_r(\bigcup_{|\lambda|\le 1}\lambda A)\in\mathfrak{B}$ for any $r\in[0,1)$, which implies that $\bigcup_{|\lambda|\le 1}\lambda A\in\omega(\mathfrak{B})$.

Thus from Definition 4.1, we know that $(X,\omega(\mathfrak{B}))$ is an $IBVS$. □

Remark 4.2 From Theorem 4.1, we know that a vector bornology $\mathfrak{B}$ can induce an I-vector bornology $\omega(\mathfrak{B})$, where $\omega(\mathfrak{B})$ is called an induced I-vector bornology, and $(X,\omega(\mathfrak{B}))$ is called an induced $IBVS$.

Lemma 4.3 *Let $(X, \mathfrak{IB})$ be an $IBVS$. Then $(X, \mathfrak{IB}^r)$ is a BVS for each $r \in [0,1)$, where $\mathfrak{IB}^r = \{\sigma_r(B^*) : B^* \in \mathfrak{IB}\}$.*

Proof We need to prove that for each $r \in [0,1)$, $\mathfrak{IB}^r$ satisfies the conditions (B1)–(B5) in Definition 2.3.

(B1) Obviously, when $(X, \mathfrak{IB})$ is an $IBVS$, from (IB1), we have for each $x \in X$ and $r \in [0,1)$, there exists $B^* \in \mathfrak{IB}$, such that $B^*(x) > r$, hence $x \in \sigma_r(B^*)$, which implies that $X \subset \bigcup_{B^* \in \mathfrak{B}^*} \sigma_r(B^*)$ $(r \in [0,1))$, and so $X = \bigcup_{B^* \in \mathfrak{IB}} \sigma_r(B^*)$.

(B2) Let $B_1 \subset B_2$ and $B_2 \in \mathfrak{IB}^r$. Then there exists $B_2^* \in \mathfrak{IB}$, such that $B_2 = \sigma_r(B_2^*)$. When $r \in (0,1)$, we put

$$B_1^*(x) = \begin{cases} (B_2^*(x) + r)/2, & x \in B_1; \\ r - \varepsilon_0, & x \in B_2 \setminus B_1; \\ B_2^*(x), & x \in X \setminus B_2, \end{cases}$$

where $\varepsilon_0 \in (0, r)$. If $x \in B_1$, it is easy to know that $B_2^*(x) > (B_2^*(x) + r)/2 = B_1^*(x)$. If $x \in B_2 \setminus B_1$, $B_2^*(x) > r > r - \varepsilon_0$. If $x \in X \setminus B_2$, $B_1^*(x) = B_2^*(x)$. Therefore, $B_1^*(x) \leq B_2^*(x)$ for any $x \in X$, so $B_1^* \subset B_2^*$. By (IB2), we obtain that $B_1^* \in \mathfrak{IB}$. Meanwhile, it is not difficult to prove that $B_1 = \sigma_r(B_1^*)$. In fact, for each $x \in B_1$, we have $B_1^*(x) = (B_2^*(x) + r)/2 > r$, which means $x \in \sigma_r(B_1^*)$, thus $B_1 \subset \sigma_r(B_1^*)$; Conversely, for each $x \in \sigma_r(B_1^*)$, we have $B_1^*(x) > r$, which implies that $x \in B_1$, or if $x \overline{\in} B_1$, by the definition of B_1^*, $B_1^*(x) < r$. So $\sigma_r(B_1^*) \subset B_1$.

When $r = 0$, we put

$$B_1^*(x) = \begin{cases} B_2^*(x)/2, & x \in B_1 \\ 0, & x \in X \setminus B_1. \end{cases}$$

Repeat the above proof, we could obtain same conclusion, that is $B_1 \in \mathfrak{IB}^r$ for $r \in [0,1)$.

(B3) Let $B_1, B_2 \in \mathfrak{IB}^r$. Then there exist $B_1^*, B_2^* \in \mathfrak{IB}$, such that $B_1 = \sigma_r(B_1^*), B_2 = \sigma_r(B_2^*)$. By (IB3), we have $B_1^* \cup B_2^* \in \mathfrak{IB}$. Meanwhile, by Remark 2.4, we have $B_1 \cup B_2 = \sigma_r(B_1^*) \cup \sigma_r(B_2^*) = \sigma_r(B_1^* \cup B_2^*)$, so we have $B_1 \cup B_2 \in \mathfrak{IB}^r$.

(B4) Let $B_1, B_2 \in \mathfrak{IB}^r$. Then there exist $B_1^*, B_2^* \in \mathfrak{IB}$, such that $B_1 = \sigma_r(B_1^*), B_2 = \sigma_r(B_2^*)$. By Remark 2.4, we have $\sigma_r(B_1^*) + \sigma_r(B_2^*) = \sigma_r(B_1^* + B_2^*)$. Considering $B_1^* + B_2^* \in \mathfrak{IB}$, thus we obtain that $B_1, +B_2 \in \mathfrak{IB}^r$.

(B5) Let $B \in \mathfrak{IB}^r$. Then there exists $B^* \in \mathfrak{IB}$, such that $B = \sigma_r(B^*)$. For any $\lambda \in \mathbb{K}$, by Remark 2.4 and (IB5), we obtain that $\lambda B = \lambda \sigma_r(B^*) = \sigma_r(\lambda B^*)$, and $\lambda B^* \in \mathfrak{IB}$, which implies that $\lambda B \in \mathfrak{IB}^r$.

Likewise, by Remark 2.4, we have

$$\bigcup_{|\lambda| \leq 1} \lambda B = \bigcup_{|\lambda| \leq 1} \lambda \sigma_r(B^*) = \bigcup_{|\lambda| \leq 1} \sigma_r(\lambda B^*) = \sigma_r(\bigcup_{|\lambda| \leq 1} \lambda B^*).$$

Note that $\bigcup_{|\lambda| \leq 1} \lambda B^* \in \mathfrak{IB}$, so $\bigcup_{|\lambda| \leq 1} \lambda B \in \mathfrak{IB}^r$.

Therefore, for each $r \in [0,1)$, $(X, \mathfrak{IB}^r)$ is a BVS. $\square$

Then, we will show the characterization theorem of the induced $IBVS$.

Theorem 4.4 *Let $(X, \mathfrak{IB})$ be an $IBVS$, $\mathfrak{B}$ be a vector bornology, and $\omega(\mathfrak{B})$ be an induced*

I-vector bornology. Then $\mathfrak{IB} = \omega(\mathfrak{B})$ if and only if $\mathfrak{B} \subset \mathfrak{IB}$ and $\mathfrak{IB}^r = \mathfrak{B}$ for each $r \in [0,1)$.

Proof Necessary. Let $\mathfrak{IB} = \omega(\mathfrak{B})$. Now, we prove $\mathfrak{B} \subset \mathfrak{IB}$. In fact, if $B \in \mathfrak{B}$, notice that B is a crisp set, then $\sigma_r(B) = B \in \mathfrak{B}$ for each $r \in [0,1)$, which means $B \in \omega(\mathfrak{B})$. Hence, $\mathfrak{B} \subset \mathfrak{IB} = \omega(\mathfrak{B})$.

Next, we need to prove $\mathfrak{IB}^r = \mathfrak{B}$ for each $r \in [0,1)$. In fact, it is obvious from the above proof that $\mathfrak{B} \subset \mathfrak{IB}^r$. Conversely, if $B \in \mathfrak{IB}^r$, then there exists an I-vector bornological subset $B^* \in \mathfrak{IB}$ such that $B = \sigma_r(B^*)$. Considering that $\mathfrak{IB} = \omega(\mathfrak{B})$, then $B \in \mathfrak{B}$, so $\mathfrak{IB}^r \subset \mathfrak{B}$ for each $r \in [0,1)$.

Sufficiency. If $B \in \mathfrak{IB}$, from Lemma 4.3 and $\mathfrak{IB}^r = \mathfrak{B}$, we have $\sigma_r(B) \in \mathfrak{B}$ for each $r \in [0,1)$. It implies that $B \in \omega(\mathfrak{B})$, so $\mathfrak{IB} \subset \omega(\mathfrak{B})$. If $B \in \omega(\mathfrak{B})$, we know that $\sigma_r(B) \in \mathfrak{B} \subset \mathfrak{IB}$ for each $r \in [0,1)$. Meanwhile, from the decomposition theorem of a fuzzy set and Remark 2.4 (see [15]), we have

$$B = \bigcup_{r\in[0,1]} [r^* \cap \sigma_r(B)] \subset \bigcup_{r\in[0,1]} \sigma_r(B) = \sigma_0(B) \cup \bigcup_{r\in(0,1]} r\sigma_r(\frac{1}{r}B).$$

Noting that $\omega(\mathfrak{B})$ is an I-vector bornology. So for $r \in (0,1)$, $\frac{1}{r}B \in \omega(\mathfrak{B})$, naturally, $\sigma_r(\frac{1}{r}B) \in \mathfrak{B} \subset \mathfrak{IB}$. Thus, by (IB5) and (IB2) in Definition 3.1, $B \in \mathfrak{IB}$, which means $\omega(\mathfrak{B}) \subset \mathfrak{IB}$. So, $\mathfrak{IB} = \omega(\mathfrak{B})$. $\square$

5. An IBVS generated by I-bornological linear map

In this section, we will give the other method for constructing new $IBVS$ by using I-bornological linear maps. For this, we will introduce the definition of fuzzy linear map due to Katsaras and Liu [17] firstly.

Definition 5.1 *Let X, Y be two vector spaces over $\mathbb{K}$. Then f is called a fuzzy linear map from X into Y (where f is a fuzzy map extended by the general map $\widetilde{f}$, see [20]), if for any $A, B \in I^X$, $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B), \alpha, \beta \in \mathbb{K}$.*

By using the definition of fuzzy linear map, we give:

Definition 5.2 *Let $(X, \mathfrak{IB})$ and $(Y, \mathfrak{ID})$ be two I-vector bornological spaces. If f is a fuzzy linear map of X into Y and for each $B^* \in \mathfrak{IB}$, $f(B^*) \in \mathfrak{ID}$. Then f is called an I-bornological linear map.*

Obviously, the identity map of any $IBVS$ is an I-bornological linear map. As another example of the I-bornological linear map, and from [1] and Theorem 4.1, we have:

Example 5.3 The linear map from $IBVS$ $(X, \mathfrak{IB})$ into the scalar field $\mathbb{K}$ is an I-bornological linear map, where the scalar field $\mathbb{K}$ is endowed with the I-vector bornology defined by $\mathscr{U}$,

$$\mathscr{U} = \{A : \mathbb{K} \longrightarrow [0,1] | \sigma_r(A) \subset \mathbb{K} \ \text{ and } \ \sup\{|b| : b \in \sigma_r(A)\} < \infty\}.$$

Theorem 5.4 *Let $(X, \mathfrak{IB})$, $(Y, \mathfrak{ID})$ and $(Z, \mathfrak{IC})$ be three I-vector bornological spaces, and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two I-bornological linear maps. Then $g \circ f$ is an I-bornological* *linear map.*

An I-vector bornology $\mathfrak{IB}_1$ on X is a Finer Bornology than an I-vector bornology $\mathfrak{IB}_2$ on X (or $\mathfrak{IB}_2$ is a Coarser Bornology than $\mathfrak{IB}_1$) if $\mathfrak{IB}_1 \subset \mathfrak{IB}_2$. It is not difficult to prove the following result:

Theorem 5.5 *The identity map $(X, \mathfrak{IB}_1) \longrightarrow (X, \mathfrak{IB}_2)$ is an I-bornological linear map if and only if $\mathfrak{IB}_1 \subset \mathfrak{IB}_2$.*

It is easy to prove or obtain from [20] the following result:

Lemma 5.6 *Let X, Y be two non-empty sets, $f : X \to Y$ and $A, B \in I^X$. Then*

(1) $A \subset B \Rightarrow f(A) \subset f(B)$;

(2) $f(\bigcup_{\alpha\in\mathscr{A} A_\alpha}) = \bigcup_{\alpha\in\mathscr{A}} f(A_\alpha)$.

Then we show the method for obtaining a new $IBVS$:

Theorem 5.7 *Let $(X_\alpha, \mathscr{U}_\alpha)(\alpha \in \mathscr{A})$ be a family of I-vector bornological spaces, and X be a vector space. Suppose that, for every $\alpha \in \mathscr{A}$, $f_\alpha : X \longrightarrow X_\alpha$ is a linear map. If the family $\mathscr{U}$ of all fuzzy subsets A on X has the following property: Every $A \in \mathscr{U}$ iff $f_\alpha(A) \in \mathscr{U}_\alpha$ for each $\alpha \in \mathscr{A}$. Then $\mathscr{U}$ is an I-vector bornology on X.*

Proof Using the conditions (IB1)–(IB5) in Definition 3.1 and Lemma 5.6, we can easily obtain the result, so we just prove the conditions (IB1) and (IB3).

(IB1) For each $x \in X$ and $\alpha \in \mathscr{A}$, $f_\alpha(x) \in X_\alpha$. Since $\mathscr{U}_\alpha$ is an I-vector bornology, by (IB1) in Definition 3.1, we know that $f_\alpha(x) \in \mathscr{U}_\alpha$, thus $\{x\} \in \mathscr{U}$, which means $X = \bigcup_{A\in\mathscr{U}} A$.

(IB3) Let $A_1, A_2 \in \mathscr{U}$. By Lemma 5.6, we know $f_\alpha(A_1 \cup A_2) = f_\alpha(A_1) \cup f_\alpha(A_2)$. Since $f_\alpha(A_1), f_\alpha(A_2) \in \mathscr{U}_\alpha$, $f_\alpha(A_1) \cup f_\alpha(A_2) \in \mathscr{U}_\alpha$, which implies $A_1 \cup A_2 \in \mathscr{U}$.

It follows that the theorem is proved. $\Box$

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