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A Note on Almost Completely Regular Spaces and c-Semistratifiable Spaces

Lianhua FANG[1], **Lihong XIE**[2,*], **Kedian LI**[3]
1. *Department of Public-courses Teaching, Quanzhou Institute of Information Engineering, Fujian* 362000*, P. R. China;*
2. *School of Mathematics and Computational Science, Wuyi University, Guangdong* 529020*, P. R. China;*
3. *Department of Mathematics, Minnan Normal University, Fujian* 363000*, P. R. China*

Abstract In this paper, we give some characterizations of almost completely regular spaces and c-semistratifiable spaces (CSS) by semi-continuous functions. We mainly show that: (1) Let X be a space. Then the following statements are equivalent:

(i) X is almost completely regular.

(ii) Every two disjoint subsets of X, one of which is compact and the other is regular closed, are completely separated.

(iii) If $g, h : X \to \mathbb{I}$, g is compact-like, h is normal lower semicontinuous, and $g \leq h$, then there exists a continuous function $f : X \to \mathbb{I}$ such that $g \leq f \leq h$;

and (2) Let X be a space. Then the following statements are equivalent:

(a) X is CSS;

(b) There is an operator U assigning to a decreasing sequence of compact sets $(F_j)_{j\in\mathbb{N}}$, a decreasing sequence of open sets $(U(n,(F_j)))_{n\in N}$ such that

(b1) $F_n \subseteq U(n,(F_j))$ for each $n \in \mathbb{N}$;

(b2) $\bigcap_{n\in\mathbb{N}} U(n,(F_j)) = \bigcap_{n\in\mathbb{N}} F_n$;

(b3) Given two decreasing sequences of compact sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n,(F_j)) \subseteq U(n,(E_j))$ for each $n \in \mathbb{N}$;

(c) There is an operator $\Phi : \mathrm{LCL}(X,\mathbb{I}) \to \mathrm{USC}(X,\mathbb{I})$ such that, for any $h \in \mathrm{LCL}(X,\mathbb{I})$, $0 \leqslant \Phi(h) \leqslant h$, and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$.

Keywords almost completely regular spaces; CSS; semi-continuous functions

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1. Introduction

By a space we mean a topological space. Since 1920's, the problem that, for a given pair of real-valued functions (g, h) on a space X, under what conditions there is a continuous function f such that $g \leqslant f \leqslant h$ has been investigated extensively. A canonical example is provided by the

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* Corresponding author
E-mail address: lotus35@163.com (Lianhua FANG); yunli198282@126.com (Lihong XIE); likd56@126.com (Kedian LI)

Katětov-Tong-Hahn insertion theorem for normal spaces [1–3]. Recall that, given a topological space X, a function $f : X \to \mathbb{R}$ is lower [upper] semi-continuous if $f^{-1}(t,\infty)[f^{-1}(-\infty,t)]$ is open for each $t \in \mathbb{R}$.

Theorem 1.1 (Katětov-Tong-Hahn) *Let X be a space. Then the following statements are equivalent:*

(i) X is normal;

(ii) If $g,h : X \to \mathbb{R}$, g is upper semi-continuous, h is lower semi-continuous, and $g \le h$, then there exists a continuous function $f : X \to \mathbb{R}$ such that $g \le f \le h$.

More examples can be seen in [4]. In 2006, Yan and Yang [5] investigated the relations between the insertion of semi-continuous functions and semi-stratifiable structure of spaces and presented some characterizations of semi-stratifiable spaces and perfect spaces. More about the relations between semi-continuous functions and sequences of sets can be found in [6].

In 2007, Sandwich-type characterization of completely regular spaces was given in [7]. In this note we give an insertion-type characterization of almost completely regular spaces. Recall that a space X is called almost completely regular if any point x and regular closed set F such that $x \notin F$ is completely separated. Also, recall that two subsets A and B of a space X are called completely separated if there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$. Finally, insertion-type characterization of CSS by lower compact-like functions is given.

Before stating the specific results of this paper, we introduce some notions. The following list of notations seems to be convenient though they may be found in the references: $C(X,\mathbb{I}), \mathrm{USC}(X,\mathbb{I})$ and $\mathrm{LSC}(X,\mathbb{I})$ represent the set of all continuous, upper semi-continuous and lower semi-continuous functions, respectively, from X to $\mathbb{I} = [0,1]$. By "$g \leqslant h$ $(g < h)$", we mean $g(x) \leqslant h(x)$ $(g(x) < h(x))$ for all $x \in X$. For any function $f : X \to \mathbb{R}$ and $t \in \mathbb{R}$, denote by $[f \ge t] = \{x \in X | f(x) \ge t\}$ (resp., $[f \le t] = \{x \in X | f(x) \le t\}$).

Let X be a space. For any subset $A \subseteq X$, we write χ_A for the characteristic function on A, that is, a function $\chi_A : X \to [0,1]$ defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

2. Main results

Recall that an open set U of X is regular open, if $\mathrm{Int}\,\overline{U} = U$. Similarly, a closed set F of X is regular closed, if $\overline{\mathrm{Int}}\,F = F$. To characterize almost completely regular spaces, we need the following definition.

Definition 2.1 *Let X be a space. A function $f : X \to \mathbb{R}$ is normal lower [normal upper] semi-continuous if $f^{-1}(t,\infty)$ $[f^{-1}(-\infty,t)]$ is a regular open set for each $t \in \mathbb{R}$.*

Proposition 2.2 *A space X is almost completely regular if and only if, whenever $U \subseteq X$ is a* *regular open set, there exists an open cover $\mathcal{V}$ of U with the property that for every $V \in \mathcal{V}$ there is an $f_V \in C(X, \mathbb{I})$ such that $\chi_V \leq f_V \leq \chi_U$.*

Proof The only if part: by almost complete regularity, we have $\chi_x \leq g_x \leq \chi_U$ for each $x \in U$, where $g_x \in C(X, \mathbb{I})$. Let $V_x = g_x^{-1}((1/2, 1])$ and $f_{V_x} = \min\{1, 2g_x\}$. Then $\mathcal{V} = \{V_x : x \in U\}$ is an open cover of U and $\chi_{V_x} \leq f_{V_x} \leq \chi_U$. The if part is evident. □

In what follows, $\mathbf{t}$ stands for the constant map on X taking the value $t \in \mathbb{I}$. All the infs and the sups of families of functions are pointwise.

Recall that given a space X, a function $f : X \to \mathbb{I}$ is called compact-like [7] if, given a $t \in \mathbb{I}\backslash\{0\}$ and $\mathcal{K} \subseteq \mathrm{USC}(X, \mathbb{I})$ with $\min(f, \inf \mathcal{K}) < \mathbf{t}$, there exists a finite $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $\min(f, \inf \mathcal{K}_0) < \mathbf{t}$.

Lemma 2.3 ([4,8]) *Let X be a space and $g, h : X \to \mathbb{I}$, be two functions. Then the following statements are equivalent:*

(i) There exists a continuous function $f : X \to \mathbb{I}$ such that $g \leq f \leq h$.

(ii) If $s < t$ in $\mathbb{I}$, then $[g \geq t]$ and $[h \leq s]$ are completely separated.

Theorem 2.4 *Let X be a space. Then the following statements are equivalent:*

(i) X is almost completely regular;

(ii) Every two disjoint subsets of X, one of which is compact and the other is regular closed, are completely separated;

(iii) If $g, h : X \to \mathbb{I}$, g is compact-like, h is normal lower semicontinuous, and $g \leq h$, then there exists a continuous function $f : X \to \mathbb{I}$ such that $g \leq f \leq h$.

Proof (i) ⇒ (ii). Let A be a compact subset of X and B a regular closed subset of X with $A \cap B = \emptyset$. By almost complete regularity, there exist an open cover $\mathcal{U}$ of $X \setminus B$ and a family $\{f_U : U \in \mathcal{U}\} \subseteq C(X, \mathbb{I})$ such that $\chi_U \leq f_U \leq \chi_{X \setminus B}$ by Proposition 2.2. Since A is compact, $A \subseteq \bigcup \mathcal{U}_0$ for a finite $\mathcal{U}_0$ of $\mathcal{U}$. Then $\chi_A \leq g = \sup\{f_U : U \in \mathcal{U}_0\} \leq \chi_{X \setminus B}$. The continuous g completely separates A and B by Lemma 2.3.

(ii) ⇒ (iii). Let $g \leq h$ be as in (iii). For any $s, t \in \mathbb{I}$ with $s < t$ one has $[g \geq t] \cap [h \leq s] = \emptyset$, where $[g \geq t]$ is compact and $[h \leq s]$ is a regular closed. By Lemma 2.3, there is a continuous $f \in C(X, \mathbb{I})$ such that $g \leq f \leq h$.

(iii) ⇒ (i). It is obvious. Indeed, if $x \in U$ with U a regular open, then $\chi_x \leq \chi_U$. Clearly, χ_x is compact-like and χ_U is regular lower semicontinuous. Hence there is a continuous function $f \in C(X, \mathbb{I})$ such that $\chi_x \leq f \leq \chi_U$. This implies that x and $X \setminus U$ are completely separated by Lemma 2.3. Indeed, we just observe that $x \in [\chi_x \geq 1]$ and $X \setminus U \subseteq [\chi_U \leq 0]$. □

Corollary 2.5 *Let X be an almost Tychonoff space, let $A \subseteq X$ be compact such that A is a regular closed set, and let $f : A \to \mathbb{R}$ be continuous. Then there exists a continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof Let A be a compact subset of X and $f : A \to \mathbb{R}$ a continuous function. Clearly, the set $f(A)$ is bounded. Thus we can assume that $f(A) \subseteq \mathbb{I}$. Now, define $g, h : X \to \mathbb{I}$ as follows: $g = f = h$ on A, $g = \mathbf{0}$, and $h = \mathbf{1}$ on $X \setminus A$. Since A is regular closed, h is normal lower semi-continuous. Also, if $t > 0$, then $[g \geq t] = [f \geq t]$ is closed in A, hence compact in X. By Theorem 2.4, there exists a continuous function $F : X \to \mathbb{I}$ with $g \leq F \leq h$. Clearly, F extends f to X. $\square$

Definition 2.6 *Let X be a space. A function $f : X \to \mathbb{I}$ is called lower compact-like [7] if, given a $t \in \mathbb{I}\backslash\{1\}$ and $\mathcal{K} \subseteq \mathrm{LSC}(X, \mathbb{I})$ with $\max(f, \sup \mathcal{K}) > \mathbf{t}$, there exists a finite $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $\max(f, \sup \mathcal{K}_0) > \mathbf{t}$.*

Proposition 2.7 *Let X be a space. Then the following statements hold:*

(i) $f : X \to \mathbb{I}$ is lower compact-like iff $[f \leq t]$ is compact for all $t \in \mathbb{I} \setminus \{1\}$;

(ii) $A \subseteq X$ is compact iff $1_{X\setminus A}$ is lower compact-like;

(iii) If X is compact, then $\mathrm{LSC}(X, \mathbb{I})$ consists of lower compact-like functions;

(iv) If X is Hausdorff and $f : X \to \mathbb{I}$ is lower compact-like, then $f \in \mathrm{LSC}(X, \mathbb{I})$.

Proof For (i), let $\mathcal{U}$ be an open cover of $[f \leq t]$ with $t < 1$, then $\max(f, \sup\{1_U : U \in \mathcal{U}\}) > \mathbf{t}$. However, the finite subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ for which $\max(f, \sup\{1_U : U \in \mathcal{U}_0\}) > \mathbf{t}$ yields $[f \leq t] \subseteq \bigcup \mathcal{U}_0$. Conversely, let $\max(f, \sup \mathcal{K}) > \mathbf{t}$ with $t < 1$ and $\mathcal{K} \subseteq \mathrm{LSC}(X, \mathbb{I})$. Then $\emptyset = [\max(f, \sup \mathcal{K}) \leq t] = [f \leq t] \cap \bigcap_{k\in\mathcal{K}}[k \leq t]$. By the finite intersection property, there exists a finite $\mathcal{K}_0 \subseteq \mathcal{K}$ with $\emptyset = \max(f, \sup \mathcal{K}_0) \leq t] = [f \leq t] \cap \bigcap_{k\in\mathcal{K}_0}[k \leq t]$. This translates into $\max(f, \sup \mathcal{K}_0) > t$ and proves (i). Finally, (ii) follows from (i), while (iii) and (iv) are obvious. $\square$

Definition 2.8 *Let X be a space. X is called a c-semistratifiable space (CSS) [9] if for each compact subset K of X and each $n \in \omega$ there is an open set $G(n, K)$ in X such that:*

(i) $\bigcap_{n\in\omega} G(n, K) = K$;

(ii) $G(n + 1, K) \subset G(n, K)$ for each $n \in \omega$;

(iii) If for any compact subsets K, L of X with $K \subset L$, then $G(n, K) \subset G(n, L)$ for each $n \in \omega$.

In what follows, $\mathrm{LCL}(X, \mathbb{I})$ stands for all lower compact-like functions from X to $\mathbb{I}$.

Theorem 2.9 *Let X be a space. Then the following statements are equivalent:*

(a) X is CSS;

(b) There is an operator U assigning to a decreasing sequence of compact sets $(F_j)_{j\in\mathbb{N}}$, a decreasing sequence of open sets $(U(n, (F_j)))_{n\in N}$ such that

(b1) $F_n \subseteq U(n, (F_j))$ for each $n \in \mathbb{N}$;

(b2) $\bigcap_{n\in\mathbb{N}} U(n, (F_j)) = \bigcap_{n\in\mathbb{N}} F_n$;

(b3) Given two decreasing sequences of compact sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n, (F_j)) \subseteq U(n, (E_j))$ for each $n \in \mathbb{N}$;

(c) There is an operator $\Phi : \mathrm{LCL}(X, \mathbb{I}) \to \mathrm{USC}(X, \mathbb{I})$ such that, for any $h \in \mathrm{LCL}(X, \mathbb{I})$, $0 \leqslant \Phi(h) \leqslant h$, and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$.

Proof (c)$\Rightarrow$ (b). Let $(F_j)_{j\in\mathbb{N}}$ be a decreasing sequence of compact sets in X. Define a function $h_{(F_j)}: X \to \mathbb{R}$ by

$$h_{(F_j)}(x) = \begin{cases} 1, & x \in X - F_1, \\ \frac{1}{n+1}, & x \in F_n - F_{n+1}, \\ 0, & x \in \bigcap_{j\in\mathbb{N}} F_j. \end{cases} \tag{*}$$

By Proposition 2.7 one can easily show that $h_{(F_j)} \in \mathrm{LCL}(X, \mathbb{I})$. By hypothesis, there is an upper semi-continuous function $\Phi(h_{(F_j)}) : X \to \mathbb{I}$ such that, for any $h \in \mathrm{LCL}(X, \mathbb{I})$, $0 \leqslant \Phi(h) \leqslant h$, and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$. Thus we can define an operator U assigning to each decreasing sequence of compact sets $(F_j)_{j\in\mathbb{N}}$ by

$$U((F_j)) = (U(n, (F_j)))_{n\in\mathbb{N}},$$

where $U(n, (F_j)) = \{x \in X | \Phi(h_{(F_j)})(x) < \frac{1}{n}\}$ for each $n \in \mathbb{N}$.

We assert that the operator U satisfies (b1), (b2) and (b3) of (b).

By $\Phi(h_{(F_j)}) \leqslant h_{(F_j)}$, it is easy to see that $U(n, (F_j)) \supseteq F_n$ for each $n \in \mathbb{N}$. This shows that the operator U satisfies (b1).

Take any $x \notin \bigcap_{n\in\mathbb{N}} F_n$. Then we have $h_{(F_j)}(x) > 0$, and therefore, $\Phi(h_{(F_j)})(x) > 0$. This implies that $x \notin \bigcap_{n\in\mathbb{N}} U(n, (F_j))$. Thus we show that the operator U satisfies (b2), i.e., $\bigcap_{n\in\mathbb{N}} U(n, (F_j)) = \bigcap_{n\in\mathbb{N}} U(n, (F_j))$.

Take two decreasing sequences of compact sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ in X such that $F_j \subseteq E_j$ for each $j \in \mathbb{N}$, clearly $h_{(F_j)} \geqslant h_{(E_j)}$, where $h_{(F_j)}$ and $h_{(E_j)}$ are defined by (*). By hypothesis, $\Phi(h_{(F_j)}) \geqslant \Phi(h_{(E_j)})$. Furthermore,

$$U(n, (F_j)) = \{x \in X | \Phi(h_{(F_j)})(x) < \frac{1}{n}\} \subseteq \{x \in X | \Phi(h_{(E_j)})(x) < \frac{1}{n}\} = U(n, (E_j))$$

for each $n \in \mathbb{N}$. This shows that the operator U satisfies (b3).

(b) $\Rightarrow$ (a). Let U_0 be an operator satisfying (b1), (b2) and (b3) in (b). For each compact set F in X, let $F_n = F$ for each $n \in \mathbb{N}$. Clearly, $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact sets. By hypothesis, we can define an operator U assigning to each compact set, a decreasing sequence of open sets by

$$U(F) = (U_0(j, (F_n)))_{j\in\mathbb{N}}.$$

One can easily verify that the operator U satisfies (b1), (b2) and (b3) of Definition 2.8.

(a)$\Rightarrow$ (c). Take $h \in \mathrm{LCL}(X, \mathbb{I})$. Then, by Proposition 2.7, $F_n^h = \{x \in X | h(x) \leq \frac{1}{n+2}\}$ is a compact set for each $n \in \mathbb{N}$. Since X is CSS, there is a decreasing sequence of open sets $(U(n, F_j^h))_{n\in\mathbb{N}}$ for each F_j such that $\bigcap_{n\in\mathbb{N}} U(n, F_j^h) = F_j^h$ and $U(n, F_j^h) \subseteq U(n, F_i^h)$ whenever $i < j$. Then one can easily show that $\bigcap_{n\in\mathbb{N}} U(n, F_n^h) = \bigcap_{j\in\mathbb{N}} F_j^h$. Define a function $\Phi(h) : X \to \mathbb{R}$ by

$$\Phi(h)(x) = \begin{cases} \frac{1}{2}, & x \in X - U(0, F_0^h), \\ \frac{1}{n+3}, & x \in U(n, F_n^h) - U(n+1, F_{n+1}^h), \\ 0, & x \in \bigcap_{n\in\mathbb{N}} U(n, F_n^h). \end{cases} \tag{**}$$

Then one can easily show that $\Phi(h)$ is an upper semi-continuous function and the operator $\Phi : \mathrm{LCL}(X, \mathbb{I}) \to \mathrm{USC}(X, \mathbb{I})$ defined by (**) satisfies conditions of (c). $\square$

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