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An Extension of the Rényi Formula

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Abstract In this paper, as a natural extension of the Rényi formula which counts labeled connected unicyclic graphs, we present a formula for the number of labeled $(k+1)$-uniform $(p,\ q)$-unicycles as follows:

$$U_{p,\ q}^{(k+1)} = \begin{cases} \dfrac{p!}{2[(k-1)!]^q} \cdot \displaystyle\sum_{t=2}^{q} \dfrac{q^{q-t-1} \cdot \mathrm{sgn}(tk-2)}{(q-t)!}, & p = qk, \\ 0, & p \neq qk, \end{cases}$$

where $k,\ p,\ q$ are positive integers.

Keywords Labeled hypergraph; $(p,\ q)$-unicycles; $(k+1)$-uniform; Rényi formula

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1. Introduction

Let $p,\ q$ be positive integers. Let $X = \{x_1, x_2, \ldots, x_p\}$ be a finite set, and let $\mathscr{E} = \{E_i | i = 1, 2, \ldots, q\}$ be a family of subsets of X. Denote by $|X|$ the number of the elements in X. If $E_i \neq \emptyset$ $(1 \leq i \leq q)$, then the couple $H = (X, \mathscr{E})$ is called a hypergraph. Usually, $|X| = p$ is called the order of H, the elements of X are called the vertices of H, and the sets $E_1, E_2, \ldots, E_q$ are called the hyperedges. A hypergraph $H = (X, \mathscr{E})$ with $|X| = p$ and $|\mathscr{E}| = q$ is called a (p, q)-hypergraph.

It is known that a hypergraph $H = (X, \mathscr{E})$ with $|X| = p$ and $|\mathscr{E}| = q$ is corresponding to a bipartite graph $G(H) = (Y_1, Y_2, E)$, where vertex $x_i \in Y_1 = X$ $(i = 1, 2, \ldots, p)$ and vertex $E_j \in Y_2 = \mathscr{E}$ $(j = 1, 2, \ldots, q)$ is adjacent in $G(H)$ if and only if $x_i \in E_j$ in H.

In a hypergraph $H = (X, \mathscr{E})$, two vertices are said to be adjacent if there is a hyperedge E_i that contains both of these vertices; $E_i \in \mathscr{E}$ with $|E_i| = 1$ is called a loop; if $E_i \in \mathscr{E}$ with $|E_i| \geq 2$, and there is only one vertex $v \in E_i$ shared with other hyperedges, then E_i is called a pendant hyperedge; a chain of length t is defined to be a sequence $(x_1, E_1, x_2, E_2, \ldots, E_t, x_{t+1})$ such that

(1) $x_1, x_2, \ldots, x_t$ are all distinct vertices of H,

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(2) $E_1, E_2, \ldots, E_t$ are all distinct hyperedges of H,

(3) $x_i, x_{i+1} \in E_i$ for $i = 1, 2, \ldots, t$.

Moreover, if $t > 1$ and $x_1 = x_{t+1}$, then this chain is called a cycle of length t. If there is a chain in the hypergraph that starts at vertex x and terminates at vertex y, then we shall write $x \equiv y$. It is not difficult to verify that the relation $x \equiv y$ is an equivalence relation, whose classes are called the "connected components" of the hypergraph [1]. A hypergraph with exactly one connected component is called a connected hypergraph.

For a hypergraph $H = (X, \mathscr{E})$, H is called k-uniform if $\forall E_i \in \mathscr{E}$, $|E_i| = k$, where k is a positive integer. If H is connected and contains no cycles, then H is called a hypertree; besides, H is called a unicycle if H is connected and contains exactly one cycle.

The theory of hypergraphs is generalized from graph theory [1–3]. For example, 2-uniform hypertrees (resp., unicycles) are the usual trees (resp., connected unicyclic graphs) from graph theory [2]. Graphical enumeration is interesting, and many mathematicians have studied labeled enumeration problems and obtained a lot of results [4]. However, there are not many results about the enumeration of labeled hypergraphs [5]. In 1980, Hegde and Sridharan [6] presented formulas for the number of labeled k-colored hypergraphs, labeled connected hypergraphs without loops, labeled even hypergraphs, respectively. Furthermore, Liu [7] obtained counting formulas for hypergraphs of order p (resp., (p, q)-hypergraphs) with exactly k vertices of odd degree, which are generalizations of the results in [6]. On the other hand, Mao [8] studied the properties of hypertrees, and conjectured a counting formula for $(k+1)$-uniform (p, q)-hypertrees. In 1988, Liu [9] gave a proof of Mao's conjecture, and concluded that when $k = 1$, such formula is the famous Cayley formula which counts labeled trees [10]. In recent years, there are some papers concerning enumeration of labeled information hypergraphs (which is slightly different from labeled hypergraphs) [11–14].

In this paper, we study the properties of unicycles, and as a natural extension of the Rényi formula which counts labeled connected unicyclic graphs [15], we present a formula for the number of labeled $(k+1)$-uniform (p, q)-unicycles.

2. Counting labeled unicycles

To begin with, we investigate the properties of unicycles.

Lemma 2.1 *A hypergraph $H = (X, \mathscr{E})$ is a unicycle if and only if its corresponding bipartite graph $G(H)$ is a connected unicyclic graph.*

Proof Note that $(x_1, E_1, x_2, E_2, \ldots, x_t, E_t, x_1)$ $(t > 1)$ is the unique cycle of H, if and only if, $(x_1, E_1, x_2, E_2, \ldots, x_t, E_t, x_1)$ (E_i can be seen as a vertex, where $i = 1, 2, \ldots, t$) is the unique cycle of $G(H)$. Moreover, it is easy to see that H is connected if and only if $G(H)$ is connected. $\square$

Lemma 2.2 ([1]) *A connected (p, q)-hypergraph $H = (X, \mathscr{E})$ is a unicycle if and only if $\sum_{i=1}^{q} |E_i| = p + q$.*

Corollary 2.3 *If a connected (p, q)-hypergraph $H = (X, \mathscr{E})$ is a $(k+1)$-uniform unicycle, then $p = qk$.*

Proof It is obvious that $|E_i| = k+1$ $(i = 1, 2, \ldots, q)$. Combining this with Lemma 2.2, we have $p + q = q(k+1)$, and the proof is completed. $\square$

Lemma 2.4 *Let $H = (X, \mathscr{E})$ be a unicycle with $|X| = p$ and $|\mathscr{E}| = q$.*

(1) For any $E_i,\ E_j \in \mathscr{E}$ $(i \neq j)$, we have $|E_i \cap E_j| \leq 2$.

(2) If $|\mathscr{E}| \geq 3$ and there are $E_i, E_j \in \mathscr{E}$ $(i \neq j)$ with $|E_i \cap E_j| = 2$, then there exist pendant hyperedges in H; Conversely, if H contains no pendant hyperedges and $|\mathscr{E}| \geq 3$, then $|E_i \cap E_j| \leq 1$ for any $E_i, E_j \in \mathscr{E}$ $(i \neq j)$.

(3) If H contains no pendant hyperedges, then for any $E_j \in \mathscr{E}$ $(j = 1, 2, \ldots, q)$, $|E_j \cap (\bigcup_{\substack{1 \leq k \leq q \\ k \neq j}} E_k)| = 2$.

Proof (1) By contradiction. Suppose there exist $E_i,\ E_j \in \mathscr{E}$ $(i \neq j)$ such that $|E_i \cap E_j| \geq 3$. Let x, y, z be three different vertices of $E_i \cap E_j$. Then (x, E_i, y, E_j, x) and (x, E_i, z, E_j, x) are two different cycles, a contradiction.

(2) By contradiction. If H contains no pendant hyperedges, then every hyperedge shares at least two vertices with all other hyperedges. Since $|\mathscr{E}| \geq 3$ and H is connected, E_i (or E_j) shares at least three vertices with all other hyperedges. Therefore, we have

$$\begin{aligned}\sum_{x=1}^{q} |E_x| &= |\bigcup_{x=1}^{q} E_x| + \sum_{y=1}^{q-1} |E_y \cap (\bigcup_{z=y+1}^{q} E_z)| \\ &= |\bigcup_{x=1}^{q} E_x| + \frac{1}{2} \sum_{y=1}^{q} |E_y \cap (\bigcup_{\substack{1 \leq z \leq q \\ z \neq y}} E_z)| \\ &\geq p + \frac{1}{2}[2(q-1) + 3] = p + q + \frac{1}{2}.\end{aligned}$$

By Lemma 2.2, the unicycle H satisfies $\sum_{x=1}^{q} |E_x| = p + q$, a contradiction. Hence there exist pendant hyperedges in H. The converse can be proved similarly.

(3) If H contains no pendant hyperedges, then $\forall E_j \in \mathscr{E}$ $(j = 1, 2, \ldots, q)$, $|E_j \cap (\bigcup_{\substack{1 \leq k \leq q \\ k \neq j}} E_k)| \geq 2$. It follows that

$$p + q = \sum_{i=1}^{q} |E_i| = |\bigcup_{i=1}^{q} E_i| + \frac{1}{2} \sum_{j=1}^{q} |E_j \cap (\bigcup_{\substack{1 \leq k \leq q \\ k \neq j}} E_k)| \geq p + \frac{1}{2} \cdot (2q) = p + q,$$

and the equality holds if and only if $|E_j \cap (\bigcup_{\substack{1 \leq k \leq q \\ k \neq j}} E_k)| = 2$ $(j = 1, 2, \ldots, q)$. $\square$

It is known that labeled 2-uniform unicycles are the usual labeled, connected, unicyclic graphs whose counting formula is the Rényi formula [15]. As a natural extension, we shall obtain a counting formula for labeled $(k+1)$-uniform (p, q)-unicycles, where k is a positive integer. Let $U_{p,q}^{(k+1)}$ (resp., $\overline{U}_{p,q}^{(k+1)}$) denote the number of labeled $(k+1)$-uniform (p, q)-unicycles (resp., whose hyperedges are also labeled). It is obvious that $\overline{U}_{p,q}^{(k+1)} = q! U_{p,q}^{(k+1)}$. Moreover, by Corollary 2.3, if $p \neq qk$, then $U_{p,q}^{(k+1)} = 0$; if $p = qk$, then we have the following results.

Lemma 2.5 *Let $q,\ t$ be positive integers. If $q>t$, then*

$$\sum_{j=0}^{q-t}(-1)^{j+1}\binom{q-t}{j}(q-j)^{q-t-1}=0.$$

Proof Let $\Delta^q O^t=\sum_{j=0}^{q}(-1)^j\binom{q}{j}(q-j)^t$. If $q>t$, then $\Delta^q O^t=0$ (see [16]). Hence

$$\begin{aligned}&\sum_{j=0}^{q-t}(-1)^{j+1}\binom{q-t}{j}(q-j)^{q-t-1}\\&=\sum_{j=0}^{q-t}(-1)^{j+1}\binom{q-t}{j}\sum_{i=0}^{q-t-1}\binom{q-t-1}{i}(q-t-j)^i t^{q-t-1-i}\\&=-\sum_{i=0}^{q-t-1}\binom{q-t-1}{i}t^{q-t-1-i}\sum_{j=0}^{q-t}(-1)^j\binom{q-t}{j}(q-t-j)^i\\&=-\sum_{i=0}^{q-t-1}\binom{q-t-1}{i}t^{q-t-1-i}\Delta^{q-t}O^i=0,\end{aligned}$$

where the last equality holds since $\Delta^{q-t}O^i=0$ $(q-t>i)$. $\Box$

Lemma 2.6 *If $p=qk$, then $\overline{U}_{p,\ q}^{(k+1)}$ satisfies the following recurrence:*

$$\overline{U}_{p,\ q}^{(k+1)}=\frac{p!(q-1)!}{2[(k-1)!]^q}+\sum_{j=1}^{q}(-1)^{j+1}\binom{p}{jk}\binom{q}{j}\frac{(jk)!}{(k!)^j}(p-jk)^j\overline{U}_{p-jk,\ q-j}^{(k+1)}. \tag{2.1}$$

Proof By Lemma 2.1, let S be the set of labeled, connected, bipartite unicyclic graphs $G(H)=(Y_1,Y_2,E)$ (corresponding to the set of labeled $(k+1)$-uniform (p,q)-unicycle $H=(X,\mathscr{E})$ whose hyperedges are also labeled). Then $|S|=\overline{U}_{p,q}^{(k+1)}$, $|Y_1|=p$, $|Y_2|=q$, and the degree of each vertex in Y_2 is $(k+1)$.

In $G(H)$, the vertex in Y_2 representing a pendant hyperedge of H is joined to k pendant vertices in Y_1. For convenience, such vertices in Y_2 are called pendant-hyperedge vertices, and those pendant vertices in Y_1 are called match vertices. It is easy to see that the number of match vertices in Y_1 is multiples of k. Denote the vertices of Y_2 by $v_1,v_2,\ldots,v_q$. Let A_i $(i=1,2,\ldots,q)$ be the set of $G(H)$ with v_i as a pendant-hyperedge vertex.

Now we give two methods to calculate the number of $G(H)$ without pendant-hyperedge vertices in Y_2 (equivalent to $G(H)$ without match vertices in Y_1).

On the one hand, by the principle of Inclusion-Exclusion [16], the number of $G(H)$ without pendant-hyperedge vertices in Y_2 is

$$|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_q}|=|S-\cup_{i=1}^{q}A_i|=|S|+\sum_{j=1}^{q}(-1)^j\sum_{1\le i_1<i_2<\cdots<i_j\le q}|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_j}|.$$

Note that $A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_j}$ $(1\le i_1<i_2<\cdots<i_j\le q)$ is the set of $G(H)$ with $v_{i_1},v_{i_2},\ldots,v_{i_j}$ as pendant-hyperedge vertices. Observe that v_{i_t} $(t=1,2,\ldots,j)$ is joined to k match vertices in Y_1, then we shall choose jk match vertices from Y_1, and there are $\binom{p}{jk}\frac{(jk)!}{(k!)^j}$ ways for choosing and joining. Since the pendant-hyperedge vertices $v_{i_1},v_{i_2},\ldots,v_{i_j}$ are all of $(k+1)$ degree, then

each of them should be joined to one of the remaining $(p-jk)$ labeled vertices in Y_1, and there are $(p-jk)^j$ different ways. Moreover, the remaining $(p-jk)$ labeled vertices in Y_1 and $(q-j)$ labeled vertices in Y_2 shall construct labeled, connected, bipartite unicyclic graphs, corresponding to labeled $(k+1)$-uniform $(p-jk, q-j)$-unicycles whose hyperedges are labeled, and the total number is $\overline{U}^{(k+1)}_{p-jk,q-j}$. Therefore,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}| = \binom{p}{jk} \frac{(jk)!}{(k!)^j} (p-jk)^j \overline{U}^{(k+1)}_{p-jk,\ q-j}.$$

And the number of $G(H)$ without pendant-hyperedge vertices in Y_2 is

$$\begin{aligned} &|S| + \sum_{j=1}^{q} (-1)^j \sum_{1 \le i_1 < i_2 < \cdots < i_j \le q} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}| \\ &= \overline{U}^{(k+1)}_{p,\ q} + \sum_{j=1}^{q} (-1)^j \binom{q}{j} \binom{p}{jk} \frac{(jk)!}{(k!)^j} (p-jk)^j \overline{U}^{(k+1)}_{p-jk,\ q-j} \\ &= \sum_{j=0}^{q} (-1)^j \binom{q}{j} \binom{p}{jk} \frac{(jk)!}{(k!)^j} (p-jk)^j \overline{U}^{(k+1)}_{p-jk,\ q-j}. \end{aligned}$$

On the other hand, if $G(H)$ contains no match vertices in Y_1, then its corresponding labeled $(k+1)$-uniform (p,q)-unicycle $H = (X, \mathscr{E})$ whose hyperedges are labeled contains no pendant hyperedges. By Lemma 2.4, $\forall E_j \in \mathscr{E}$ $(j = 1, 2, \ldots, q)$, we have $|E_j \cap (\bigcup_{\substack{1 \le k \le q \\ k \ne j}} E_k)| = 2$. Consequently, $G(H)$ should be isomorphic to the following graph (see Figure 1, the solid points and hollow points represent the vertices in Y_1 and Y_2, respectively). The number of ways to label such graph (we shall give the vertices of Y_1 and Y_2 different types of labels) is

$$\frac{(q-1)!}{2} \cdot \frac{p!}{[(k-1)!]^q}.$$

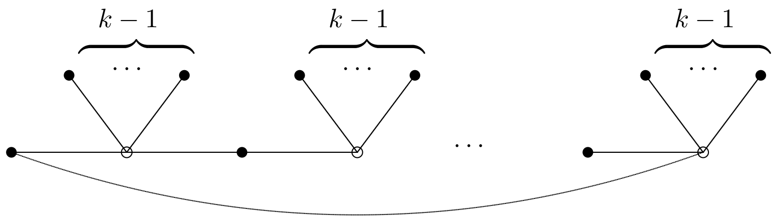

Figure 1 $G(H)$

Therefore,

$$\sum_{j=0}^{q} (-1)^j \binom{p}{jk} \binom{q}{j} \frac{(jk)!}{(k!)^j} (p-jk)^j \overline{U}^{(k+1)}_{p-jk,\ q-j} = \frac{(q-1)!}{2} \cdot \frac{p!}{[(k-1)!]^q}.$$

To transpose the terms of $j \ne 0$ in the above equality, we get Eq. (2.1). □

Theorem 2.7 *Let* $\operatorname{sgn}(x) = \begin{cases} 0, & x = 0, \\ 1, & x > 0, \\ -1, & x < 0. \end{cases}$ *Then the number of labeled* $(k+1)$*-uniform* (p,q)*-*

unicycles is

$$U_{p,q}^{(k+1)}=\begin{cases}\dfrac{p!}{2[(k-1)!]^q}\cdot\displaystyle\sum_{t=2}^{q}\dfrac{q^{q-t-1}\cdot\mathrm{sgn}(tk-2)}{(q-t)!}, & p=qk,\\ 0, & p\neq qk,\end{cases}\tag{2.2}$$

where p,q,k are positive integers.

Proof If $p\neq qk$, then we obtain the result as desired. If $p=qk$, we first prove the following equality by induction on q:

$$\overline{U}_{p,q}^{(k+1)}=\frac{p!q!}{2[(k-1)!]^q}\cdot\sum_{t=2}^{q}\frac{q^{q-t-1}\cdot\mathrm{sgn}(tk-2)}{(q-t)!}.\tag{2.3}$$

Combining this with $U_{p,q}^{(k+1)}=\frac{1}{q!}\overline{U}_{p,\ q}^{(k+1)}$, the theorem is proved.

If $q=1$, then $\overline{U}_{p,1}^{(k+1)}=0$. Note that there is no labeled $(k+1)$-uniform $(p,1)$-unicycles, then Eq. (2.3) holds. If $q=2$, then

$$\overline{U}_{p,\ 2}^{(k+1)}=\frac{(2k)!}{[(k-1)!]^2}\cdot\frac{\mathrm{sgn}(2k-2)}{2}=\begin{cases}0, & k=1,\\ \dfrac{(2k)!}{2[(k-1)!]^2}, & k>1.\end{cases}$$

It is not difficult to see that there is no labeled 2-uniform $(p,2)$-unicycles (namely, connected unicyclic graphs of order 2), and the number of labeled $(k+1)$-uniform $(k>1)$ $(p,2)$-unicycles is the number of ways to label the following graph G (see Figure 2, the solid points and hollow points shall use different types of labels), that is,

$$\frac{(2k)!}{2[(k-1)!]^2}.$$

Hence Eq. (2.3) holds.

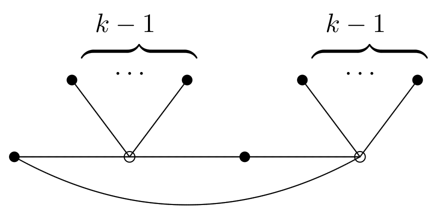

Figure 2 G

Suppose that Eq. (2.3) holds when the number of hyperedges is less than q. By Lemma 2.6 and induction hypothesis,

$$\begin{aligned}\overline{U}_{p,q}^{(k+1)}=&\frac{p!(q-1)!}{2[(k-1)!]^q}+\sum_{j=1}^{q}(-1)^{j+1}\binom{p}{jk}\binom{q}{j}\frac{(jk)!}{(k!)^j}(p-jk)^j\overline{U}_{p-jk,\ q-j}^{(k+1)}\\ =&\frac{(qk)!(q-1)!}{2[(k-1)!]^q}+\sum_{j=1}^{q}(-1)^{j+1}\binom{qk}{jk}\binom{q}{j}\frac{(jk)!}{(k!)^j}(qk-jk)^j\cdot\\ &\frac{(qk-jk)!(q-j)!}{2[(k-1)!]^{q-j}}\cdot\sum_{t=2}^{q-j}\frac{(q-j)^{q-j-t-1}\cdot\mathrm{sgn}(tk-2)}{(q-j-t)!}\end{aligned}$$

$$= \frac{(qk)!q!}{2[(k-1)!]^q} \cdot [\frac{1}{q} + \sum_{t=2}^{q-1} \frac{\mathrm{sgn}(tk-2)}{(q-t)!} \sum_{j=1}^{q-t} (-1)^{j+1} \binom{q-t}{j} (q-j)^{q-t-1}].$$

Combining this with Lemma 2.5 gives

$$\begin{aligned} \overline{U}_{p,q}^{(k+1)} &= \frac{(qk)!q!}{2[(k-1)!]^q} [\frac{1}{q} + \sum_{t=2}^{q-1} \frac{\mathrm{sgn}(tk-2)}{(q-t)!} q^{q-t-1}] \\ &= \frac{(qk)!q!}{2[(k-1)!]^q} \sum_{t=2}^{q} \frac{\mathrm{sgn}(tk-2)}{(q-t)!} q^{q-t-1} \\ &= \frac{p!q!}{2[(k-1)!]^q} \cdot \sum_{t=2}^{q} \frac{q^{q-t-1} \cdot \mathrm{sgn}(tk-2)}{(q-t)!}. \end{aligned}$$

All in all, this completes the proof of Theorem 2.7. □

Remark 2.8 By Eq. (2.2), when $k = 1$, we get the Rényi formula which counts labeled connected unicyclic graphs of order p (see [15]):

$$U_{p,\ p}^{(2)} = \frac{1}{2} \sum_{t=3}^{p} \frac{(p-1)! \cdot p^{p-t}}{(p-t)!}.$$

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References

[1] C. BERGE. *Graphs and Hypergraphs*. North-Holland Pub., Amsterdam, 1973.

[2] J. A. BONDY, U. S. R. MURTY. *Graph Theory with Applications*. Macmillan, London, 1976.

[3] C. BERGE. *Hypergraphs*. North-Holland Pub., Amsterdam, 1989.

[4] F. HARARY, E. M. PALMER. *Graphical Enumeration*. Academic Press Inc., New York, 1973.

[5] T. ANDRIAMAMPIANINA, V. RAVELOMANANA. *Enumeration of connected uniform hypergraphs*. Proceedings of FPSAC, 2005.

[6] M. HEGDE, M. R. SRIDHARAN. *Enumeration of labeled digraphs and hypergraphs*. Lecture Notes in Math., 1980, **885**: 267–283.

[7] Bolian LIU. *On the enumeration of labeled digraphs and hypergraphs*. Chinese Quarterly J. Math., 1989, **4**: 27–32.

[8] Jingzhong MAO. *On the tree of hypergraph: hypertree*. J. Central China Normal University, 1982, **S1**: 45–49.

[9] Bolian LIU. *Enumeration of hypertrees*. Appl. Math. A Journal of Chinese Universities, 1988, **3**: 359–363.

[10] A. CAYLEY. *A theorem on trees*. Quart. J. Math. Oxford, 1889, **23**: 376–378.

[11] Muhuo LIU, Bolian LIU. *Counting strict (d)-connected acyclic hypergraphs*. Acta Math. Sinica (Chin. Ser.), 2007, **50**(6): 1305–1310. (in Chinese)

[12] Zhilong SHAN, Bolian LIU. *Counting strict non-uniform linear hypertrees*. Acta Math. Appl. Sin., 2002, **25**(3): 455–459. (in Chinese)

[13] Zhilong SHAN, Bolian LIU. *A counting formula for uniformly symmetric linear acyclic hypergraphs of order* $(k+1)$. Kexue Tongbao (Chinese), 2000, **45**(16): 1705–1709. (in Chinese)

[14] Jianfang WANG, Haizhu LI. *Counting acyclic hypergraphs*. Sci. China Ser. A, 2001, **44**(2): 220–224.

[15] A. RÉNYI. *On connected graphs (I)*. Magyar Tud. Akad. Mat. Kutató Int. Közl, 1959, **4**: 385–388.

[16] L. COMTET. *Advanced Combinatorics*. Reidel, Dordrechet, 1974.