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Strong Differential Subordination and Superordination Properties of Phi-Like Functions

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Abstract Strong differential subordination and superordination properties are determined for Phi-like functions in the open unit disk by investigating appropriate classes of admissible functions. New strong differential sandwich-type results are also obtained.

Keywords strong differential subordination; strong differential superordination; analytic functions; Phi-like functions; admissible functions

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all normalized analytic functions $f(z)$ in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$.

Let Φ be an analytic function in a domain containing $f(\mathbb{U})$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. The function $f \in \mathcal{A}$ is called Φ-like function if

$$\Re\Big(\frac{zf'(z)}{\Phi(f(z))}\Big) > 0, \quad z \in \mathbb{U}.$$

This concept was introduced by Brickman [1] and he established that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ. When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the function f is starlike and spirallike of type arg λ, respectively. In a later investigation, Ruscheweyh [2] introduced and studied the general class of Φ-like functions.

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U}$. For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \big\{f : f \in \mathcal{H}(\mathbb{U}) \ \text{ and } \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\big\}$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$.

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Let f and F be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F, or (equivalently) F is said to be superordinate to f, if there exists a Schwarz function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = F(w(z))$, $z \in \mathbb{U}$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then we have

$$f \prec F \Longleftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $H(z,\zeta)$ be analytic in $\mathbb{U} \times \overline{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then the function $H(z,\zeta)$ is said to be strongly subordinate to $f(z)$, or $f(z)$ is said to be strongly superordinate to $H(z,\zeta)$, written as $H(z,\zeta) \prec\prec f(z)$, if, for $\zeta \in \overline{\mathbb{U}}$, $H(z,\zeta)$ as a function of z is subordinate to $f(z)$. We note that

$$H(z,\zeta) \prec\prec f(z) \Longleftrightarrow H(0,\zeta) = f(0) \quad \text{and} \quad H(\mathbb{U} \times \overline{\mathbb{U}}) \subset f(\mathbb{U}).$$

Definition 1.1 ([3,4]) *Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following (second-order) strong differential subordination:*

$$\phi(p(z), zp'(z), z^2p''(z); z, \zeta) \prec\prec h(z), \tag{1.1}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the strong differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.1) is said to be the best dominant.

Recently, Oros [5] introduced the following notion of strong differential superordination as the dual concept of strong differential subordination.

Definition 1.2 ([5,6]) *Let $\varphi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If*

$$p(z) \quad \text{and} \quad \varphi(p(z), zp'(z), z^2p''(z); z, \zeta)$$

are univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \zeta), \tag{1.2}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the strong differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.2). A univalent subordinant $\tilde{q}(z)$ that satisfies

$$q(z) \prec \tilde{q}(z)$$

for all subordinants $q(z)$ of (1.2) is said to be the best subordinant.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{\xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty\right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.3 ([4]) *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N}$. The class of admissible functions*

$\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z, \zeta) \notin \Omega$, whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \Re(\frac{t}{s} + 1) \geq k\,\Re\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\},$$

$$z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq n.$$

We simply write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Definition 1.4 ([5]) *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; \xi, \zeta) \in \Omega$, whenever*

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \text{and} \quad \Re(\frac{t}{s} + 1) \leq \frac{1}{m}\Re\{\frac{zq''(z)}{q'(z)} + 1\},$$

$$z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq n \geq 1.$$

In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, Oros and Oros [4] proved the following result.

Lemma 1.5 ([4]) *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega,$$

then $p(z) \prec q(z)$ $(z \in \mathbb{U})$.

Oros [5], on the other hand proved Lemma 1.6.

Lemma 1.6 ([5]) *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi(p(z), zp'(z), z^2p''(z); z, \zeta)$ is univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$, then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\}$$

implies the following subordination relationship:

$$q(z) \prec p(z), \quad z \in \mathbb{U}.$$

Recently, Ali et al. [7,8] have obtained certain second order differential subordination and superordination results, by considering certain suitable class of admissible functions. More recently, Cho et al. [9] have considered second order strong subordination and strong superordination results for multivalent meromorphic functions involving the Liu-Srivastava operator. Also, several authors obtained many interesting results in subordination, strong differential subordination and superordination [4–16].

Motivated by the afore mentioned works, in this present investigation, by making use of the strong differential subordination results and the strong superordination results of Oros and Oros [4,5], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and strong differential superordination properties of Φ-like functions. As an application, several interesting examples are considered. New strong differential sandwich-type results are also obtained.

2. The main strong subordination results

We first define the following class of admissible functions that are required in our first result.

Definition 2.1 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_P[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; z, \zeta) \notin \Omega$, whenever*

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \quad q(\xi) \neq 0,$$

and

$$\Re\{\frac{w+v^2}{v}\} \geq k\,\Re\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\}, \quad z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq 1.$$

Theorem 2.2 *Let $\phi \in \Phi_P[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\Big\{\phi\big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\big) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\Big\} \subset \Omega, \tag{2.1}$$

then $\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$.

Proof Define the function p in $\mathbb{U}$ by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))}. \tag{2.2}$$

A simple calculation yields

$$\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))} = \frac{zp'(z)}{p(z)}. \tag{2.3}$$

Further computations show that

$$z\Big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\Big]' = \frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - \big(\frac{zp'(z)}{p(z)}\big)^2. \tag{2.4}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}. \tag{2.5}$$

Let

$$\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi\big(r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z, \zeta\big). \tag{2.6}$$

The proof will make use of Lemma 1.5. Using (2.2), (2.3), and (2.4), from (2.6) we obtain

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \\ &= \phi\Big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big). \end{aligned} \tag{2.7}$$

Hence (2.1) becomes $\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega$. A computation using (2.5) yields

$$\frac{t}{s} + 1 = \frac{w+v^2}{v}.$$

Thus the admissibility condition for $\phi \in \Phi_P[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.1

$$p(z) \prec q(z)$$

or, equivalently, $\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$, which evidently completes the proof of Theorem 2.2. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_P[h(\mathbb{U}), q]$ is written as $\Phi_P[h, q]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3 *Let $\phi \in \Phi_P[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\Big(\frac{zf'(z)}{\Phi(f(z))}, \big(1+\frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1+\frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big) \prec\prec h(z), \quad (2.8)$$

then $\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$.

Our next result in an extension of Theorem 2.2 to the case in which the behavior of q on $\partial\mathbb{U}$ is not known.

Theorem 2.4 *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfy one of the following conditions:*

(i) $\phi \in \Phi_P[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(ii) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_P[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.8), then $\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$.

Proof The proof of Theorem 2.4 is similar to that of a known result [3, Theorem 2.3d, page 30] and so it is omitted here. □

Our next theorem yields the best dominant of the strong differential subordination (2.8).

Theorem 2.5 *Let h be univalent in $\mathbb{U}$, and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation*

$$\phi\big(q(z), \frac{zq'(z)}{q(z)}, \frac{q(z)(z^2q''(z) + zq'(z)) - (zq'(z))^2}{(q(z))^2}; z, \zeta\big) = h(z) \quad (2.9)$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

(i) $q \in \mathcal{Q}_1$ and $\phi \in \Phi_P[h, q]$,

(ii) q is univalent in $\mathbb{U}$ and $\phi \in \Phi_P[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(iii) q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_P[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.8), then $\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$, and q is the best dominant.

Proof Following the same arguments as in [3, Theorem 2.3e, page 31], we deduce that q is a dominant from Theorems 2.3 and 2.4. Since q satisfies (2.9), it is also a solution of (2.8) and therefore q will be dominated by all dominants. Hence q is the best dominant.

We will apply Theorem 2.2 to a specific case for $q(z) = 1 + Mz$, $M > 0$.

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_P[\Omega, q]$, denoted by $\Phi_P[\Omega, M]$, is described below.

Definition 2.6 *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_P[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that*

$$\phi\big(1 + Me^{i\theta}, \frac{kMe^{i\theta}}{1 + Me^{i\theta}}, \frac{(M + e^{-i\theta})(Le^{-i\theta} + kM) - k^2M^2}{(M + e^{-i\theta})^2}; z, \zeta\big) \notin \Omega, \tag{2.10}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\{Le^{-i\theta}\} \geq (k-1)kM$ for all θ, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Corollary 2.7 *Let $\phi \in \Phi_P[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\Big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big) \in \Omega$$

then $|\frac{zf'(z)}{\Phi(f(z))} - 1| < M$.

For the special case $\Omega = q(\mathbb{U}) = \{w : |w - 1| < M\}$, the class $\Phi_P[\Omega, M]$ is simply denoted by $\Phi_P[M]$.

Corollary 2.8 *Let $\phi \in \Phi_P[M]$. If $f \in \mathcal{A}$ satisfies*

$$\Big|\phi\big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\big) - 1\Big| < M,$$

then $|\frac{zf'(z)}{\Phi(f(z))} - 1| < M$.

Example 2.9 If $f \in \mathcal{A}$ satisfies

$$\left|\Big\{\frac{zf'(z)}{f(z)}\big(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\big)\Big\} - 1\right| < 1,$$

then $|\frac{zf'(z)}{f(z)} - 1| < 1$.

Proof This follows from Corollary 2.7 by taking $\phi(u, v, w; z, \zeta) = uv$, $\Phi(w) = w$, $M = 1$ and $\Omega = h(\mathbb{U})$ where $h(z) = z$. This result was obtained in [8, Example 2.2, page 1775]. $\square$

Example 2.10 If $M > 0$ and $f \in \mathcal{A}$ satisfies

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < M,$$

then $|\frac{zf'(z)}{f(z)} - 1| < M$.

Proof This follows from Corollary 2.7 by taking $\phi(u, v, w; z, \zeta) = 1 + v/u$, $\Phi(w) = w$. This result was obtained in [8, Example 2.3, page 1775]. $\square$

Example 2.11 If $\delta \geq 0$ and $f \in \mathcal{A}$ satisfies

$$\left|\Big\{\delta\big(1 + \frac{zf''(z)}{f'(z)}\big) + (1 - \delta)\frac{zf'(z)}{f(z)}\Big\} - 1\right| < 1,$$

then $|\frac{zf'(z)}{f(z)} - 1| < 1$.

Proof Let $\phi(u, v, w; z, \zeta) = u + \delta v$ for all $\delta \geq 0$, $\Phi(w) = w$, $M = 1$ and $\Omega = h(\mathbb{U})$ where

$h(z) = (1+\delta/2)z$. To use Corollary 2.7, we need to show that $\Phi_P[\Omega, M] \equiv \Phi_P[\mathbb{U}]$, that is, the admissibility condition (2.10) is satisfied. This follows since

$$\Big|\phi\big(1+Me^{i\theta}, \frac{kMe^{i\theta}}{1+Me^{i\theta}}, \frac{(M+e^{-i\theta})(Le^{-i\theta}+kM)-k^2M^2}{(M+e^{-i\theta})^2} : z, \zeta\big)\Big|$$
$$= \big|Me^{i\theta} + \delta\frac{kMe^{i\theta}}{1+Me^{i\theta}}\big| \geq 1+\delta/2, \quad k \geq 1. \ \square$$

Example 2.12 If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

$$\big|\frac{z^2 f''(z)}{f(z)} + \lambda\big(\frac{zf'(z)}{f(z)} - 1\big)\big| < M(\lambda + 2 - M),$$

where $0 < M \leq 1$ and $\lambda \geq 2(M-1)$, then $|\frac{zf'(z)}{f(z)} - 1| < M$.

Proof By considering the function $\phi(u, v, w; z, \zeta) = u(u+v-1) + \lambda(u-1) + 1$, and $\Phi(w) = w$ with $0 < M \leq 1$, $\lambda + 2 - M \geq 0$, it follows from Corollary 2.7. This result was obtained in [17, Corollary 2, page 583]. $\square$

Example 2.13 If $M > 0$ and $f \in \mathcal{A}$ satisfies

$$\big|(1+\frac{zf''(z)}{f'(z)}) - 2\frac{zf'(z)}{f(z)}\big| < \frac{M^2}{1+M},$$

then $|\frac{zf'(z)}{f(z)} - 1| < M$.

Proof This follows from Corollary 2.7 by taking $\phi(u, v, w; z, \zeta) = v - u + 1$, $\Phi(w) = w$ and $\Omega = h(\mathbb{U})$ where $h(z) = \frac{M^2}{1+M}z$. To use Corollary 2.7, we need to show that $\Phi_P[\Omega, M] \equiv \Phi_P[\mathbb{U}]$, that is, the admissibility condition (2.10) is satisfied. This follows since

$$|\phi(u, v, w; z, \zeta)| = \big|\frac{kMe^{i\theta}}{1+Me^{i\theta}} - Me^{i\theta}\big| = M\big|\frac{k}{1+M} - 1\big| \geq M\big|\frac{1}{1+M} - 1\big| = \frac{M^2}{1+M},$$
$$z \in \mathbb{U},\ \theta \in \mathbb{R},\ k \neq 1+M,\ k \geq 1.$$

Hence by Corollary 2.7, we deduce the required result. $\square$

Corollary 2.14 *Let $\beta, \gamma \in \mathbb{C}$, and let h be convex in $\mathbb{U}$, with $h(0) = 1$ and $\Re[\beta h(z) + \gamma] > 0$. Let $\phi \in \Phi_P[h, q]$. If $f \in \mathcal{A}$, and $\frac{zf'(z)}{\Phi(f(z))}$ is analytic in $\mathbb{U}$, then*

$$\frac{zf'(z)}{\Phi(f(z))} + \frac{(1+\frac{zf''(z)}{f'(z)})\Phi(f(z)) - z[\Phi(f(z))]'}{\beta zf'(z) + \gamma\Phi(f(z))} \prec\prec h(z)$$

implies $\frac{zf'(z)}{\Phi(f(z))} \prec h(z)$.

Proof This follows from Theorem 2.3 by taking $\phi(u, v, w; z, \zeta) = u + \dfrac{v}{\beta u + \gamma}$. $\square$

3. Strong superordination and sandwich-type results

In this section, we investigate the dual problem of strong differential subordination (that is, strong differential superordination). For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1 *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}$. The class of admissible functions $\Phi'_P[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition: $\phi(u, v, w; \xi, \zeta) \in \Omega$, whenever

$$u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0,$$

and

$$\Re\{\frac{w + v^2}{v}\} \leq \frac{1}{m}\Re\{\frac{zq''(z)}{q'(z)} + 1\}, \quad z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq 1.$$

Theorem 3.2 *Let $\phi \in \Phi'_P[\Omega, q]$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\Phi(f(z))} \in \mathcal{Q}_1$ and*

$$\phi\big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\big)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \Big\{\phi\big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\big) : z, \zeta\Big\} \quad (3.1)$$

implies

$$q(z) \prec \frac{zf'(z)}{\Phi(f(z))}. \quad (3.2)$$

Proof With $p(z) = \dfrac{zf'(z)}{\Phi(f(z))}$ and

$$\psi(r, s, t; z, \zeta) = \phi(r, \frac{s}{r}, \frac{r(t + s) - s^2}{r^2}; z, \zeta) = \phi(u, v, w; \xi, \zeta),$$

equation (2.7) and (3.1) yields

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\}.$$

Since $\frac{t}{s} + 1 = \frac{w+v^2}{v}$, the admissibility condition for $\phi \in \Phi'_P[\Omega, q]$ in Definition 3.1 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.6, $q(z) \prec p(z)$ or $q(z) \prec \frac{zf'(z)}{\Phi(f(z))}$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω with $\Phi'_P[h(\mathbb{U}), q]$ as $\Phi'_P[h, q]$, Theorem 3.2 can be written in the following form.

Theorem 3.3 *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_P[h, q]$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\Phi(f(z))} \in \mathcal{Q}_1$ and*

$$\phi\Big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\Big(\frac{zf'(z)}{\Phi(f(z))}, \big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[\big(1 + \frac{zf''(z)}{f'(z)}\big) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big) \quad (3.3)$$

implies $q(z) \prec \frac{zf'(z)}{\Phi(f(z))}$.

Corollary 3.4 *Let $\beta, \gamma \in \mathbb{C}$, and let h be convex in $\mathbb{U}$, with $h(0) = 1$. Suppose that the* *differential equation* $q(z) + \frac{zq'(z)}{\beta q(z)+\gamma} = h(z)$ *has a univalent solution* q *that satisfies* $q(0) = 1$, *and* $q(z) \prec h(z)$. *Let* $\phi \in \Phi'_P[h, q]$. *If* $\frac{zf'(z)}{\Phi(f(z))} \in \mathcal{H} \cap \mathcal{Q}_1$ *and*

$$\frac{zf'(z)}{\Phi(f(z))} + \frac{(1 + \frac{zf''(z)}{f'(z)})\Phi(f(z)) - z[\Phi(f(z))]'}{\beta zf'(z) + \gamma\Phi(f(z))}$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec\prec \frac{zf'(z)}{\Phi(f(z))} + \frac{(1 + \frac{zf''(z)}{f'(z)})\Phi(f(z)) - z[\Phi(f(z))]'}{\beta zf'(z) + \gamma\Phi(f(z))}$$

implies $q(z) \prec \frac{zf'(z)}{\Phi(f(z))}$.

Proof This follows from Theorem 3.3 by taking $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u+\gamma}$. $\square$

Theorems 3.2 and 3.3 can only be used to obtain subordinants of strong differential superordination of the form (3.1) or (3.3). The following theorem proves the existence of the best subordinant of (3.3) for an appropriate ϕ.

Theorem 3.5 *Let* h *be analytic in* $\mathbb{U}$ *and* $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\Big(q(z), \frac{zq'(z)}{q(z)}, \frac{q(z)(z^2q''(z) + zq'(z)) - (zq'(z))^2}{(q(z))^2}; z, \zeta\Big) = h(z)$$

has a solution $q \in \mathcal{Q}_1$. *If* $\phi \in \Phi'_P[h, q]$, $f \in \mathcal{A}$, $\frac{zf'(z)}{\Phi(f(z))} \in \mathcal{Q}_1$ *and*

$$\phi\Big(\frac{zf'(z)}{\Phi(f(z))}, (1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[(1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec\prec \phi\Big(\frac{zf'(z)}{\Phi(f(z))}, (1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[(1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big)$$

implies $q(z) \prec \frac{zf'(z)}{\Phi(f(z))}$ *and* q *is the best subordinant.*

Proof The proof is similar to that of Theorem 2.5, and so it is being omitted here. $\square$

By combining Theorems 2.3 and 3.3, we obtain the following sandwich-type theorem.

Corollary 3.6 *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{U}$, h_2 *be univalent function in* $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \Phi_P[h_2, q_2] \cap \Phi'_P[h_1, q_1]$. *If* $f \in \mathcal{A}$, $\frac{zf'(z)}{\Phi(f(z))} \in \mathcal{H} \cap \mathcal{Q}_1$ *and*

$$\phi\Big(\frac{zf'(z)}{\Phi(f(z))}, (1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[(1 + \frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec\prec \phi\Big(\frac{zf'(z)}{\Phi(f(z))}, (1+\frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}, z\big[(1+\frac{zf''(z)}{f'(z)}) - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}\big]'; z, \zeta\Big) \prec\prec h_2(z)$$

implies $q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z)$.

By combining Corollaries 2.14 and 3.4 we obtain the following sandwich-type corollary.

Corollary 3.7 *Let* $\beta, \gamma \in \mathbb{C}$, *and let* h_i *be convex in* $\mathbb{U}$, *with* $h_i(0) = 1$, *for* $i = 1, 2$. *Suppose* *that the differential equations* $q_i(z)+\dfrac{zq_i'(z)}{\beta q_i(z)+\gamma}=h_i(z)$ *have a univalent solution* q_i *that satisfies* $q_i(0)=1$ *and* $q_i(z)\prec h_i(z)$, *for* $i=1,2$. *Let* $\phi\in\Phi_P[h_2,q_2]\cap\Phi_P'[h_1,q_1]$. *If* $\frac{zf'(z)}{\Phi(f(z))}\in\mathcal{H}\cap\mathcal{Q}_1$ *and*

$$\frac{zf'(z)}{\Phi(f(z))}+\frac{(1+\frac{zf''(z)}{f'(z)})\Phi(f(z))-z[\Phi(f(z))]'}{\beta zf'(z)+\gamma\Phi(f(z))}$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z)\prec\prec\frac{zf'(z)}{\Phi(f(z))}+\frac{(1+\frac{zf''(z)}{f'(z)})\Phi(f(z))-z[\Phi(f(z))]'}{\beta zf'(z)+\gamma\Phi(f(z))}\prec\prec h_2(z)$$

implies $q_1(z)\prec\frac{zf'(z)}{\Phi(f(z))}\prec q_2(z)$.

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References

[1] L. BRICKMAN. *Φ-like analytic functions (I).* Bull. Amer. Math. Soc., 1973, **79**(3): 555–558.
[2] ST. RUSCHEWEYH. *A subordination theorem for Φ-like functions.* J. London Math. Soc., 1976, **13**(2): 275–280.
[3] S. S. MILLER, P. T. MOCANU. *Differential Subordination: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, New York and Basel, 2000.
[4] G. I. OROS, G. OROS. *Strong differential subordination.* Turkish J. Math., 2009, **33**: 249–257.
[5] G. I. OROS. *Strong differential superordination.* Acta Univ. Apulensis Math. Inform., 2009, **19**: 101–106.
[6] S. S. MILLER, P. T. MOCANU. *Subordinates of differential superordinations.* Complex Var. Theory Appl., 2003, **48**(10): 815–826.
[7] R. M. ALI, V. RAVICHANDRAN, N. SEENIVASAGAN. *Subordination and superordination on Schwarzian derivatives.* J. Inequal. Appl., 2008, Art. ID 712328, 18 pp.
[8] R. M. ALI, V. RAVICHANDRAN, N. SEENIVASAGAN. *Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava linear operator.* J. Franklin Inst., 2010, **347**(9): 1762–1781.
[9] N. E. CHO, O. S. KWON, H. M. SRIVASTAVA. *Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu-Srivastava operator.* Integral Transforms Spec. Funct., 2010, **21**(7-8): 589–601.
[10] J. A. ANTONINO, S. ROMAGUERA. *Strong differential subordination to Briot-Bouquet differential equations.* J. Differential Equations, 1994, **114**(1): 101–105.
[11] J. A. ANTONINO. *Strong differential subordination and applications to univalency conditions.* J. Korean Math. Soc., 2006, **43**(2): 311–322.
[12] M. P. JEYARAMAN, T. K. SURESH. *Strong differential subordination and superordination of analytic functions.* J. Math. Anal. Appl., 2012, **385**(2): 854–864.
[13] S. P. GOYAL, P. GOSWAMI, H. SILVERMAN. *Subordination and superordination results for a class of analytic multivalent functions.* Int. J. Math. Math. Sci., 2008, Art. ID 561638, 12 pp.
[14] S. P. GOYAL, P. GOSWAMI, Zhigang WANG. *Subordination and superordination results involving certain analytic functions.* J. Appl. Math. Stat. Inform., 2011, **7**(2): Article 7, 75–86.
[15] G. I. OROS. *First order strong differential superordination.* Gen. Math., 2007, **15**(2-3): 77–87.
[16] G. I. OROS, G. OROS. *First order linear strong differential subordinations.* Gen. Math., 2007, **15**(2): 98–107.
[17] Neng XU, Dinggong YANG. *Some criteria for starlikeness and strongly starlikeness.* Bull. Korean Math. Soc., 2005, **42**(3): 579–590.