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On the Stability of the Functional Equation in Matrix β-Normed Spaces

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Abstract In this paper we introduce the matrix β-normed space and study the stability of the additive-quadratic type functional equation and the Pexider type functional equation in this type of spaces.

Keywords matrix β-normed spaces; stability; additive-quadratic functional equation; Pexider equation

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1. Introduction

In 1940, Ulam [1] proposed the following stability problem: Given a metric group $G(\cdot,\rho)$, a number $\varepsilon > 0$ and a mapping $f: G \to G$ which satisfies the inequality $\rho(f(x\cdot y), f(x)\cdot f(y)) < \varepsilon$ for all x, y in G, does there exist an automorphism a of G and a constant $k > 0$, depending only on G, such that $\rho(a(x), f(x)) \le k\varepsilon$ for all x in G? If the answer is affirmative, we call the equation $a(x\cdot y) = a(x)\cdot a(y)$ of automorphism stable. One year later, Hyers [2] provided a positive partial answer to Ulam's problem. In 1978, a generalized version of Hyers' result was proved by Rassias in [3]. Since then, the stability problems of several functional equations have been extensively investigated by a number of authors [4–12]. In fact, we also refer the readers to the books [13–16].

A function $f: X \to Y$ between real vector spaces is said to be quadratic if it satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

for all $x, y \in X$, and a function $B: X \times X \to Y$ is said to be bi-quadratic if B is quadratic for each fixed variable [17]. Skof [18] was the first person to prove the Hyers-Ulam stability of the

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quadratic function equation (1.1). Cholewa [8] demonstrated that Skof's theorem is also valid if X is replaced with an Abelian group G.

In this paper we will use the following notations.

(1) M_n is the set of all $n \times n$-matrices in X which is a real vector space;

(2) $e_j \in M_{1,n}(\mathbb{C})$ means that jth component is 1 and the other components are zero;

(3) $E_{ij} \in M_n(\mathbb{C})$ means that (i,j)-component is 1 and the other components are zero;

(4) $E_{ij} \otimes x \in M_n(X)$ means that (i,j)-component is x and the other components are zero.

Next we extend general matrix normed spaces [19] to matrix β–normed spaces.

Definition 1.1 *Let X be a real vector space, $M_n = M_n(\mathbb{C})$, $M_n(X) = X \otimes M_n$. A function $\|\cdot\|_{\beta,n} : M_n(X) \to [0,\infty)$ is called a β-norm, where $0 < \beta \leq 1$, if for all $n \in \mathbb{N}$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$*

(i) $\|x\|_{\beta,n} \geq 0$; $\|x\|_{\beta,n} = 0$ *if and only if* $x = O$;

(ii) $\|\alpha x\|_{\beta,n} = |\alpha|^\beta \|x\|_{\beta,n}$, *for all* $\alpha \in \mathbb{R}$;

(iii) $\|x+y\|_{\beta,n} \leq \|x\|_{\beta,n} + \|y\|_{\beta,n}$.

The pair $(M_n(X), \|\cdot\|_{\beta,n})$ is called a β-normed space. When $\beta = 1$, $(M_n(X), \|\cdot\|_{1,n})$ is a normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Now, we introduce matrix β-normed spaces and related properties.

Definition 1.2 *Let X be a real vector space and $(X, \|\cdot\|_\beta)$ be a β-normed space. $(X, \{\|\cdot\|_{\beta,n}\})$ is called a matrix β-normed space if for each positive integer n, $(M_n(X), \|\cdot\|_{\beta,n})$ is a β-normed space, $\|E_{ij}\|_{\beta,n} = 1$ and $\|AxB\|_{\beta,k} \leq \|A\|_\beta \|B\|_\beta \|x\|_{\beta,n}$ for all $A \in M_{k,n}(\mathbb{C})$, $x = [x_{ij}] \in M_{k,n}(X)$ and $B \in M_{n,k}(\mathbb{C})$.*

Lemma 1.3 *Let $(X, \{\|\cdot\|_{\beta,n}\})$ be a matrix β-normed space.*

(1) $\|E_{kl} \otimes x\|_{\beta,n} = \|x\|_\beta$, *for all* $x \in X$;

(2) $\|x_{kl}\|_\beta \leq \|[x_{ij}]\|_{\beta,n} \leq \sum_{i,j=1}^n \|x_{ij}\|_\beta$;

(3) $\lim_{m\to\infty} x_m = x$ *if and only if* $\lim_{m\to\infty} x_{mij} = x_{ij}$, *for* $x_m = [x_{mij}], x = [x_{ij}] \in M_n(X)$.

Proof (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\|_\beta = \|e_l\|_\beta = 1$, $\|E_{kl} \otimes x\|_{\beta,n} \leq \|e_k^*\|_\beta \|e_l\|_\beta \|x\|_\beta = \|x\|_\beta$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, we have $\|E_{kl} \otimes x\|_{\beta,n} \geq \|x\|_\beta$.

(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k^*\|_\beta = \|e_l\|_\beta = 1$, $\|x_{kl}\|_\beta \leq \|[x_{ij}]\|_{\beta,n}$, $[x_{ij}] = \sum_{i,j=1}^n E_{ij} \otimes x_{ij}$,

$$\|[x_{ij}]\|_{\beta,n} = \|\sum_{i,j=1}^n E_{ij} \otimes x_{ij}\|_{\beta,n} \leq \sum_{i,j=1}^n \|E_{ij} \otimes x_{ij}\|_{\beta,n} = \sum_{i,j=1}^n \|x_{ij}\|_\beta.$$

(3) By (2), we have

$$\|x_{mkl}-x_{kl}\|_\beta \le \|[x_{mij}-x_{ij}]\|_{\beta,n} = \|[x_{mij}]-[x_{ij}]\|_{\beta,n} \le \sum_{i,j=1}^{m} \|x_{mij}-x_{ij}\|_\beta.$$

We obtain the results. □

Jung [20] investigated the stability of the mixed additive-quadratic functional equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x) \tag{1.2}$$

and Jun, Shin and Kim [21] investigated the stability problem of the Pexider equation

$$f(x+y)=g(x)+h(y). \tag{1.3}$$

We prove the Hyers-Ulam stability of the mixed additive-quadratic functional equation (1.2) in matrix β-normed spaces for an odd mapping in Section 2 and for an even mapping in Section 3. The Hyers-Ulam stability of the Pexider equation (1.3) in matrix β-normed spaces is proved in Section 4.

Throughout this paper, assume that $(X,\{\|\cdot\|_{\beta,n}\})$ is a matrix β-normed space and $(Y,\{\|\cdot\|_{\beta,n}\})$ is a complete matrix β-normed space.

2. Stability of additive-quadratic functional equations: an odd mapping case

In this section, we will investigate the stability of the additive-quadratic functional equation for the odd case in matrix β -normed space.

For a mapping $f: X\to Y$, define $Df: X^3\to Y$ by

$$Df(a,b,c):=f(a+b+c)+f(a)+f(b)+f(c)-f(a+b)-f(b+c)-f(a+c)$$

and define $Df_n: M_n(X^3)\to M_n(Y)$ by

$$\begin{aligned}Df_n([x_{ij}],[y_{ij}],[z_{ij}]) :=& f_n([x_{ij}+y_{ij}+z_{ij}])+f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])-\\ & f_n([x_{ij}+y_{ij}])-f_n([y_{ij}+z_{ij}])-f_n([x_{ij}+z_{ij}])\end{aligned}$$

for all $a,b,c\in X$ and all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$.

Theorem 2.1 *Let $f: X\to Y$ be an odd mapping and $\phi: X^3\to[0,\infty)$ be a function such that*

$$\Phi(a,b,c)=\sum_{k=0}^{\infty} 2^{-(k+1)\beta}\phi(2^k a, 2^k b, 2^k c) < +\infty, \tag{2.1}$$

$$\|Df_n([x_{ij}],[y_{ij}],[z_{ij}])\|_{\beta,n} \le \sum_{i,j=1}^{n} \phi(x_{ij},y_{ij},z_{ij}) \tag{2.2}$$

for all $a,b,c\in X$ and all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $A: X\to Y$ such that

$$\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n} \le \sum_{i,j=1}^{n} \Phi(x_{ij},x_{ij},-x_{ij}) \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$. Then (2.2) is equivalent to

$$\|Df(a,b,c)\|_\beta \le \phi(a,b,c) \tag{2.4}$$

for all $a, b, c \in X$. Letting $b = a$, $c = -a$ in (2.4), and multiplying both sides by $2^{-\beta}$, we get

$$\|2^{-1}f(2a) - f(a)\|_\beta \le 2^{-\beta}\phi(a,a,-a) \tag{2.5}$$

for all $a \in X$. Applying an induction argument to l, we will prove

$$\|2^{-l}f(2^l a) - f(a)\|_\beta \le \sum_{k=0}^{l-1} 2^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a) \tag{2.6}$$

for all $a \in X$ and $l \in \mathbb{N}$. Indeed,

$$\|2^{-(l+1)}f(2^{l+1}a) - f(a)\|_\beta \le \|2^{-(l+1)}f(2^{l+1}a) - 2^{-1}f(2a)\|_\beta + \|2^{-1}f(2a) - f(a)\|_\beta$$

and by (2.5) and (2.6), we obtain

$$\begin{aligned}
&\|2^{-(l+1)}f(2^{l+1}a) - f(a)\|_\beta \\
&\quad \le 2^{-\beta}\sum_{k=0}^{l-1} 2^{-(k+1)\beta}\phi(2^{k+1}a, 2^{k+1}a, -2^{k+1}a) + 2^{-\beta}\phi(a,a,-a) \\
&\quad = 2^{-\beta}\sum_{k=1}^{l} 2^{-k\beta}\phi(2^k a, 2^k a, -2^k a) + 2^{-\beta}\phi(a,a,-a) \\
&\quad = \sum_{k=0}^{l} 2^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a)
\end{aligned}$$

for all $a \in X$ and $l \in \mathbb{N}$, which ends the proof of (2.6).

We will present that the sequence $\{2^{-l}f(2^l a)\}$ is a Cauchy sequence. For $l > m > 0$, we have

$$\begin{aligned}
&\|2^{-l}f(2^l a) - 2^{-m}f(2^m a)\|_\beta = 2^{-m\beta}\|2^{-(l-m)}f(2^{l-m}\cdot 2^m a) - f(2^m a)\|_\beta \\
&\quad \le 2^{-m\beta}\sum_{k=0}^{l-m-1} 2^{-(k+1)\beta}\phi(2^{k+m}a, 2^{k+m}a, -2^{k+m}a) \\
&\quad = \sum_{k=m}^{l-1} 2^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a)
\end{aligned}$$

for all $a \in X$ and $l, m \in \mathbb{N}$. From (2.1), we obtain the sequence $\{2^{-l}f(2^l a)\}$ is a Cauchy sequence. Since Y is complete, the sequence converges to some $A(a) \in Y$. So one can define the mapping $A : X \to Y$ by $A(a) = \lim_{l\to\infty} 2^{-l}f(2^l a)$ for all $a \in X$ and $l \in \mathbb{N}$.

It follows from (2.4) that $\|Df(2^l a, 2^l b, 2^l c)\|_\beta \le \phi(2^l a, 2^l b, 2^l c)$ for all $a, b, c \in X$ and $l \in \mathbb{N}$. Therefore,

$$\|2^{-l}Df(2^l a, 2^l b, 2^l c)\|_\beta \le 2^{-l\beta}\phi(2^l a, 2^l b, 2^l c) \tag{2.7}$$

for all $a,b,c\in X$ and $l\in\mathbb{N}$. It follows from (2.1) that

$$\lim_{l\to\infty}2^{-l\beta}\phi(2^la,2^lb,2^lc)=0$$

for all $a,b,c\in X$ and $l\in\mathbb{N}$. Thus, (2.7) implies that $DA(a,b,c)=0$. Since $f:X\to Y$ is odd, $A:X\to Y$ is odd. So the mapping $A:X\to Y$ is additive.

Taking the limit in (2.6) as $l\to\infty$, we obtain

$$\|A(a)-f(a)\|_\beta\le\sum_{k=0}^{\infty}2^{-(k+1)\beta}\phi(2^ka,2^ka,-2^ka)=\Phi(a,a,-a)\tag{2.8}$$

for all $a\in X$.

It remains to show that A is uniquely defined. Let $A':X\to Y$ be another additive function satisfying (2.8). Then we get

$$\begin{aligned}\|A(a)-A'(a)\|_\beta&=\|2^{-l}A(2^la)-2^{-l}A'(2^la)\|_\beta\\&\le\|2^{-l}A(2^la)-2^{-l}f(2^la)\|_\beta+\|2^{-l}f(2^la)-2^{-l}A'(2^la)\|_\beta\\&\le 2\cdot2^{-l\beta}\Phi(2^la,2^la,-2^la)\\&=2\sum_{k=l}^{\infty}2^{-(k+1)\beta}\phi(2^ka,2^ka,-2^ka)\end{aligned}$$

for all $a\in X$ and $l\in\mathbb{N}$. Taking the limit in the above inequality as $l\to\infty$, we get $A(a)=A'(a)$ for all $a\in X$.

By Lemma 1.3 and (2.8),

$$\begin{aligned}\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}=\|[f(x_{ij})-A(x_{ij})]\|_{\beta,n}&\le\sum_{i,j=l}^{n}\|f(x_{ij})-A(x_{ij})\|_\beta\\&\le\sum_{i,j=1}^{n}\Phi(x_{ij},x_{ij},-x_{ij})\end{aligned}$$

for all $x=[x_{ij}]\in M_n(X)$. $\square$

Corollary 2.2 *Let r,θ and β be positive real numbers with $r<1$, $0<\beta\le1$ and $f:X\to Y$ be an odd mapping such that*

$$\|Df_n([x_{ij}],[y_{ij}],[z_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\theta(\|x_{ij}\|_\beta^r+\|y_{ij}\|_\beta^r+\|z_{ij}\|_\beta^r)\tag{2.9}$$

for all $x=[x_{ij}],y=[y_{ij}],z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\frac{3}{2^\beta-2^{\beta r}}\theta\|x_{ij}\|_\beta^r$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof Letting $\phi(a,b,c)=\theta(\|a\|_\beta^r+\|b\|_\beta^r+\|c\|_\beta^r)$ in Theorem 2.1, we get the result. $\square$

Theorem 2.3 *Let $f:X\to Y$ be an odd mapping and $\phi:X^3\to[0,\infty)$ be a function satisfying*

(2.2) and

$$\Phi(a,b,c)=\sum_{k=1}^{\infty}2^{(k-1)\beta}\phi(2^{-k}a,2^{-k}b,2^{-k}c)<+\infty$$

for all $a,b,c\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\Phi(x_{ij},x_{ij},-x_{ij})$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof The proof is similar to the proof of Theorem 2.1. □

Corollary 2.4 *Let r,θ and β be positive real numbers with $r>1$, $0<\beta\le1$ and $f:X\to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\frac{3}{2^{\beta r}-2^{\beta}}\theta\|x_{ij}\|_\beta^r$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof Letting $\phi(a,b,c)=\theta(\|a\|_\beta^r+\|b\|_\beta^r+\|c\|_\beta^r)$ in Theorem 2.3, we get the result. □

3. Stability of additive-quadratic functional equations: an even mapping case

In this section, we will investigate the stability of the additive-quadratic functional equation for the even case in matrix β-normed space.

Theorem 3.1 *Let $f:X\to Y$ be an even mapping with $f(0)=0$ and $\phi:X^3\to[0,\infty)$ be a function satisfying (2.2) and*

$$\Phi(a,b,c)=\sum_{k=0}^{\infty}4^{-(k+1)\beta}\phi(2^{k}a,2^{k}b,2^{k}c)<+\infty \tag{3.1}$$

for all $a,b,c\in X$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\Phi(x_{ij},x_{ij},-x_{ij}) \tag{3.2}$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof Let $n=1$. Then (2.2) is equivalent to

$$\|Df(a,b,c)\|_\beta\le\phi(a,b,c) \tag{3.3}$$

for all $a,b,c\in X$. Letting $b=a$, $c=-a$ in (3.3), and multiplying both sides by $4^{-\beta}$, we get

$$\|4^{-1}f(2a)-f(a)\|_\beta\le4^{-\beta}\phi(a,a,-a) \tag{3.4}$$

for all $a \in X$. Applying an induction argument to l, we will prove

$$\|4^{-l}f(2^l a) - f(a)\|_\beta \le \sum_{k=0}^{l-1} 4^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a) \tag{3.5}$$

for all $a \in X$ and $l \in \mathbb{N}$. Indeed,

$$\|4^{-(l+1)}f(2^{l+1}a) - f(a)\|_\beta \le \|4^{-(l+1)}f(2^{l+1}a) - 4^{-1}f(2a)\|_\beta + \|4^{-1}f(2a) - f(a)\|_\beta$$

and by (3.4) and (3.5), we obtain

$$\begin{aligned}
&\|4^{-(l+1)}f(2^{l+1}a) - f(a)\|_\beta \\
&\quad \le 4^{-\beta}\sum_{k=0}^{l-1} 4^{-(k+1)\beta}\phi(2^{k+1}a, 2^{k+1}a, -2^{k+1}a) + 4^{-\beta}\phi(a, a, -a) \\
&\quad = 4^{-\beta}\sum_{k=1}^{l} 4^{-k\beta}\phi(2^k a, 2^k a, -2^k a) + 4^{-\beta}\phi(a, a, -a) \\
&\quad = \sum_{k=0}^{l} 4^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a)
\end{aligned}$$

for all $a \in X$ and $l \in \mathbb{N}$, which ends the proof of (3.5).

We will present that the sequence $\{4^{-l}f(2^l a)\}$ is a Cauchy sequence. For $l > m > 0$, we have

$$\begin{aligned}
&\|4^{-l}f(2^l a) - 4^{-m}f(2^m a)\|_\beta = 4^{-m\beta}\|4^{-(l-m)}f(2^{l-m}\cdot 2^m a) - f(2^m a)\|_\beta \\
&\quad \le 4^{-m\beta}\sum_{k=0}^{l-m-1} 4^{-(k+1)\beta}\phi(2^{k+m}a, 2^{k+m}a, -2^{k+m}a) \\
&\quad = \sum_{k=m}^{l-1} 4^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a)
\end{aligned}$$

for all $a \in X$ and $l, m \in \mathbb{N}$. From (3.1), we obtain the sequence $\{4^{-l}f(2^l a)\}$ is a Cauchy sequence. Since Y is complete, the sequence converges to some $Q(a) \in Y$. So one can define the mapping

$$Q(a) = \lim_{l\to\infty} 4^{-l}f(2^l a)$$

for all $a \in X$ and $l \in \mathbb{N}$.

It follows from (3.3) that $\|Df(2^l a, 2^l b, 2^l c)\|_\beta \le \phi(2^l a, 2^l b, 2^l c)$ for all $a, b, c \in X$ and $l \in \mathbb{N}$. Therefore,

$$\|4^{-l}Df(2^l a, 2^l b, 2^l c)\|_\beta \le 4^{-l\beta}\phi(2^l a, 2^l b, 2^l c) \tag{3.6}$$

for all $a, b, c \in X$ and $l \in \mathbb{N}$. It follows from (3.1) that

$$\lim_{l\to\infty} 4^{-l\beta}\phi(2^l a, 2^l b, 2^l c) = 0$$

for all $a, b, c \in X$ and $l \in \mathbb{N}$. Thus, (3.6) implies that $DQ(a, b, c) = 0$, since $f : X \to Y$ is even and $f(0) = 0$, we know $Q : X \to Y$ is even and $Q(0) = 0$. So the mapping $Q : X \to Y$ is quadratic.

Taking the limit in (3.6) as $l \to \infty$, we obtain

$$\|Q(a) - f(a)\|_\beta \le \sum_{k=0}^{\infty} 4^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a) = \Phi(a, a, -a) \tag{3.7}$$

for all $a \in X$.

It remains to show that Q is uniquely defined. Let $Q' : X \to Y$ be another quadratic function satisfying (3.7). Then we get

$$\begin{aligned}\|Q(a) - Q'(a)\|_\beta &= \|4^{-l}Q(2^l a) - 4^{-l}Q^{'}(2^l a)\|_\beta \\ &\le \|4^{-l}Q(2^l a) - 4^{-l}f(2^l a)\|_\beta + \|4^{-l}f(2^l a) - 4^{-l}Q'(2^l a)\|_\beta \\ &\le 2 \cdot 4^{-l\beta}\Phi(2^l a, 2^l a, -2^l a) \\ &= 2\sum_{k=l}^{\infty} 4^{-(k+1)\beta}\phi(2^k a, 2^k a, -2^k a)\end{aligned}$$

for all $a \in X$ and $l \in \mathbb{N}$. Taking the limit in the above inequality as $l \to \infty$, we get $Q(a) = Q'(a)$ for all $a \in X$.

By Lemma 1.3 and (3.7),

$$\begin{aligned}&\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_{\beta,n} = \|[f(x_{ij}) - Q(x_{ij})]\|_{\beta,n} \\ &\le \sum_{i,j=l}^{n} \|f(x_{ij}) - Q(x_{ij})\|_\beta \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -x_{ij})\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. □

Corollary 3.2 *Let r, θ and β be positive real numbers with $r < 2$, $0 < \beta \le 1$ and $f : X \to Y$ be an even mapping satisfying (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_{\beta,n} \le \sum_{i,j=1}^{n} \frac{3}{4^\beta - 2^{\beta r}}\theta\|x_{ij}\|_\beta^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Letting $\phi(a, b, c) = \theta(\|a\|_\beta^r + \|b\|_\beta^r + \|c\|_\beta^r)$ in Theorem 3.1, we get the result. □

Theorem 3.3 *Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and $\phi : X^3 \to [0, \infty)$ be a function satisfying (2.2) and*

$$\Phi(a, b, c) = \sum_{k=1}^{\infty} 4^{(k-1)\beta}\phi(2^{-k}a, 2^{-k}b, 2^{-k}c) < +\infty \tag{3.8}$$

for all $a, b, c \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_n([x_{ij}]) - Q_n([x_{ij}])\|_{\beta,n} \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof The proof is similar to the proof of Theorem 3.1. □

Corollary 3.4 *Let r,θ and β be positive real numbers with $r>2$, $0<\beta\le 1$ and $f:X\to Y$ be an even mapping satisfying (2.9). Then there exists a unique quadratic mapping $Q:X\to Y$ such that*

$$\|f_n([x_{ij}])-Q_n([x_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\frac{3}{2^{\beta r}-4^{\beta}}\theta\|x_{ij}\|_{\beta}^{r}$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof Letting $\phi(a,b,c)=\theta(\|a\|_{\beta}^{r}+\|b\|_{\beta}^{r}+\|c\|_{\beta}^{r})$ in Theorem 3.3, we get the result. $\Box$

Let $f_o([x_{ij}])=\frac{f([x_{ij}])-f([-x_{ij}])}{2}$ and $f_e([x_{ij}])=\frac{f([x_{ij}])+f([-x_{ij}])}{2}$. Then f_o is an odd mapping and f_e is an even mapping such that $f=f_o+f_e$. The above corollaries can be summarized as follows.

Theorem 3.5 *Let r,θ and β be positive real numbers with $r<1$ or $r>2$ and $0<\beta\le 1$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and (2.9). Then there exist a unique additive mapping $A:X\to Y$ and a unique quadratic mapping $Q:X\to Y$ such that*

$$\|f_n([x_{ij}])-A_n([x_{ij}])-Q_n([x_{ij}])\|_{\beta,n}\le 2^{1-\beta}\sum_{i,j=1}^{n}(\frac{3}{|2^{\beta}-2^{\beta r}|}+\frac{3}{|4^{\beta}-2^{\beta r}|})\theta\|x_{ij}\|_{\beta}^{r}$$

for all $x=[x_{ij}]\in M_n(X)$.

4. The Pexider equation

In this section, using the direct method, we prove the generalized Hyers-Ulam-Rassias stability of the Pexider equation (1.3) in matrix β-normed space.

Theorem 4.1 *Let $\varphi:X^2\to[0,\infty)$ be a function satisfying*

$$\Phi(a)=\sum_{k=0}^{\infty}2^{-(k+1)\beta}(\varphi(0,2^ka)+\varphi(2^ka,0)+\varphi(2^ka,2^ka))<+\infty;\tag{4.1}$$

$$\lim_{k\to\infty}2^{-k\beta}\varphi(2^ka,2^kb)=0\tag{4.2}$$

for all $a,b\in X$. If functions $f,g,h:X\to Y$ satisfy the inequality

$$\|f_n([x_{ij}+y_{ij}])-g_n([x_{ij}])-h_n([y_{ij}])\|_{\beta,n}\le\sum_{i,j=1}^{n}\varphi(x_{ij},y_{ij})\tag{4.3}$$

for all $x=[x_{ij}],y=[y_{ij}]\in M_n(X)$, there exists a unique additive function $A:X\to Y$ such that

$$\begin{aligned}\|f_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le&\frac{n^2}{2^{\beta}-1}(\|g(0)\|_{\beta}+\|h(0)\|_{\beta})+\sum_{i,j=1}^{n}\Phi(x_{ij}),\\ \|g_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le&\frac{n^2}{2^{\beta}-1}\|g(0)\|_{\beta}+\frac{2^{\beta}}{2^{\beta}-1}\|h(0)\|_{\beta}+\\ &\sum_{i,j=1}^{n}(\varphi(x_{ij},0)+\Phi(x_{ij})),\qquad(4.4)\\ \|h_n([x_{ij}])-A_n([x_{ij}])\|_{\beta,n}\le&\frac{n^2}{2^{\beta}-1}\|h(0)\|_{\beta}+\frac{2^{\beta}}{2^{\beta}-1}\|g(0)\|_{\beta}+\end{aligned}$$

$$\sum_{i,j=1}^{n}(\varphi(0,x_{ij})+\Phi(x_{ij})).$$

Proof Let $n=1$. Then (4.3) is equivalent to

$$\|f(a+b)-g(a)-h(b)\|_\beta \le \varphi(a,b) \tag{4.5}$$

for all $a,b\in X$. If we put $b=a$ in (4.5), then we have

$$\|f(2a)-g(a)-h(a)\|_\beta \le \varphi(a,a) \tag{4.6}$$

for all $a\in X$. Putting $b=0$ in (4.5) yields that

$$\|f(a)-g(a)-h(0)\|_\beta \le \varphi(a,0) \tag{4.7}$$

for all $a\in X$. It follows from (4.7) that

$$\|g(a)-f(a)\|_\beta \le \|h(0)\|_\beta+\varphi(a,0) \tag{4.8}$$

for all $a\in X$. If we put $a=0$ in (4.5), then we get

$$\|f(b)-g(0)-h(b)\|_\beta \le \varphi(0,b) \tag{4.9}$$

for all $b\in X$. Thus, we obtain

$$\|h(a)-f(a)\|_\beta \le \|g(0)\|_\beta+\varphi(0,a) \tag{4.10}$$

for all $a\in X$. Using the inequalities (4.6), (4.7) and (4.10), we have

$$\begin{aligned}\|f(2a)-2f(a)\|_\beta &\le \|f(2a)-g(a)-h(a)\|_\beta+\|g(a)-f(a)\|_\beta+\|h(a)-f(a)\|_\beta\\ &\le \varphi(a,a)+\|h(0)\|_\beta+\varphi(a,0)+\|g(0)\|_\beta+\varphi(0,a)=:u(a)\end{aligned} \tag{4.11}$$

for all $a\in X$. Multiplying both sides by $2^{-\beta}$ in (4.11), we get

$$\|2^{-1}f(2a)-f(a)\|_\beta \le 2^{-\beta}u(a) \tag{4.12}$$

for all $a\in X$. Replace a with $2^l a$ in (4.11) and multiplying both sides by $2^{-l\beta}$, we get

$$\|2^{-(l+1)}f(2^{l+1}a)-2^{-l}f(2^l a)\|_\beta \le 2^{-(l+1)\beta}u(2^l a) \tag{4.13}$$

for all $a\in X$ and $l\in\mathbb{N}$. Now, we get

$$\begin{aligned}\|2^{-l}f(2^l a)-f(a)\|_\beta &\le \|2^{-l}f(2^l a)-2^{-(l-1)}f(2^{l-1}a)\|_\beta+\cdots+\|2^{-1}f(2a)-f(a)\|_\beta\\ &\le 2^{-l\beta}u(2^{l-1}a)+\cdots+2^{-\beta}u(a)=\sum_{k=0}^{l-1}2^{-(k+1)\beta}u(2^k a)\end{aligned} \tag{4.14}$$

for all $a\in X$ and $l\in\mathbb{N}$. Moreover, if $l>m>0$, then it follows from (4.13) that

$$\begin{aligned}&\|2^{-l}f(2^l a)-2^{-m}f(2^m a)\|_\beta\\ &\quad\le \|2^{-l}f(2^l a)-2^{-(l-1)}f(2^{l-1}a)\|_\beta+\cdots+\|2^{-(m+1)}f(2^{m+1}a)-2^{-m}f(2^m a)\|_\beta\\ &\quad\le 2^{-l\beta}u(2^{l-1}a)+\cdots+2^{-(m+1)\beta}u(2^m a)\\ &\quad=\sum_{k=m}^{l-1}2^{-(k+1)\beta}u(2^k a)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=m}^{l-1} 2^{-(k+1)\beta}(\|g(0)\|_\beta + \|h(0)\|_\beta + \varphi(0, 2^k a) + \varphi(2^k a, 0) + \varphi(2^k a, 2^k a)) \\
&\le 2^{-m\beta}(2^\beta - 1)^{-1}(\|g(0)\|_\beta + \|h(0)\|_\beta) + \\
&\quad \sum_{k=m}^{l-1} 2^{-(k+1)\beta}(\varphi(0, 2^k a) + \varphi(2^k a, 0) + \varphi(2^k a, 2^k a))
\end{aligned}
$$

which tends to 0 as $m \to \infty$ for all $a \in X$ and $l, m \in \mathbb{N}$. Hence, $2^{-l} f(2^l a)$ is a Cauchy sequence for every $a \in X$. Since Y is complete, the sequence converges to some $A(a) \in Y$. So one can define a function $A : X \to Y$ by $A(a) = \lim_{l\to\infty} 2^{-l} f(2^l a)$ for all $a \in X$ and $l \in \mathbb{N}$. In view of (4.5), we obtain

$$\|2^{-l} f(2^l a + 2^l b) - 2^{-l} g(2^l a) - 2^{-l} h(2^l b)\|_\beta \le 2^{-l\beta} \varphi(2^l a, 2^l b)$$

for all $a, b \in X$ and $l \in \mathbb{N}$. It follows from (4.8) that

$$\|2^{-l} g(2^l a) - 2^{-l} f(2^l a)\|_\beta \le 2^{-l\beta}(\|h(0)\|_\beta + \varphi(2^l a, 0)) \tag{4.15}$$

for all $a \in X$ and $l \in \mathbb{N}$. Since $2^{-l\beta}\varphi(2^l a, 0) \to 0$ as $n \to \infty$ for all $a \in X$, one has

$$\lim_{l\to\infty} 2^{-l} g(2^l a) = \lim_{l\to\infty} 2^{-l} f(2^l a) = A(a) \tag{4.16}$$

for all $a \in X$. Also, by (4.10), we have

$$\|2^{-l} h(2^l a) - 2^{-l} f(2^l a)\|_\beta \le 2^{-l\beta}(\|g(0)\|_\beta + \varphi(0, 2^l a)) \tag{4.17}$$

for all $a \in X$ and $l \in \mathbb{N}$. Similarly, it follows from (4.17) that

$$\lim_{l\to\infty} 2^{-l} h(2^l a) = \lim_{l\to\infty} 2^{-l} f(2^l a) = A(a) \tag{4.18}$$

for all $a \in X$. Thus we get

$$0 = \|\lim_{l\to\infty}(2^{-l} f(2^l a + 2^l b) - 2^{-l} g(2^l a) - 2^{-l} h(2^l b))\|_\beta = \|A(a+b) - A(a) - A(b)\|_\beta$$

for all $a, b \in X$. Taking the limit in (4.14) as $l \to \infty$ yields

$$
\begin{aligned}
\|A(a) - f(a)\|_\beta &\le \lim_{l\to\infty} \sum_{k=0}^{l-1} 2^{-(k+1)\beta} u(2^k a) \\
&= \lim_{l\to\infty} \sum_{k=0}^{l-1} 2^{-(k+1)\beta}(\|g(0)\|_\beta + \|h(0)\|_\beta) + \\
&\quad \lim_{l\to\infty} \sum_{k=0}^{l-1} 2^{-(k+1)\beta}(\varphi(0, 2^k a) + \varphi(2^k a, 0) + \varphi(2^k a, 2^k a)) \\
&\le \frac{1}{2^\beta - 1}(\|g(0)\|_\beta + \|h(0)\|_\beta) + \Phi(a)
\end{aligned} \tag{4.19}
$$

for all $a \in X$. So, we can obtain

$$\|g(a) - A(a)\|_\beta \le \frac{1}{2^\beta - 1}\|g(0)\|_\beta + \frac{2^\beta}{2^\beta - 1}\|h(0)\|_\beta + \varphi(a, 0) + \Phi(a), \tag{4.20}$$

$$\|h(a) - A(a)\|_\beta \le \frac{1}{2^\beta - 1}\|h(0)\|_\beta + \frac{2^\beta}{2^\beta - 1}\|g(0)\|_\beta + \varphi(0, a) + \Phi(a) \tag{4.21}$$

for all $a \in X$.

It remains to prove the uniqueness of A. Assume that $A' : X \to Y$ is another additive function which satisfies the inequalities in (4.19). Then we have

$$\begin{aligned}
\|A(a) - A'(a)\|_\beta &\le \|2^{-l}A(2^l a) - 2^{-l}f(2^l a)\|_\beta + \|2^{-l}f(2^l a) - 2^{-l}A'(2^l a)\|_\beta \\
&\le \frac{2}{2^{l\beta}(2^\beta - 1)}(\|g(0)\|_\beta + \|h(0)\|_\beta) + \frac{2}{2^{l\beta}}\Phi(2^l a) \\
&= \frac{2}{2^{l\beta}(2^\beta - 1)}(\|g(0)\|_\beta + \|h(0)\|_\beta) + 2\sum_{k=l}^{\infty} 2^{-(k+1)\beta}(\varphi(0, 2^k a) + \varphi(2^k a, 0) + \varphi(2^k a, 2^k a))
\end{aligned}$$

which tends to 0 as $l \to \infty$ for all $a \in X$, which implies that $A(a) = A'(a)$. By Lemma 1.3, (4.19)–(4.21), we have (4.4). □

Corollary 4.2 *Let r, θ and β be positive real numbers with $r < 1$, $0 < \beta \le 1$ and functions $f, g, h : X \to Y$ satisfy the inequality*

$$\|f_n([x_{ij} + y_{ij}]) - g_n([x_{ij}]) - h_n([y_{ij}])\|_{\beta,n} \le \sum_{i,j=1}^{n} \theta(\|x_{ij}\|_\beta^r + \|y_{ij}\|_\beta^r)$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$, then there exists a unique additive function $A : X \to Y$ such that

$$\begin{aligned}
\|f_n([x_{ij}]) - A_n([x_{ij}])\|_{\beta,n} &\le \frac{n^2}{2^\beta - 1}(\|g(0)\|_\beta + \|h(0)\|_\beta) + \sum_{i,j=1}^{n} \frac{4\theta}{2^\beta - 2^{\beta r}}\theta\|x_{ij}\|_\beta^r, \\
\|g_n([x_{ij}]) - A_n([x_{ij}])\|_{\beta,n} &\le \frac{n^2}{2^\beta - 1}\|g(0)\|_\beta + \frac{2^\beta}{2^\beta - 1}\|h(0)\|_\beta + \sum_{i,j=1}^{n} \theta(1 + \frac{4\theta}{2^\beta - 2^{\beta r}})\|x_{ij}\|_\beta^r, \\
\|h_n([x_{ij}]) - A_n([x_{ij}])\|_{\beta,n} &\le \frac{n^2}{2^\beta - 1}\|h(0)\|_\beta + \frac{2^\beta}{2^\beta - 1}\|g(0)\|_\beta + \sum_{i,j=1}^{n} \theta(1 + \frac{4\theta}{2^\beta - 2^{\beta r}})\|x_{ij}\|_\beta^r
\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$.

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