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A Note on Invertible Weighted Composition Operators on the Fock Space of $\mathbb{C}^N$

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Abstract This paper shows that a bounded invertible weighted composition operator on the Fock space of $\mathbb{C}^N$ is nonzero multiples of a unitary operator, which is an addition to the recent result on invertible weighted composition operator on the Fock space of $\mathbb{C}^N$ and an extension to the corresponding result on the Fock space of $\mathbb{C}$.

Keywords Fock space; weighted composition operator; invertible; unitary

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1. Introduction

In [1] the author gave a characterization for bounded invertible weighted composition operators on the Fock space of $\mathbb{C}^N$. By analyzing the conditions satisfied by invertible weighted composition operators carefully, this paper shows that a bounded invertible weighted composition operator is nonzero multiples of a unitary operator on the Fock space of $\mathbb{C}^N$, which extends a corresponding result on the Fock space of $\mathbb{C}$ (see [2]) also.

Firstly, we recall some basic knowledge on Fock space.

The Fock space $\mathcal{F}^2$ is the space of analytic functions f on $\mathbb{C}^N$ for which

$$\|f\|^2=\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N}|f(z)|^2e^{-\frac{|z|^2}{2}}\mathrm{d}m_{2N}(z),$$

where $\mathrm{d}m_{2N}$ is the usual Lebesgue measure on $\mathbb{C}^N$ and $|z|$ denotes the norm for $z\in\mathbb{C}^N$. $\frac{1}{(2\pi)^N}e^{-\frac{|z|^2}{2}}\mathrm{d}m_{2N}(z)$ is called Gaussian measure on $\mathbb{C}^N$ also.

It is well-known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with reproducing kernel functions

$$K_w(z)=\exp(\frac{\langle z,w\rangle}{2}),\quad w,z\in\mathbb{C}^N$$

and inner product

$$\langle f,g\rangle=\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N}f(z)\overline{g(z)}e^{-\frac{|z|^2}{2}}\mathrm{d}m_{2N}(z),\quad f,g\in\mathcal{F}^2.$$

Note that we use $\langle z,w\rangle$ to denote the inner product for $z,w\in\mathbb{C}^N$ also. $|z|^2=\langle z,z\rangle$.

Let ψ be an analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. Define the weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ as $C_{\psi,\varphi}f=\psi(f\circ\varphi)$, $f\in\mathcal{F}^2$.

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The main result reads as follows.

Theorem 1.1 *Let ψ be an analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathcal{F}^2$ if and only if $C_{\psi,\varphi}$ is nonzero multiples of a unitary weighted composition operator on $\mathcal{F}^2$.*

A complete characterization of unitary weighted composition operators and their spectrum on the Fock space of $\mathbb{C}^N$ has been obtained in [3]. In [4,5], the bounded, compact and Hilbert-Schmidt weighted composition operators on the Fock space of $\mathbb{C}$ were characterized respectively, the corresponding results can be extended to the Fock space of $\mathbb{C}^N$ word by word. In [6], relative simple conditions were given for bounded and compact weighted composition operators on the Fock space of $\mathbb{C}$, as applications, normal and isometric weighted composition operators on the Fock space of $\mathbb{C}$ are characterized completely. The characterization of self-adjoint and Fredholm weighted composition operators on the Fock space of $\mathbb{C}$ appeared in [7] and [2], respectively. The book [8] is a good reference for the Fock space of $\mathbb{C}$ and operators. We refer to [9,10] for other studies on weighted composition operators between different Fock spaces.

2. Proof of the main result

For an operator A on $\mathbb{C}^N$, denote by $|A|$ the norm of A. The proof of our main result is based on the following results.

Lemma 2.1 ([1, Theorem 1]) *Let ψ be an analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded invertible operator on $\mathcal{F}^2$ if and only if*

$$\varphi(z) = Az + b$$

for some invertible operator A on $\mathbb{C}^N$ with $|A| = 1$, $b \in \mathbb{C}^N$, and there exist positive constants M, L such that

$$L \le |\psi(z)|^2 \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}) \le M, \quad z \in \mathbb{C}^N.$$

Let k_w be the normalization of K_w, i.e.,

$$k_w(z) = \exp(\frac{\langle z, w\rangle}{2} - \frac{|w|^2}{4}), \quad z, w \in \mathbb{C}^N.$$

Lemma 2.2 ([3, Theorem 1.1]) *Let ψ be an analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a unitary operator on $\mathcal{F}^2$ if and only if there exists a unitary operator A on $\mathbb{C}^N$, $b \in \mathbb{C}^N$ and a constant α with $|\alpha| = 1$ such that*

$$\varphi(z) = Az + b, \quad \psi(z) = \alpha k_{-A^*b}(z), \ z \in \mathbb{C}^N.$$

In the following, we give a deep analysis of the conditions presented in Lemma 2.1.

Lemma 2.3 *Let ψ be an analytic function and A be an operator on $\mathbb{C}^N$ with $|A| \le 1$, $b \in \mathbb{C}^N$. Let $\varphi(z) = Az + b$, $z \in \mathbb{C}^N$. If there exists a positive constant L such that*

$$L \le |\psi(z)|^2 \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}), \quad z \in \mathbb{C}^N, \tag{2.1}$$

then there exists a nonzero constant s *such that*

$$\psi(z) = sk_{-A^*b}(z), \quad z \in \mathbb{C}^N$$

and A *is a unitary operator on* $\mathbb{C}^N$.

Proof By (2.1), ψ has no zeros in $\mathbb{C}^N$. Since $|A| \le 1$, we have

$$|\varphi(z)|^2 - |z|^2 = |Az+b|^2 - |z|^2 = |Az|^2 - |z|^2 + \langle Az, b\rangle + \langle b, Az\rangle + |b|^2 \le \langle Az, b\rangle + \langle b, Az\rangle + |b|^2.$$

It follows from (2.1) that

$$L \le |\psi(z)|^2 \exp(\frac{\langle Az, b\rangle + \langle b, Az\rangle + |b|^2}{2}) = \big|\psi(z)\exp(\frac{\langle Az, b\rangle}{2} + \frac{|b|^2}{4})\big|^2,$$

i.e.,

$$\Big|\frac{1}{\psi(z)\exp(\frac{\langle Az,\ b\rangle}{2} + \frac{|b|^2}{4})}\Big|^2 \le \frac{1}{L}.$$

By Liouville Theorem, there exists a constant t such that $\frac{1}{\psi(z)\exp(\frac{\langle Az,\ b\rangle}{2} + \frac{|b|^2}{4})} = t$. Obviously $t \ne 0$. Let $s = \frac{1}{t}\exp(\frac{|A^*b|^2 - |b|^2}{4})$. Then

$$\psi(z) = sk_{-A^*b}(z), \ z \in \mathbb{C}^N.$$

Substituting the equation above into (2.1), we obtain that there exists a positive constant l such that

$$\exp(\frac{|Az|^2 - |z|^2}{2}) \ge l, \quad z \in \mathbb{C}^N.$$

i.e., $|Az|^2 - |z|^2 \ge \log l^2$, $z \in \mathbb{C}^N$. Since $|A| \le 1$, $0 < l \le 1$. Let $\delta = -\log l^2$. Then $\delta \ge 0$ and

$$|z|^2 - \delta \le |Az|^2, \quad z \in \mathbb{C}^N.$$

For $t \in \mathbb{R}^+$, we have $|tz|^2 - \delta \le |Atz|^2$, $z \in \mathbb{C}^N$. i.e.,

$$|z|^2 - \frac{1}{t^2}\delta \le |Az|^2, \quad z \in \mathbb{C}^N.$$

Let $t \to \infty$. Then $|z|^2 \le |Az|^2$, $z \in \mathbb{C}^N$.

On the other hand, $|A| \le 1$. So we have $|Az| = |z|$, $z \in \mathbb{C}^N$, which implies that A is a unitary operator on $\mathbb{C}^N$. □

Combining Lemmas 2.1–2.3, we obtain Theorem 1.1. Explicitly, we have the following corollary.

Corollary 2.4 *Let* ψ *be an analytic function and* φ *be an analytic self-mapping on* $\mathbb{C}^N$. *Then* $C_{\psi,\varphi}$ *is a bounded invertible operator on* $\mathcal{F}^2$ *if and only if there exist a unitary operator* A *on* $\mathbb{C}^N$, $b \in \mathbb{C}^N$ *and a nonzero constant* s *such that*

$$\varphi(z) = Az + b, \ \psi(z) = sk_{-A^*b}(z), \ z \in \mathbb{C}^N.$$

In the following, we give another application of Lemma 2.3.

Recall an operator T on a Hilbert space $\mathcal{H}$ is called co-isometric if TT^* is the identity on $\mathcal{H}$. The following lemma is part of [1, Proposition 7].

Lemma 2.5 *Let ψ be a nonzero analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then there exists an operator A on $\mathbb{C}^N$ with $|A| \le 1$ and $b \in \mathbb{C}^N$ such that $\varphi(z) = Az + b$, $z \in \mathbb{C}^N$.*

Corollary 2.6 *Let ψ be an analytic function and φ be an analytic self-mapping on $\mathbb{C}^N$. $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$. Then $C_{\psi,\varphi}$ is a co-isometric operator on $\mathcal{F}^2$ if and only if $C_{\psi,\varphi}$ is a unitary operator.*

Proof The sufficiency is obvious.

If $C_{\psi,\varphi}$ is co-isometric on $\mathcal{F}^2$, then $C^*_{\psi,\varphi}$ is bounded below. There exists a positive constant L such that $\|C^*_{\psi,\varphi}K_z\|^2 \ge L\|K_z\|^2$, $z \in \mathbb{C}^N$. In fact, $L = 1$ since $C_{\psi,\varphi}$ is co-isometric. So we have

$$1 \le |\psi(z)|^2 \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}), \quad z \in \mathbb{C}^N.$$

On the other hand, since $C_{\psi,\varphi}$ is bounded, by Lemma 2.5,

$$\varphi(z) = Az + b, \quad z \in \mathbb{C}^N$$

for some operator A on $\mathbb{C}^N$ with $|A| \le 1$ and $b \in \mathbb{C}^N$.

By Lemma 2.3, A is a unitary operator on $\mathbb{C}^N$ and there is a non-zero constant s such that $C_{\psi,\varphi}$ can be represented by

$$(C_{\psi,\varphi}f)(z) = sk_{-A^*b}(z) \cdot f(Az + b), \quad f \in \mathbb{C}^N.$$

Put $Uf(z) = k_{-A^*b}(z) \cdot f(Az + b)$, $z \in \mathbb{C}^N$. Then $C_{\psi,\varphi} = sU$ and U is a unitary operator on $\mathcal{F}^2$ by Lemma 2.2. Since $C_{\psi,\varphi}$ is co-isometric, $|s| = 1$. Hence $C_{\psi,\varphi}$ is unitary on $\mathcal{F}^2$ also. □

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