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Expansive Behaviours for Homeomorphisms of tvs-Cone Metric Spaces

Xun GE
School of Mathematical Sciences, Soochow University, Jiangsu 215006*, P. R. China*

Abstract Let f be a homeomorphism of a compact tvs-cone metric space. In this paper, we show that f is tvs-cone expansive if and only if f has a generator. Further, it is proved that if f is tvs-cone expansive, then the set of points having converging semiorbits under f is a countable set. Results of this paper improve some expansive homeomorphisms theorems in topological dynamics, which will help to research dynamical properties for homeomorphisms of tvs-cone metric spaces.

Keywords tvs-cone metric space; tvs-cone expansive homeomorphism; generator; converging semiorbit

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1. Introduction

Ordered normed spaces and cones have many applications in applied mathematics, for instance, in using Newton's approximation method [1–4] and in optimization theory [5]. By using an ordered Banach space instead of the set of real numbers as the codomain for a metric, Kantorovich [2] introduced K-metric and K-normed spaces in the mid-20th century [3,4,6]. Such spaces under the name of cone metric spaces were re-introduced by Huang and Zhang in [7], and many relevant results have been obtained [7–11]. In particular, Khani and Pourmahdian [11] proved that each cone metric space is metrizable, which results in that many results around cone metric spaces were trivial. However, just as stated in [11], "considering certain topological groups in place of Banach spaces may result in the construction of new spaces which are not in general metrizable. This can serve as a topic for further studies". In [12], Du introduced and investigated tvs-cone metric spaces by replacing Banach spaces with topological vector spaces in the definition of cone metric spaces. Over the past years, tvs-cone metric spaces had aroused many mathematical scholars' interests and the following question was investigated [12–16].

Question 1.1 Can what results on metric spaces be generalized to tvs-cone metric spaces?

As expansive behaviours for homeomorphism of spaces, the following results play an important role in study of dynamical systems.

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E-mail address: gexun@suda.edu.cn

Proposition 1.2 ([17]) *Let f be a homeomorphism of a compact metric space. Then f is expansive if and only if f has a generator.*

Proposition 1.3 ([18]) *Let f be an expansive homeomorphism of a compact metric space. Then the set of points having converging semiorbits under f is a countable set.*

In this paper, we investigate Question 1.1 combining the above two propositions. As the main results of this paper, we prove that "metric" and "expansive" in Proposition 1.2 (resp., Proposition 1.3) can be relaxed to "tvs-cone metric space" and "tvs-cone expansive", respectively. Results of this paper improve some expansive homeomorphisms theorems in topological dynamics, which will be of help to research dynamical properties for homeomorphisms of tvs-cone metric spaces.

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^*$ denote the set of all natural numbers, the set of all integral numbers, the set of all positive real numbers and the set of all nonnegative real numbers, respectively.

2. tvs-Cone metric spaces

Definition 2.1 ([12]) *Let E be a topological vector space with its zero vector θ. A subset P of E is called a tvs-cone in E if the following are satisfied.*

(i) P is a closed subset in E with a nonempty interior.

(ii) $\alpha, \beta \in P$ and $a, b \in \mathbb{R}^* \Longrightarrow a\alpha + b\beta \in P$.

(iii) $\alpha, -\alpha \in P \Longrightarrow \alpha = \theta$.

Remark 2.2 Let P be a tvs-cone in a topological vector space E. The interior of P is always denoted by P°. It is known that $\theta \in P - P^\circ$ (see [16]).

Definition 2.3 ([12]) *Let E be a topological vector space with a tvs-cone P. Some partial orderings $\leq$, $<$ and $\ll$ on E with respect to P are defined as follows, respectively. Let $\alpha, \beta \in E$.*

(i) $\alpha \leq \beta$ if $\beta - \alpha \in P$.

(ii) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

(iii) $\alpha \ll \beta$ if $\beta - \alpha \in P^\circ$.

Then pair (E, P) is called an ordered topological vector space.

Remark 2.4 ([16]) For the sake of conveniences, we also use notations "$\geq$", "$>$" and "$\gg$" on E with respect to P. The meanings of these notations are clear and the following hold.

(i) $\alpha \geq \beta \Longleftrightarrow \alpha - \beta \geq \theta \Longleftrightarrow \alpha - \beta \in P$.

(ii) $\alpha > \beta \Longleftrightarrow \alpha - \beta > \theta \Longleftrightarrow \alpha - \beta \in P - \{\theta\}$.

(iii) $\alpha \gg \beta \Longleftrightarrow \alpha - \beta \gg \theta \Longleftrightarrow \alpha - \beta \in P^\circ$.

(iv) $\alpha \gg \beta \Longrightarrow \alpha > \beta \Longrightarrow \alpha \geq \beta$.

Lemma 2.5 ([16]) *Let (E, P) be an ordered topological vector space. Then the following hold.*

(i) If $\alpha \gg \theta$, then $r\alpha \gg \theta$ for each $r \in \mathbb{R}^+$.

(ii) If $\alpha_1 \gg \beta_1$ and $\alpha_2 \geq \beta_2$, then $\alpha_1 + \alpha_2 \gg \beta_1 + \beta_2$.

(iii) If $\alpha \gg \theta$ and $\beta \gg \theta$, then there is $\gamma \gg \theta$ such that $\gamma \ll \alpha$ and $\gamma \ll \beta$.

Definition 2.6 ([12]) *Let (E, P) be an ordered topological vector space and let X be a non-empty set. A mapping $d : X \times X \longrightarrow E$ is called a tvs-cone metric and (X, d) is called a tvs-cone metric space if the following are satisfied.*

(i) $d(x, y) \geq \theta$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark 2.7 It is clear that every metric is a tvs-cone metric and a tvs-cone metric need not to be a metric. In [14,15], it was obtained that every tvs-cone metric space is metrizable under assumption that the topological vector space is locally convex and Hausdorff. Recently, it was proved that the above assumption can be omitted [19, Theorem 3.2].

In this paper, we always suppose that

(i) A tvs-cone metric space (X, d) deals with an ordered topological vector space (E, P);

(ii) A tvs-cone metric space (X, d) is a topological space with the topology $\mathscr{T}$ described in the following Proposition 2.8.

Proposition 2.8 ([16]) *Let (X, d) be a tvs-cone metric space. For $x \in X$ and $\varepsilon \gg 0$, put $B(x, \varepsilon) = \{y \in X : d(x, y) \ll \varepsilon\}$. Put $\mathscr{B} = \{B(x, \varepsilon) : x \in X$ and $\varepsilon \gg \theta\}$, and put $\mathscr{T} = \{U \subseteq X$: there is $\mathscr{B}' \subseteq \mathscr{B}$ such that $U = \bigcup \mathscr{B}'\}$. Then $\mathscr{T}$ is a topology on X and $\mathscr{B}$ is a base for $\mathscr{T}$.*

Definition 2.9 *Let (X, d) be a tvs-cone metric space and $\mathscr{U}$ be an open cover of X. $\varepsilon \gg \theta$ is called a Lebesgue element of $\mathscr{U}$ if for every $A \subseteq X$, $d(A) \ll \varepsilon$ implies $A \subseteq U$ for some $U \in \mathscr{U}$.*

Lemma 2.10 *Let (X, d) be a compact tvs-cone metric space. Then every open cover of X has a Lebesgue element.*

Proof Let $\mathscr{U}$ be an open cover of X. For every $x \in X$, there are $\varepsilon_x \gg \theta$ and $U_x \in \mathscr{U}$ such that $B(x, \varepsilon_x) \subseteq U_x$. Since X is compact, there is a finite subset F of X such that $\{B(x, \varepsilon_x/2) : x \in F\}$ covers X. By Lemma 2.5(iii), there is $\varepsilon \gg \theta$ such that $\varepsilon \ll \varepsilon_x/2$ for every $x \in F$. Let $A \subseteq X$ satisfying $d(A) \ll \varepsilon$. Pick $y \in A$, then there is $x \in F$ such that $y \in B(x, \varepsilon_x/2)$. For every $z \in A$, $d(z, x) \leq d(z, y) + d(y, x) \ll \varepsilon + \varepsilon_x/2 \ll \varepsilon_x$, so $z \in B(x, \varepsilon_x) \subseteq U_x$. This proves that $A \subseteq U_x$, and it follows that ε is a Lebesgue element of $\mathscr{U}$. □

3. Expansive behaviours of tvs-cone metric spaces

Definition 3.1 ([19]) *Let $f : X \longrightarrow X$ be a homeomorphism of a space X. f is called tvs-cone expansive if there are a tvs-cone metric d on X and $\varepsilon \gg \theta$ such that $x, y \in X$ with $x \neq y$ implies $d(f^n(x), f^n(y)) \gg \varepsilon$ for some $n \in \mathbb{Z}$. Here, ε is called a tvs-cone expansive constant for f.*

Remark 3.2 If "tvs-cone metric", "$\gg$" and "θ" in Definition 3.1 are replaced by "metric", "$>$" and "0", respectively, then the definition of expansive homeomorphism of a space X is obtained

[20].

Definition 3.3 ([17]) *Let $f : X \longrightarrow X$ be a homeomorphism of a space X. A finite open cover $\mathscr{F}$ of X is called a generator for f if for every bisequence $\{F_n : n \in \mathbb{Z}\}$ consisting of members of $\mathscr{F}$, $\bigcap\{f^{-n}(\overline{F}_n) : n \in \mathbb{Z}\}$ is at most one point.*

Definition 3.4 ([18]) *Let $f : X \longrightarrow X$ be a homeomorphism of a space X, and $x \in X$. α-limit set $\alpha(x)$ and ω-limit set $\omega(x)$ are defined as follows.*

$$\alpha(x) = \{y = \lim_{k\to+\infty} f^{n_k}(x) : \{n_k\} \text{ is a strictly decreasing sequence in } \mathbb{Z}\};$$

$$\omega(x) = \{y = \lim_{k\to+\infty} f^{n_k}(x) : \{n_k\} \text{ is a strictly increasing sequence in } \mathbb{Z}\}.$$

x is called to have converging semiorbits under f if $\alpha(x)$ and $\omega(x)$ each consists of a single point.

The following theorem gives a necessary and sufficient condition such that a homeomorphism of a compact tvs-cone metric space is tvs-cone expansive.

Theorem 3.5 *Let f be a homeomorphism of a compact tvs-cone metric space X. Then the following are equivalent.*

(i) f is tvs-cone expansive.

(ii) f has a generator.

Proof (i) $\Longrightarrow$ (ii). Assume that (i) holds. Let $\varepsilon \gg \theta$ be a tvs-cone expansive constant for f with respect to a tvs-cone metric d. Pick $\theta \ll \alpha \ll \varepsilon/2$ and put $\mathscr{U} = \{B(x,\alpha) : x \in X\}$. Then $\mathscr{U}$ is an open cover of X. Since X is compact, $\mathscr{U}$ has a finite subcover $\mathscr{F}$ of $\mathscr{U}$. It suffices to prove that $\mathscr{F}$ is a generator for f. If $\mathscr{F}$ is not a generator for f, then there are a bisequence $\{F_n : n \in \mathbb{Z}\}$ and $x, y \in \bigcap\{f^{-n}(\overline{F}_n) : n \in \mathbb{Z}\}$, where $F_n \in \mathscr{F}$ for every $n \in \mathbb{Z}$ and $x \neq y$. Since ε is a tvs-cone expansive constant for f, there is $i \in \mathbb{Z}$ such that $d(f^i(x), f^i(y)) \gg \varepsilon$. We can write $F_i = B(z,\alpha)$ for some $z \in X$. Put $\beta = d(f^i(x), f^i(y)) - \varepsilon$, then $\beta \gg \theta$. Since $x, y \in f^{-i}(\overline{B(z,\alpha)})$, $f^i(x), f^i(y) \in \overline{B(z,\alpha)}$. Pick $\theta \ll \gamma \ll \beta/2$, then there is $u \in B(f^i(x),\gamma) \cap B(z,\alpha) \neq \emptyset$, and then $d(f^i(x), z) \leq d(f^i(x), u) + d(u, z) \ll \gamma + \alpha$. By the same way, we have $d(f^i(y), z) \ll \gamma + \alpha$. It follows that $d(f^i(x), f^i(y)) \leq d(f^i(x), z) + d(f^i(y), z) \ll 2\gamma + 2\alpha \ll \beta + \varepsilon = d(f^i(x), f^i(y))$. This contradicts that $d(f^i(x), f^i(y)) = d(f^i(x), f^i(y))$.

(ii)$\Longrightarrow$(i). It holds by Remark 2.7 and Proposition 1.2. $\square$

Now we discuss the set of points having converging semiorbits under a tvs-cone expansive homeomorphism.

Lemma 3.6 *Let f be a homeomorphism of a compact tvs-cone metric space X. If f is tvs-cone expansive, then f has at most finitely many fixed points.*

Proof Let f be tvs-cone expansive. By Theorem 3.5, f has a generator $\mathscr{F}$. Note that $\mathscr{F}$ is finite. If f has infinitely many fixed points, then there is $A \in \mathscr{F}$ such that A contains at least two fixed points x, y. Thus, for every $n \in \mathbb{Z}$, $f^n(x) = x, f^n(y) = y \in A$, i.e., $x, y \in f^{-n}(A)$. It follows that $x, y \in \bigcap\{f^{-n}(A) : n \in \mathbb{Z}\}$. This contradicts that $\mathscr{F}$ is a generator.

Lemma 3.7 *Let f be a homeomorphism of a compact space X. If x has converging semiorbits under f, then $\alpha(x)$ and $\omega(x)$ each consists of a fixed point.*

Proof Let x have converging semiorbits under f. If $u \in \alpha(x)$, then $u = \lim_{k\to+\infty} f^{n_k}(x)$ for some strictly decreasing sequence $\{n_k\}$ in $\mathbb{Z}$. Note that $\{n_k + 1\}$ is a strictly decreasing sequence in $\mathbb{Z}$ and $f(u) = \lim_{k\to+\infty} f^{n_k+1}(x)$. So $f(u) \in \alpha(x)$. Since $\alpha(x)$ consists of a single point, $f(u) = u$, i.e., $\alpha(x) = \{u\}$ and u is a fixed point. Similarly, $\omega(x) = \{v\}$ for some $v \in X$ and v is a fixed point. □

Lemma 3.8 *Let f be a homeomorphism of a compact tvs-cone metric space (X, d). Then, for any $\varepsilon \gg \theta$ and any $k \in \mathbb{N}$, there is $\delta \gg \theta$ such that $d(x, y) \ll \delta$ implies $d(f^n(x), f^n(y)) \ll \varepsilon$ for all $x, y \in X$ and all $n \in \mathbb{Z}$ with $|n| \leq k$.*

Proof Let $\varepsilon \gg \theta$ and $k \in \mathbb{N}$. If $n \in \mathbb{Z}$ with $|n| \leq k$, then f^n is continuous. So, for every $x \in X$, there is $\delta_n(x) \gg \theta$ such that $y \in B(x, \delta_n(x))$ implies $f^n(y) \in B(f^n(x), \varepsilon/2)$. Thus, $\mathscr{U} = \{B(x, \delta_n(x)) : x \in X\}$ is an open cover of X, so there is a Lebesgue element δ_n of $\mathscr{U}$ from Lemma 2.10. If $x, y \in X$ with $d(x, y) \ll \delta_n$, then there is $z \in X$ such that $x, y \in B(z, \delta_n(z))$, hence $f^n(x), f^n(y) \in B(f^n(z), \varepsilon/2)$, i.e., $d(f^n(x), f^n(z)) \ll \varepsilon/2$ and $d(f^n(y), f^n(z)) \ll \varepsilon/2$. It follows that $d(f^n(x), f^n(y)) \leq d(f^n(x), f^n(z)) + d(f^n(y), f^n(z)) \ll \varepsilon$. By Lemma 2.5(iii), there is $\delta \gg \theta$ such that $\delta \ll \delta_n$ for every $n \in \mathbb{Z}$ with $|n| \leq k$. Consequently, for all $x, y \in X$ and all $n \in \mathbb{Z}$ with $|n| \leq k$, if $d(x, y) \ll \delta$, then $d(x, y) \ll \delta_n$, hence $d(f^n(x), f^n(y)) \ll \varepsilon$. □

Theorem 3.9 *Let f be a tvs-cone expansive homeomorphism of a compact tvs-cone metric space X. Then the set A of points having converging semiorbits under f is a countable set.*

Proof Let $\varepsilon \gg \theta$ be a tvs-cone expansive constant for f with respect to a tvs-cone metric d. By Lemma 3.6, f has at most finitely many fixed points, say $a_1, a_2, \ldots, a_t$. Suppose that A is uncountable. For every $x \in A$, $\alpha(x)$ and $\omega(x)$ each consists of a fixed point from Lemma 3.7. Put $A(i, j) = \{x \in A : \alpha(x) = \{a_i\}$ and $\omega(x) = \{a_j\}\}$. It is easy to see that $A = \bigcup\{A(i, j) : i, j = 1, 2, \ldots, t\}$. Thus, $A(i_0, j_0)$ is uncountable for some $i_0, j_0 = 1, 2, \ldots, t$. For every $k \in \mathbb{N}$, put $A(k) = \{x \in A(i_0, j_0) : d(f^n(x), a_{j_0}) \ll \varepsilon/2$ and $d(f^{-n}(x), a_{i_0}) \ll \varepsilon/2$ for all $n > k\}$. We claim that $A(i_0, j_0) = \bigcup\{A(k) : k \in \mathbb{N}\}$. In fact, if $x \in A(i_0, j_0)$, then $\alpha(x) = \{a_{i_0}\}$ and $\omega(x) = \{a_{j_0}\}$, hence $\lim_{n\to+\infty} f^n(x) = a_{j_0}$ and $\lim_{n\to+\infty} f^{-n}(x) = a_{i_0}$. So there is $k \in \mathbb{N}$ such that $d(f^n(x), a_{j_0}) \ll \varepsilon/2$ and $d(f^{-n}(x), a_{i_0}) \ll \varepsilon/2$ for all $n > k$. It follows that $x \in A(k)$. This shows that $A(i_0, j_0) \subseteq \bigcup\{A(k) : k \in \mathbb{N}\}$. Note that $A(i_0, j_0) \supseteq \bigcup\{A(k) : k \in \mathbb{N}\}$. So $A(i_0, j_0) = \bigcup\{A(k) : k \in \mathbb{N}\}$. Thus, $A(k_0)$ is infinite for some $k_0 \in \mathbb{N}$. Let $\delta \gg \theta$ be described as in Lemma 3.9 (for $k = k_0$). By compactness of X, there are $x, y \in A(k_0)$ such that $x \neq y$ and $d(x, y) \ll \delta$. Since ε is a tvs-cone expansive constant for f, there is $n_0 \in \mathbb{Z}$ such that $d(f^{n_0}(x), f^{n_0}(y)) \gg \varepsilon$, i.e., $d(f^{n_0}(x), f^{n_0}(y)) - \varepsilon \gg \theta$. On the other hand, we can obtain $d(f^{n_0}(x), f^{n_0}(y)) \ll \varepsilon$. In fact, if $|n_0| \leq k_0$, then $d(f^{n_0}(x), f^{n_0}(y)) \ll \varepsilon$ from Lemma 3.8; if $n_0 > k_0$, then $d(f^{n_0}(x), a_{j_0}) \ll \varepsilon/2$ and $d(f^{n_0}(y), a_{j_0}) \ll \varepsilon/2$, hence $d(f^{n_0}(x), f^{n_0}(y)) \ll \varepsilon$; if $n_0 < -k_0$, then $d(f^{n_0}(x), a_{i_0}) \ll \varepsilon/2$ and $d(f^{n_0}(y), a_{i_0}) \ll \varepsilon/2$, hence $d(f^{n_0}(x), f^{n_0}(y)) \ll \varepsilon$. Thus, $\varepsilon - d(f^{n_0}(x), f^{n_0}(y)) \gg \theta$. By

Lemma 2.5(ii), $\theta = (d(f^{n_0}(x), f^{n_0}(y)) - \varepsilon) + (\varepsilon - d(f^{n_0}(x), f^{n_0}(y))) \gg \theta$ i.e., $\theta \in P^\circ$. This contradicts Remark 2.2. $\square$

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References

[1] L. V. KANTOROVICH. *The majorant principle and Newton's method.* Dokl. Akad. Nauk SSSR (NS), 1951, **76**: 17–20.
[2] L. V. KANTOROVICH. *On some further applications of the Newton approximation method.* Vestn. Leningr. Univ. Ser. Mat. Mekh. Astron., 1957, **12**(7): 68–103.
[3] J. S. VANDERGRAFT. *Newton's method for convex operators in partially ordered spaces.* SIAM J. Numer. Anal., 1967, **4**: 406–432.
[4] P. P. ZABREIKO. *K-metric and K-normed spaces: survey.* Collect. Math., 1997, **48**(4-6): 825–859.
[5] K. DEIMLING. *Nonlinear Functional Analysis.* Springer-Verlag, 1985.
[6] C. D. ALIPRANTIS, R. TOURKY. *Cones and Duality.* American Mathematical Society, Providence, Rhode Island, 2007.
[7] Longguang HUANG, Xian ZHANG. *Cone metric spaces and fixed point theorems of contractive mappings.* J. Math. Anal. Appl., 2007, **332**(2): 1468–1476.
[8] A. AMINI-HARANDI, M. FAKHAR. *Fixed point theory in cone metric spaces obtained via the scalarization method.* Comput. Math. Appl., 2010, **59**(11): 3529–3534.
[9] A. SÖNMEZ, *On paracompactness in cone metric spaces.* Appl. Math. Lett., 2010, **23**(4): 494–497.
[10] D. TURKOGLU, M. ABULOHA. *Cone metric spaces and fixed point theorems in diametrically contractive mappings.* Acta Math. Sin. (Engl. Ser.), 2010, **26**(3): 489–496.
[11] M. KHAMI, M. POURMAHDIAN. *On the metrizability of cone metric space.* Topology Appl., 2011, **158**(2): 190–193.
[12] W. S. DU, *A note on cone metric fixed point theory and its equivalence.* Nonlinear Anal., 2010, **72**(5): 2259–2261.
[13] S. RADENOVIC, S. SIMIC, N. CAKIC, et al. *A note on tvs-cone metric fixed point theory.* Math. Comput. Modelling, 2011, **54**(9-10): 2418–2422.
[14] Z. KADELBURG, S. RADENOVIC, V. RAKOCEVIC. *A note on the equivalence of some metric and cone metric fixed point results.* Appl. Math. Lett., 2011, **24**(3): 370–374.
[15] H. CAKALLI, A. SÖNMEZ, C. GENR. *On an equivalence of topological vector space valued cone metric spaces and metric spaces.* Appl. Math. Lett., 2012, **25**(3): 429–433.
[16] Shou LIN, Ying GE. *Compact-valued continuous relations on tvs-cone metric spaces.* Filomat, 2013, **27**(2): 329–335.
[17] H. KEYNESS, J. ROBERTSON. *Generators for topological entropy and expansiveness.* Math. Systems Theory, 1969, **3**: 51–59.
[18] W. RADDY. *The existence of expansive homeomorphisms on manifolds.* Duke Math. J., 1965, **32**: 627–632.
[19] Shou LIN, Kedian LI, Ying GE. *On the metrizability of TVS-cone metric spaces.* Publications de l'Institut Mathématique, 2015, **98**: 271–279.
[20] N. AOKI. *Topological Dynamics.* In: K. Morita, J. Nagata eds., Topics in General Topology, Amsterdam: North-Holland, 1989.