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Optimal $(24u, \{3, 4\}, 1, \{2/3, 1/3\})$ Optical Orthogonal Codes

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Abstract Variable-weight optical orthogonal codes (OOCs) were introduced by G. C. YANG for multimedia optical CDMA systems with multiple quality of service (QoS) requirements. In this paper, some infinite classes of optimal cyclic packing are presented. Optimal $(24u, \{3, 4\}, 1, \{2/3, 1/3\})$-OOCs for any positive integer $u > 1$ are established.

Keywords cyclic packing; optical orthogonal code; perfect cyclic packing; skew starter; variable-weight optical orthogonal code

MR(2010) Subject Classification 05B05; 05B40

1. Introduction

Optical orthogonal codes (OOCs) were introduced by Salehi, as signature sequences to facilitate multiple access in optical fibre networks [1,2]. OOCs had been found wide ranges of applications such as mobile radio, frequency-hopping spread-spectrum communications, radar, sonar, collision channel without feedback, and neuromorphic networks [3–7].

Most existing works on OOCs have assumed that all codewords have the same weight, see [3,8–27] for the examples. In general, the code size of OOCs depends upon the weights of codewords, the variable-weight OOCs can generate larger code size than that of constant-weight OOCs [28]. In 1996, Yang introduced multimedia optical CDMA communication system employing variable-weight OOCs [29]. In this CDMA system, the subscribers with different code weights will have different bit error rate(BER) performance. The codewords of low code weight can be assigned to the low-QoS (Quality of Services) applications and high code weight codewords can be assigned to high-QoS requirement applications [28]. Hence, the multi-weight property of the OOCs enables the system to meet multiple QoS requirements. The interested reader may refer to [28–40] for recent results on variable-weight OOCs.

Based on the notations of [29], throughout this paper, let W, L, and Q denote the sets $\{w_0, w_1, \ldots, w_p\}$, $\{\lambda_a^0, \lambda_a^1, \ldots, \lambda_a^p\}$ and $\{q_0, q_1, \ldots, q_p\}$, respectively. Without loss of generality, we may assume that $w_0 < w_1 < \cdots < w_p$.

A (v, W, L, λ_c, Q) variable-weight optical orthogonal code C, or (v, W, L, λ_c, Q)-OOC, is a collection of binary v-tuples such that the following three properties hold:

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(1) Weight Distribution: Every v-tuple in C has a Hamming weight contained in the set W; furthermore, there are exactly $q_i|C|$ codewords of weight w_i, *i.e.*, q_i indicates the fraction of codewords of weight w_i. It is clear that $\sum_{i=0}^{p} q_i = 1$.

(2) Periodic Auto-correlation: For any $X = (x_0, x_1, \dots, x_{v-1}) \in C$ with Hamming weight $w_i \in W$, and any integer τ, $0 < \tau < v$,

$$\sum_{t=0}^{v-1} x_t x_{t\oplus\tau} \le \lambda_a^i,$$

where the summation is carried out by treating binary symbols as reals.

(3) Periodic Cross-correlation: Similarly, for $\mathbf{x} \ne \mathbf{y}, \mathbf{x} = (x_0, x_1, \dots, x_{v-1}) \in C$, $\mathbf{y} = (y_0, y_1, \dots, y_{v-1}) \in C$, and any integer τ,

$$\sum_{t=0}^{v-1} x_t y_{t\oplus\tau} \le \lambda_c.$$

The notation (v, W, λ, Q)-OOC is used to denote a (v, W, L, λ_c, Q)-OOC with the property that $\lambda_a^0 = \lambda_a^1 = \cdots = \lambda_a^p = \lambda_c = \lambda$. The term variable-weight optical orthogonal code, or variable-weight OOC, is also used if there is no need to list the parameters.

The number of codewords of an OOC is called its size. For fixed v, W, λ, and Q, the largest size among all (v, W, λ, Q)-OOCs is denoted by $\Phi(v, W, \lambda, Q)$. Typically, when $W = \{3, 4\}$, $\lambda = 1$, and $Q = \{2/3, 1/3\}$, we get the following upper bound for the value of $\Phi(v, \{3, 4\}, 1, \{2/3, 1/3\})$ from Lemma 1 of [30].

Lemma 1.1 ([30]) *It holds that* $\Phi(v, \{3, 4\}, 1, \{2/3, 1/3\}) \le 3\lfloor \frac{v-1}{24} \rfloor$ *for any positive integer* v.

In view of Lemma 1.1, a $(v, \{3, 4\}, 1, \{2/3, 1/3\})$-OOC is said to be optimal if its size reaches the bound of $3\lfloor \frac{v-1}{24} \rfloor$.

Optimal optical orthogonal codes are closely related to some combinatorial configurations. For example, Yin [27] showed that an optimal $(v, k, 1)$-OOC is equivalent to an optimal cyclic packing CP$(k, 1; v)$. In [36], a CP$(W, 1; v)$ was also called 2-CP$(W, 1; v)$, and optimal 2-CP$(W, 1, Q; v)$s were introduced to construct optimal $(v, W, 1, Q)$-OOCs. Throughout this paper, we always denote by Z_v the additive group of integers modulo v.

For $B \subset Z_v$, the list differences from B is defined to be $\Delta B = \{x - y \pmod v : x, y \in B, x \ne y\}$. Suppose that $\mathcal{F}$ is a set of subsets (base blocks) of Z_v, and for each $B \in \mathcal{F}$, $|B| \in W$. Then $\mathcal{F}$ is called a cyclic packing CP$(W, 1; v)$ if it satisfies that $\Delta\mathcal{F} = \bigcup_{B\in\mathcal{F}} \Delta B$ covers each nonzero element of Z_v at most once, and for each $B = \{b_1, b_2, \dots, b_{|B|}\} \in \mathcal{F}$, $B + i$, $0 \le i \le v - 1$, are pairwise distinct, where $B + i = \{b_1 + i, b_2 + i, \dots, b_{|B|} + i\} \subset Z_v$. A CP$(W, 1, Q; v)$ is defined to be a CP$(W, 1; v)$ with the property that the fraction of number of blocks of size w_i is q_i, $0 \le i \le p$. From the definition, it is not difficult to see that the largest possible number of base blocks of a CP$(\{3, 4\}, 1, \{2/3, 1/3\}; v)$ is $3\lfloor \frac{v-1}{24} \rfloor$. A CP$(\{3, 4\}, 1, \{2/3, 1/3\}; v)$ is called optimal if the number of its base blocks reaches this bound.

Example 1.2 There exists an optimal CP$(\{3, 4\}, 1, \{2/3, 1/3\}; v)$ for $v \in \{48, 72\}$.

Proof For $v=48$, the $3\lfloor\frac{v-1}{24}\rfloor=3$ base blocks of an optimal CP$(\{3,4\},1,\{2/3,1/3\};48)$ are $\{0,1,3,7\}$, $\{0,5,13\}$, $\{0,9,19\}$.

For $v=72$, the $3\lfloor\frac{v-1}{24}\rfloor=6$ base blocks of an optimal CP$(\{3,4\},1,\{2/3,1/3\};72)$ are $\{0,1,3,7\}$, $\{0,5,13,22\}$, $\{0,10,21\}$, $\{0,12,26\}$, $\{0,15,31\}$, $\{0,18,37\}$. □

Suppose that $\mathcal{F}$ is a CP$(W,1,Q;v)$. The difference leave of $\mathcal{F}$, denoted by DL$(\mathcal{F})$, is defined to be the set of all nonzero integers in Z_v which are not covered by $\Delta\mathcal{F}$. A CP$(W,1,Q;v)$ $\mathcal{F}$ is called g-regular if the difference leave DL$(\mathcal{F})$ along with zero forms an additive subgroup of Z_v having order g, which must be generated by the integer v/g.

Example 1.3 There exists a $3h$-regular CP$(\{3,4\},1,\{2/3,1/3\};9h\times 3)$ for $h\in\{1,2\}$.

Proof For $h=1$, the 3 base blocks of a 3-regular CP$(\{3,4\},1,\{2/3,1/3\};9\times 3)$ are $\{0,1,4,17\}$, $\{0,2,8\}$, $\{0,5,12\}$.

For $h=2$, the 6 base blocks of a 6-regular CP$(\{3,4\},1,\{2/3,1/3\};18\times 3)$ are $\{0,1,3,31\}$, $\{0,4,10,47\}$, $\{0,5,21\}$, $\{0,8,20\}$, $\{0,13,32\}$, $\{0,14,29\}$. □

The following results were stated in [36].

Lemma 1.4 ([36]) *An optimal CP*$(W,1,Q;v)$ *is equivalent to an optimal* $(v,W,1,Q)$*-OOC.*

Lemma 1.5 ([36]) *If* $1\le g\le 24$*, then a* g*-regular CP*$(\{3,4\},1,\{2/3,1/3\};v)$ *is optimal.*

Some results of optimal $(v,\{3,4\},1,\{2/3,1/3\})$-OOCs were obtained in [32,35]. The following results come from Theorem 4 in [32].

Lemma 1.6 ([32]) *If* $v\equiv 24,120 \pmod{144}$ *is an integer, and* $v>24$*, then there exists an optimal* $(v,\{3,4\},1,\{2/3,1/3\})$*-OOC.*

The following existence results of cyclic packings were induced by checking the proof of Theorem 4 in [32]. We quote the lemma for later use.

Lemma 1.7 ([32]) *If* u *is an integer such that* $\gcd(6,u)=1$, $u>1$*, then there exists a* 24*-regular CP*$(\{3,4\},1,\{2/3,1/3\};24u)$.

In this paper, we shall investigate the existence of an optimal $(v,\{3,4\},1,\{2/3,1/3\})$-OOC. As the main result of the paper, we are to extend Lemma 1.6 to the following theorem.

Theorem 1.8 *There exists an optimal* $(24u,\{3,4\},1,\{2/3,1/3\})$*-OOC for any positive integer* $u>1$.

2. Direct constructions

In this section, we will describe two new direct constructions, which make use of skew starters, for g-regular CP$(\{3,4\},1,\{2/3,1/3\};v)$s. Let $(G,+)$ be an Abelian group of order $u>1$. A skew starter in G is a set of unordered pairs

$$S=\{\{x_i,y_i\}:1\le i\le (u-1)/2\}$$

which satisfies the following three properties:

(1) $\{x_i : 1 \le i \le (u-1)/2\} \cup \{y_i : 1 \le i \le (u-1)/2\} = G \setminus \{0\}$;
(2) $\{\pm(x_i - y_i) : 1 \le i \le (u-1)/2\} = G \setminus \{0\}$;
(3) $\{\pm(x_i + y_i) : 1 \le i \le (u-1)/2\} = G \setminus \{0\}$.

According to the definition, a skew starter in G can exist only if u is odd. Furthermore, if we write $X = \{x_i : 1 \le i \le (u-1)/2\}$ and $Y = \{y_i : 1 \le i \le (u-1)/2\}$, then we may assume, without loss of generality, that $X = -Y$, and hence we have $X \cup (-X) = Y \cup (-Y) = X \cup Y = G \setminus \{0\}$. Skew starters have been extensively investigated. We summarize the existence results on skew starters in Z_u in the following lemma.

Lemma 2.1 ([14]) *There exists a skew starter in Z_u for each positive integer u such that* $\gcd(u, 150) = 1$ *or* 25. *There does not exist any skew starter in Z_u if $u \equiv 0 \pmod 3$.*

In what follows, suppose that $\mathcal{B}$ is a set of subsets of $Z_u \times Z_h$, define the list of differences

$$D_j = \{d : (d, j) \text{ is a difference from } \mathcal{B}\}.$$

Lemma 2.2 *Let u be a positive integer such that* $\gcd(u, 150) = 1$ *or* 25. *Then there exists a* 48*-regular CP*$(\{3,4\}, 1, \{2/3, 1/3\}; 48u)$.

Proof By Lemma 2.1, there exists a skew starter $S = \{\{x_i, y_i\} : 1 \le i \le t\}$ in Z_u, where $t = (u-1)/2$. Since $\gcd(u, 48) = 1$, $Z_u \times Z_{48}$ is isomorphic to Z_{48u}. The $6(u-1)$ base blocks of a 48-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48u)$ on $Z_u \times Z_{48}$ are listed as follows.

$$\begin{array}{ll}
A_i^1 = \{(x_i,0),(y_i,0),(x_i+y_i,1),(0,25)\}, & A_i^2 = \{(x_i,0),(-y_i,2),(-x_i,10),(y_i,28)\},\\
A_i^3 = \{(-x_i,0),(y_i,2),(x_i,10),(-y_i,28)\}, & A_i^4 = \{(x_i,0),(-y_i,3),(-x_i,12),(y_i,39)\},\\
A_i^5 = \{(0,0),(x_i+y_i,3),(-x_i-y_i,14)\}, & A_i^6 = \{(y_i,0),(0,4),(-x_i,17)\},\\
A_i^7 = \{(-x_i-y_i,0),(0,5),(x_i+y_i,11)\}, & A_i^8 = \{(y_i,0),(-x_i,6),(0,19)\},\\
A_i^9 = \{(-x_i,0),(y_i-x_i,7),(y_i,14)\}, & A_i^{10} = \{(x_i,0),(-y_i,15),(0,19)\},\\
A_i^{11} = \{(0,0),(2x_i+2y_i,16),(x_i+y_i,21)\}, & A_i^{12} = \{(-x_i-y_i,0),(0,17),(x_i+y_i,32)\},
\end{array}$$

where $1 \le i \le t$. Since $D_s = -D_{48-s}$ for $25 \le s \le 47$, we only need to consider the differences D_s for $0 \le s \le 24$. Then we get

$$D_s = \begin{cases} \bigcup_{i=1}^{t} \{\pm(x_i - y_i)\}, & \text{if } s \in \{0,8,9,20\},\\ \bigcup_{i=1}^{t} \{\pm(x_i + y_i)\}, & \text{if } s \in \{2,3,5,6,14,15,17,18,21,24\},\\ \bigcup_{i=1}^{t} \{x_i, y_i\}, & \text{if } s \in \{1,7,23\}, \end{cases}$$

$$D_4 = \bigcup_{i=1}^{t} \{\pm y_i\}, \quad D_{10} = \bigcup_{i=1}^{t} \{\pm 2x_i\}, \quad D_{11} = D_{16} = \bigcup_{i=1}^{t} \{\pm(2x_i + 2y_i)\},$$
$$D_{12} = \bigcup_{i=1}^{t} \{-2x_i, -2y_i\}, \quad D_{13} = \bigcup_{i=1}^{t} \{\pm x_i\}, \quad D_{19} = \bigcup_{i=1}^{t} \{-x_i, -y_i\},$$
$$D_{22} = \bigcup_{i=1}^{t} \{\pm 2y_i\}.$$

.

Let $\mathcal{F} = \{A_i^j : 1 \le i \le t, 1 \le j \le 12\}$. Then $\Delta\mathcal{F}$ covers each element of $(Z_u \times Z_{48}) \setminus (\{0\} \times Z_{48})$ exactly once, while any element of the additive subgroup $\{0\} \times Z_{48}$ is not covered at all. Therefore, $\mathcal{F}$ forms the desired 48-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48u)$. □

Lemma 2.3 *Let u be a positive integer such that* $\gcd(u, 150) = 1$ *or* 25. *Then there exists a 72-regular CP*$(\{3,4\}, 1, \{2/3, 1/3\}; 72u)$.

Proof By Lemma 2.1, there exists a skew starter $S = \{\{x_i, y_i\} : 1 \le i \le t\}$ in Z_u, where $t = (u-1)/2$. Since $\gcd(u, 72) = 1$, $Z_u \times Z_{72}$ is isomorphic to Z_{72u}. The $9(u-1)$ base blocks of a 72-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72u)$ on $Z_u \times Z_{72}$ are listed as follows.

$$
\begin{array}{ll}
A_i^1 = \{(x_i,0),(y_i,0),(0,1),(x_i+y_i,37)\}, & A_i^2 = \{(-x_i,0),(y_i,2),(x_i,25),(-y_i,40)\},\\
A_i^3 = \{(x_i,0),(-y_i,2),(-x_i,25),(y_i,40)\}, & A_i^4 = \{(-x_i,0),(y_i,3),(x_i,27),(-y_i,44)\},\\
A_i^5 = \{(x_i,0),(-y_i,3),(-x_i,27),(y_i,44)\}, & A_i^6 = \{(-x_i,0),(-y_i,4),(0,26),(-x_i-y_i,46)\},\\
A_i^7 = \{(x_i,0),(y_i,4),(x_i+y_i,33)\}, & A_i^8 = \{(x_i+y_i,0),(-x_i-y_i,5),(0,56)\},\\
A_i^9 = \{(0,0),(2x_i+2y_i,6),(x_i+y_i,58)\}, & A_i^{10} = \{(0,0),(x_i,7),(-y_i,65)\},\\
A_i^{11} = \{(x_i+y_i,0),(0,8),(-x_i-y_i,67)\}, & A_i^{12} = \{(-y_i,0),(0,9),(x_i,62)\},\\
A_i^{13} = \{(-x_i-y_i,0),(0,10),(x_i+y_i,66)\}, & A_i^{14} = \{(y_i,0),(0,11),(-x_i,64)\},\\
A_i^{15} = \{(0,0),(y_i,12),(-x_i,30)\}, & A_i^{16} = \{(0,0),(-y_i-x_i,13),(-x_i,63)\},\\
A_i^{17} = \{(-y_i,0),(x_i,18),(0,29)\}, & A_i^{18} = \{(-x_i,0),(y_i,21),(0,33)\},
\end{array}
$$

where $1 \le i \le t$. Note that $D_s = -D_{72-s}$ for $37 \le s \le 71$, we only need to consider the differences D_s for $0 \le s \le 36$. We have

$$
D_s = \begin{cases}
\bigcup_{i=1}^{t} \{\pm(x_i - y_i)\}, & \text{if } s \in \{0,4,23,24,28,32\},\\
\bigcup_{i=1}^{t} \{\pm(x_i + y_i)\}, & \text{if } s \in \{2,3,8,10,13,14,15,16,17,18,20,21,36\},\\
\bigcup_{i=1}^{t} \{x_i, y_i\}, & \text{if } s \in \{7,9,26,29,33\},\\
\bigcup_{i=1}^{t} \{-x_i, -y_i\}, & \text{if } s \in \{1,11,35\},
\end{cases}
$$

$$
D_5 = D_6 = \bigcup_{i=1}^{t} \{\pm(2x_i + 2y_i)\}, \quad D_{12} = D_{22} = \bigcup_{i=1}^{t} \{\pm y_i\}, \quad D_{19} = D_{30} = \bigcup_{i=1}^{t} \{\pm x_i\},
$$
$$
D_{25} = D_{27} = \bigcup_{i=1}^{t} \{\pm 2x_i\}, \quad D_{31} = D_{34} = \bigcup_{i=1}^{t} \{\pm 2y_i\}.
$$

Let $\mathcal{F} = \{A_i^j : 1 \le i \le t, 1 \le j \le 18\}$. Then it is readily checked that $\Delta\mathcal{F} = (Z_q \times Z_{72}) \setminus (\{0\} \times Z_{72})$. Therefore, $\mathcal{F}$ forms the desired 72-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72u)$. □

3. Recursive constructions

Let $(G, \cdot)$ be a finite group of order v and H a subgroup of order h in G. An H-regular $(v, k; \lambda)$-incomplete difference matrix in G is a $k \times \lambda(v-h)$ martix $D = (d_{ij})$, $0 \le i \le k-1, 1 \le j \le \lambda(v-h)$, with entries from G, such that for any $0 \le i < j \le k-1$, the multiset $\{d_{il} \cdot d_{jl}^{-1} :$

$1 \le l \le \lambda(v-h)\}$ contains every element of $G \setminus H$ exactly λ times. When G is an abelian group, typically additive notation is used, so that the difference $d_{il} - d_{jl}$ is employed. In what follows, we assume that $G = Z_v$, and H is a subgroup of order h in Z_v. Then $H = \{i \cdot v/h : 0 \le i \le h-1\}$. We usually denote an H-regular $(v, k; \lambda)$-incomplete difference matrix over Z_v by h-regular $(v, k; \lambda)$-ICDM if $|H| = h$. When $H = \emptyset$ or $h = 0$, an H-regular $(v, k; \lambda)$-incomplete difference matrix over Z_v is termed as $(v, k; \lambda)$-CDM.

Lemma 3.1 ([41]) *If $m \ge 5$ is odd and $\gcd(m, 27) \ne 9$, then there exists an $(m, 4; 1)$-CDM.*

Lemma 3.2 ([13]) *There exists a 2-regular $(m, 4; 1)$-ICDM for $m \in \{12, 18\}$, or $m = 2^n$ and $n \ge 3$.*

The following two constructions were stated with Theorems 11 and 12 in [36], which were similar to the constructions in [27].

Construction 3.3 ([36]) *Suppose that both a g-regular CP$(W, 1, Q; v)$ and an optimal CP$(W, 1, Q; g)$ exist, then an optimal CP$(W, 1, Q; v)$ exists. Moreover, if the given CP$(W, 1, Q; g)$ is r-regular, then so is the derived CP$(W, 1, Q; v)$.*

Construction 3.4 ([36]) *Suppose that there exist a g-regular CP$(W, 1, Q; v)$, an $(m, w_p; 1)$-CDM, and an optimal CP$(W, 1, Q; gm)$. Then there exists an optimal CP$(W, 1, Q; mv)$. Moreover, if the given CP$(W, 1, Q; gm)$ is r-regular, then so is the derived CP$(W, 1, Q; mv)$.*

Similar to the constructions in [13] and [40], the following results of Constructions 3.5 and 3.7 are obtained.

Construction 3.5 *Let v and m be positive integers such that $\gcd(m, v) = 1$. Suppose that there exist a g-regular CP$(W, 1, Q; v)$, an h-regular $(m, w_p; 1)$-ICDM, and an hg-regular CP$(W, 1, Q; hv)$ (or a gh-regular CP$(W, 1, Q; gm)$, respectively). Then there exists a gm-regular CP$(W, 1, Q; mv)$ (or an hv-regular CP$(W, 1, Q; mv)$, respectively).*

Proof Suppose that $\mathcal{F}_1$ is the family of base blocks of the given g-regular CP$(W, 1, Q; v)$, whose difference leave plus the singleton $\{0\}$ consists of the additive subgroup $U = \{0, v/g, 2v/g, \ldots, (g-1)v/g\}$ of Z_v. Let $D = (d_{ij})$ be an h-regular $(m, w_p; 1)$-ICDM, where $d_{ij} \in Z_m$ for $0 \le i \le w_p - 1$ and $1 \le j \le m-h$. Then for $0 \le i \ne j \le w_p - 1$, the multiset $\{d_{il} - d_{jl} : 1 \le l \le m-h\} = Z_m \setminus H$, where $H = \{0, m/h, \ldots, (h-1)m/h\}$.

Let $G = Z_v \times Z_m$, $H_1 = U \times Z_m$, and $H_2 = Z_v \times H$. Since $\gcd(m, v) = 1$, G is isomorphic to Z_{mv}. Similarly, $H_1 \cong Z_{gm}$ and $H_2 \cong Z_{hv}$. Let $\mathcal{F}_2$ be the family of base blocks of the given hg-regular CP$(W, 1, Q; hv)$ (or a gh-regular CP$(W, 1, Q; gm)$, respectively) in H_2 (or H_1, respectively) whose difference leave plus the singleton $\{(0, 0)\}$ is $U \times H$. Next, construct a gm-regular CP$(W, 1, Q; mv)$ (or an hv-regular CP$(W, 1, Q; mv)$, respectively) in G so that its difference leave plus the singleton $\{(0, 0)\}$ is H_1 (or H_2, respectively) as follows:

For each base block $B = \{b_0, b_1, \ldots, b_{w_r - 1}\} \in \mathcal{F}_1$, where $w_r \in W$, $r \in \{0, 1, \ldots, p\}$, we take

$m-h$ base blocks

$$B_j = \{(b_0, d_{0j}), (b_1, d_{1j}), \ldots, (b_{w_r-1}, d_{w_r-1,j})\},$$

for $1 \le j \le m-h$.

Let $\mathcal{F} = \{B_j : B \in \mathcal{F}_1, 1 \le j \le m-h\} \cup \mathcal{F}_2$. It is readily checked that $\Delta\mathcal{F}$ covers each integer in $G \setminus H_1$ (or $G \setminus H_2$, respectively) exactly once. □

Let g be a divisor of v such that $v = gv_0$. Suppose that $\mathcal{F} = \{B_i : i = 1, 2, \ldots, t\}$ is the family of base blocks of an hg-regular CP$(W, 1, Q; hv)$, where $B_i = \{0, b_{1i}, b_{2i}, \ldots, b_{w_r-1,i}\}$ for $w_r \in W$, $r \in \{0, 1, \ldots, p\}$, and $i = 1, 2, \ldots, t$. Define

$$ele(\mathcal{F}) = \bigcup_{i=1}^{t} \{b_{1i}, b_{2i}, \ldots, b_{w_r-1,i}\}.$$

The hg-regular CP$(W, 1, Q; hv)$ is said to be h-perfect, denoted by hg-regular h-perfect CP$(W, 1, Q; hv)$, if

$$ele(\mathcal{F}) \subseteq \{a + bv : 0 \le a \le \lfloor \frac{v}{2} \rfloor, a \ne 0, v_0, 2v_0, \ldots, (g-1)v_0; b = 0, 1, \ldots, h-1\}.$$

Some useful examples of hg-regular h-perfect CP$(W, 1, Q; hv)$ are exhibited in the following example.

Example 3.6 There exists an hg-regular h-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; hv)$ for $h \in \{1, 2\}$ and $(v, g) \in \{(32, 8), (108, 12), (96, 24), (144, 24)\}$.

Proof The base blocks of an hg-regular h-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; hv)$ are listed below.

$(h, v, g) = (1, 32, 8)$:

$\{0,1,6,15\}$, $\{0,2,13\}$, $\{0,3,10\}$.

$(h, v, g) = (2, 32, 8)$:

$\{0,1,3,10\}$, $\{0,13,34,39\}$, $\{0,14,33\}$, $\{0,15,37\}$, $\{0,35,46\}$, $\{0,41,47\}$.

$(h, v, g) = (1, 108, 12)$:

$\{0,1,3,29\}$, $\{0,4,10,34\}$, $\{0,5,21,52\}$, $\{0,7,39,53\}$, $\{0,8,48\}$, $\{0,11,44\}$,
$\{0,12,49\}$, $\{0,13,51\}$, $\{0,15,50\}$, $\{0,17,42\}$, $\{0,19,41\}$, $\{0,20,43\}$.

$(h, v, g) = (2, 108, 12)$:

$\{0,1,40,110\}$, $\{0,2,43,114\}$, $\{0,3,47,116\}$, $\{0,4,132,156\}$, $\{0,5,51,129\}$,
$\{0,6,148,155\}$, $\{0,10,131,160\}$, $\{0,25,111,143\}$, $\{0,11,134\}$, $\{0,17,154\}$,
$\{0,19,52\}$, $\{0,12,49\}$, $\{0,13,48\}$, $\{0,14,133\}$, $\{0,15,151\}$,
$\{0,16,157\}$, $\{0,20,159\}$, $\{0,21,161\}$, $\{0,23,53\}$, $\{0,26,115\}$,
$\{0,28,50\}$, $\{0,31,122\}$, $\{0,34,42\}$, $\{0,38,158\}$.

$(h, v, g) = (1, 96, 24)$:

$\{0,1,3,26\}$, $\{0,5,22,43\}$, $\{0,6,35,45\}$, $\{0,7,41\}$, $\{0,9,46\}$, $\{0,11,30\}$,
$\{0,13,31\}$, $\{0,14,47\}$, $\{0,15,42\}$.

$(h, v, g) = (2, 96, 24)$:

$\{0,1,34,99\}$, $\{0,2,43,109\}$, $\{0,5,42,111\}$, $\{0,6,101,131\}$, $\{0,7,110,129\}$,
$\{0,18,47,137\}$, $\{0,9,114\}$, $\{0,10,143\}$, $\{0,11,141\}$, $\{0,13,134\}$,
$\{0,15,46\}$, $\{0,17,135\}$, $\{0,21,138\}$, $\{0,22,45\}$, $\{0,25,39\}$,
$\{0,26,139\}$, $\{0,27,142\}$, $\{0,35,38\}$.

$(h,v,g)=(1,144,24)$:

$\{0,1,32,51\}$, $\{0,2,35,55\}$, $\{0,3,37,52\}$, $\{0,4,43,65\}$, $\{0,5,62,69\}$,
$\{0,8,71\}$, $\{0,9,38\}$, $\{0,10,68\}$, $\{0,11,67\}$, $\{0,13,59\}$,
$\{0,14,41\}$, $\{0,16,44\}$, $\{0,17,40\}$, $\{0,21,47\}$, $\{0,25,70\}$.

$(h,v,g)=(2,144,24)$:

$\{0,14,51,145\}$, $\{0,15,53,148\}$, $\{0,16,55,146\}$, $\{0,17,52,151\}$, $\{0,19,59,147\}$,
$\{0,20,61,159\}$, $\{0,21,64,173\}$, $\{0,22,56,175\}$, $\{0,23,67,184\}$, $\{0,25,70,188\}$,
$\{0,27,191\}$, $\{0,28,195\}$, $\{0,29,62\}$, $\{0,32,208\}$, $\{0,46,211\}$,
$\{0,47,58\}$, $\{0,196,203\}$, $\{0,49,215\}$, $\{0,50,63\}$, $\{0,65,68\}$,
$\{0,69,172\}$, $\{0,71,177\}$, $\{0,178,187\}$, $\{0,181,183\}$, $\{0,26,57\}$,
$\{0,199,207\}$, $\{0,201,206\}$, $\{0,202,212\}$, $\{0,205,209\}$, $\{0,213,214\}$. $\Box$

Construction 3.7 *Suppose that there exist a g-regular 1-perfect CP$(W,1,Q;v)$, an hg-regular h-perfect CP$(W,1,Q;hv)$, and an h-regular $(m,w_p;1)$-ICDM, then there exists an mg-regular m-perfect CP$(W,1,Q;mv)$.*

Proof Suppose that $\mathcal{A}=\{A_i=\{0,x_{1i},x_{2i},\ldots,x_{w_r-1,i}\}:i=1,2,\ldots,t\}$, $w_r\in W$, $r\in\{0,1,\ldots,p\}$ is a g-regular 1-perfect CP$(W,1,Q;v)$. Let $\mathcal{B}=\{B_j=\{0,a_{1j}+vb_{1j},a_{2j}+vb_{2j},\ldots,a_{w_r-1,j}+vb_{w_r-1,j}\}:j=1,2,\ldots,s\}$, $w_r\in W$, $r\in\{0,1,\ldots,p\}$ be an hg-regular h-perfect CP$(W,1,Q;hv)$, where $a_{1j},a_{2j},\ldots,a_{w_r-1,j}\in\{0,1,\ldots,\lfloor v/2\rfloor\}\setminus\{0,v/g,2v/g,\ldots,(g-1)v/g\}$, and $b_{1j},b_{2j},\ldots,b_{w_r-1,j}\in\{0,1,\ldots,h-1\}$ for $1\le j\le s$.

Let $D=(d_{ij})$ be an h-regular $(m,w_p;1)$-ICDM, where $d_{ij}\in Z_m$ for $0\le i\le w_p-1$ and $1\le j\le m-h$ such that the multiset $\{d_{il}-d_{jl}:1\le l\le m-h\}=Z_m\setminus\{0,m/h,2m/h,\ldots,(h-1)m/h\}$. Now the desired mg-regular CP$(W,1,Q;mv)$ will be based on Z_{mv} whose difference leave plus the singleton $\{0\}$ forms the subgroup $H=\{i+jv:i=0,v/g,2v/g,\ldots,(g-1)v/g;j=0,1,\ldots,m-1\}$. The required base blocks come from the following two parts:

Part 1: For each base block $A_i=\{0,x_{1i},x_{2i},\ldots,x_{w_r-1,i}\}\in\mathcal{A}$, we take $m-h$ base blocks

$$A_{il}=\{0,x_{1i}+(d_{1l}-d_{0l})\cdot v,x_{2i}+(d_{2l}-d_{0l})\cdot v,\ldots,x_{w_r-1,i}+(d_{w_r-1,l}-d_{0l})\cdot v\},$$

for $l=1,2,\ldots,m-h$, where the additive operation is performed in Z_{mv}. Let $\mathcal{F}_1=\{A_{il}:1\le i\le t,1\le l\le m-h\}$. Then, by noting that $x_{1i},x_{2i},\ldots,x_{w_r-1,i}\in\{0,1,2,\ldots,\lfloor v/2\rfloor\}\setminus\{0,v/g,2v/g,\ldots,(g-1)v/g\}$ so that $-\lfloor v/2\rfloor\le x_{ei}-x_{fi}\le\lfloor v/2\rfloor$ for $1\le e\ne f\le w_r-1$ and $1\le i\le t$, it is readily checked from the property of the h-regular $(m,w_p;1)$-ICDM that

$$\Delta\mathcal{F}_1=\pm\{a+bv:0\le a\le\lfloor v/2\rfloor,a\ne 0,v/g,2v/g,\ldots,(g-1)v/g,b\in M\},$$

where $M=\{0,1,\ldots,m-1\}\setminus\{0,m/h,\ldots,(h-1)m/h\}$.

Part 2: For each base block $B_j=\{0,a_{1j}+vb_{1j},a_{2j}+vb_{2j},\ldots,a_{w_r-1,j}+vb_{w_r-1,j}\}\in\mathcal{B}$, we take a base block

$$B'_j = \{0, a_{1j} + b_{1j} \cdot mv/h, a_{2j} + b_{2j} \cdot mv/h, \dots, a_{w_r-1,j} + b_{w_r-1,j} \cdot mv/h\}.$$

Let $\mathcal{F}_2 = \{B'_j : 1 \le j \le s\}$. Then similarly it can be readily checked that

$$\Delta\mathcal{F}_2 = \pm\{a + bv : 0 \le a \le \lfloor v/2 \rfloor, a \ne 0, v/g, 2v/g, \dots, (g-1)v/g; \\ b = 0, m/h, \dots, (h-1)m/h\}.$$

The differences arising from these base blocks $\mathcal{F}_1 \cup \mathcal{F}_2$ cover each element in $Z_{mv} \setminus \{i \cdot v/g : 0 \le i \le gm-1\}$ exactly once. Therefore, this construction produces an mg-regular CP$(W, 1, Q; mv)$. It is straightforward to check that it is m-perfect. □

4. Proof of Theorem 1.8

Lemma 4.1 *There exists a 24-regular CP*$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^i)$ *for any integer* $i \ge 2$.

Proof For $i = 2$, a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^2)$ comes from Example 3.6. For $i = 3$, the 78 base blocks of a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^3)$ are listed as follows.

$\{0, 51, 181, 325\}$,	$\{0, 52, 185, 327\}$,	$\{0, 53, 190, 326\}$,	$\{0, 55, 182, 335\}$,	$\{0, 56, 196, 372\}$,
$\{0, 57, 202, 328\}$,	$\{0, 58, 197, 404\}$,	$\{0, 59, 205, 397\}$,	$\{0, 66, 204, 383\}$,	$\{0, 67, 210, 403\}$,
$\{0, 68, 223, 337\}$,	$\{0, 69, 226, 384\}$,	$\{0, 70, 201, 435\}$,	$\{0, 71, 225, 620\}$,	$\{0, 72, 231, 411\}$,
$\{0, 73, 233, 357\}$,	$\{0, 74, 235, 418\}$,	$\{0, 75, 198, 361\}$,	$\{0, 76, 240, 424\}$,	$\{0, 77, 199, 426\}$,
$\{0, 60, 194, 341\}$,	$\{0, 61, 209, 329\}$,	$\{0, 62, 211, 340\}$,	$\{0, 63, 195, 345\}$,	$\{0, 64, 215, 330\}$,
$\{0, 65, 217, 342\}$,	$\{0, 100, 206\}$,	$\{0, 101, 208\}$,	$\{0, 102, 212\}$,	$\{0, 109, 220\}$,
$\{0, 113, 232\}$,	$\{0, 116, 285\}$,	$\{0, 117, 238\}$,	$\{0, 118, 288\}$,	$\{0, 172, 173\}$,
$\{0, 175, 177\}$,	$\{0, 247, 256\}$,	$\{0, 249, 261\}$,	$\{0, 250, 263\}$,	$\{0, 252, 267\}$,
$\{0, 254, 289\}$,	$\{0, 619, 623\}$,	$\{0, 255, 292\}$,	$\{0, 257, 290\}$,	$\{0, 258, 294\}$,
$\{0, 259, 293\}$,	$\{0, 260, 298\}$,	$\{0, 78, 165\}$,	$\{0, 79, 167\}$,	$\{0, 80, 166\}$,
$\{0, 82, 171\}$,	$\{0, 83, 168\}$,	$\{0, 84, 174\}$,	$\{0, 91, 186\}$,	$\{0, 92, 188\}$,
$\{0, 93, 187\}$,	$\{0, 97, 200\}$,	$\{0, 98, 141\}$,	$\{0, 104, 295\}$,	$\{0, 105, 334\}$,
$\{0, 112, 156\}$,	$\{0, 128, 178\}$,	$\{0, 203, 214\}$,	$\{0, 218, 221\}$,	$\{0, 219, 236\}$,
$\{0, 228, 242\}$,	$\{0, 239, 246\}$,	$\{0, 241, 262\}$,	$\{0, 248, 272\}$,	$\{0, 279, 301\}$,
$\{0, 296, 343\}$,	$\{0, 599, 617\}$,	$\{0, 600, 606\}$,	$\{0, 602, 607\}$,	$\{0, 603, 622\}$,
$\{0, 608, 616\}$,	$\{0, 618, 628\}$,	$\{0, 609, 625\}$.		

Now we deal with $i = 4$. Applying Construction 3.7, we get a 24×9-regular 18-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^4)$, where the needed 12-regular 1-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 12 \times 3^2)$, 24-regular 2-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^2)$ and 2-regular $(18, 4; 1)$-ICDM come from Example 3.6 and Lemma 3.2. Start from this 24×9-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^4)$, and apply Construction 3.3 with a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^2)$ to obtain a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^4)$.

For $i \geq 5$, start from a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 3^t)$ from above with $t \equiv i \pmod 3$ and $t \in \{2,3,4\}$. Apply, recursively, Construction 3.4 with a $(3^3,4;1)$-CDM from Lemma 3.1. This gives a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 3^i)$ for $i \geq 5$. $\Box$

Lemma 4.2 *There exists a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^i)$ for any integer $i \geq 1$.*

Proof For $i=1$, the 15 base blocks of a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3)$ are

$$\begin{array}{lllll} \{0,1,3,76\}, & \{0,4,41,81\}, & \{0,5,43,87\}, & \{0,7,46,93\}, & \{0,8,53,88\}, \\ \{0,9,61\}, & \{0,10,122\}, & \{0,11,127\}, & \{0,13,123\}, & \{0,14,125\}, \\ \{0,15,128\}, & \{0,20,79\}, & \{0,23,50\}, & \{0,26,55\}, & \{0,25,74\}. \end{array}$$

For $i=2$, the 51 base blocks of a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^2)$ are

$$\begin{array}{lllll} \{0,1,3,221\}, & \{0,4,129,223\}, & \{0,5,135,222\}, & \{0,6,137,230\}, & \{0,7,139,235\}, \\ \{0,8,141,233\}, & \{0,9,143,238\}, & \{0,10,146,237\}, & \{0,11,149,247\}, & \{0,12,152,254\}, \\ \{0,13,155,239\}, & \{0,14,159,245\}, & \{0,15,163,263\}, & \{0,16,166,265\}, & \{0,17,164,267\}, \\ \{0,19,170,259\}, & \{0,20,176,264\}, & \{0,21,175\}, & \{0,22,179\}, & \{0,23,200\}, \\ \{0,42,321\}, & \{0,24,347\}, & \{0,25,186\}, & \{0,26,351\}, & \{0,27,402\}, \\ \{0,28,359\}, & \{0,31,350\}, & \{0,29,189\}, & \{0,32,352\}, & \{0,33,355\}, \\ \{0,34,362\}, & \{0,35,389\}, & \{0,39,366\}, & \{0,37,363\}, & \{0,41,394\}, \\ \{0,45,305\}, & \{0,46,174\}, & \{0,47,364\}, & \{0,60,251\}, & \{0,48,356\}, \\ \{0,65,374\}, & \{0,44,379\}, & \{0,49,171\}, & \{0,50,361\}, & \{0,52,368\}, \\ \{0,51,365\}, & \{0,55,370\}, & \{0,56,369\}, & \{0,61,373\}, & \{0,40,358\}, \\ \{0,75,349\}. & & & & \end{array}$$

Now we deal with $i=3$. Applying Construction 3.7, we get a 12×12-regular 12-perfect CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^3)$, where the needed 12-regular 1-perfect CP$(\{3,4\},1,\{2/3,1/3\};12\times 3^2)$, 24-regular 2-perfect CP$(\{3,4\},1,\{2/3,1/3\};24\times 3^2)$ and 2-regular $(12,4;1)$-ICDM comes from Example 3.6 and Lemma 3.2. Start from this 12×12-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^3)$, and apply Construction 3.3 with a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3)$ to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^3)$.

For $i \geq 4$, write $i=3s+t$ where $s\geq 1$ and $t=1,2,3$. Take a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^t)$ from above. Apply Construction 3.4 with $g=24$, $m=3^{3s}$, and $v=48\times 3^t$ to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 3^i)$ for $i\geq 4$, where the needed 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 3^{3s})$ and $(3^{3s},4;1)$-CDM exist by Lemmas 4.1 and 3.1. $\Box$

Lemma 4.3 *If $u>1$ is an integer such that $\gcd(6,u)=1$, then there exists a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};gu\times 3^i)$ for $g=48$, 72 and $i\geq 1$.*

Proof For $i\geq 1$, there exists a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};g\times 3^i)$ for $g=48$, 72 by Lemmas 4.2 and 4.1, respectively. Take a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24u)$ for $\gcd(6,u)=1$ and $u>1$ from Lemma 1.7. Then apply Construction 3.4 with a $(u,4;1)$-CDM from Lemma 3.1 to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};gu\times 3^i)$. $\Box$

Lemma 4.4 *There exists a 24-regular* CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i)$ *for any integer* $i \geq 2$.

Proof For $i = 2$, a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^2)$ exists by Example 3.6. For $i = 3, 4$, the base blocks of the desired CPs are listed as follows.

$i = 3$:

$\{0,1,60,101\}$,	$\{0,2,63,105\}$,	$\{0,3,65,102\}$,	$\{0,4,70,113\}$,	$\{0,5,73,111\}$,
$\{0,6,116,123\}$,	$\{0,10,94,145\}$,	$\{0,11,177\}$,	$\{0,12,67\}$,	$\{0,14,167\}$,
$\{0,17,165\}$,	$\{0,18,95\}$,	$\{0,19,159\}$,	$\{0,21,179\}$,	$\{0,28,74\}$,
$\{0,29,78\}$,	$\{0,30,172\}$,	$\{0,31,170\}$,	$\{0,35,169\}$,	$\{0,36,107\}$,
$\{0,45,183\}$.				

$i = 4$:

$\{0,21,150,221\}$,	$\{0,22,152,225\}$,	$\{0,23,154,222\}$,	$\{0,24,151,226\}$,	$\{0,25,157,223\}$,
$\{0,26,164,231\}$,	$\{0,27,166,228\}$,	$\{0,28,165,235\}$,	$\{0,29,155,238\}$,	$\{0,34,270,273\}$,
$\{0,35,169,278\}$,	$\{0,36,287,295\}$,	$\{0,30,170,242\}$,	$\{0,31,167,241\}$,	$\{0,33,168,237\}$,
$\{0,45,319\}$,	$\{0,46,323\}$,	$\{0,47,326\}$,	$\{0,49,325\}$,	$\{0,50,321\}$,
$\{0,51,171\}$,	$\{0,52,344\}$,	$\{0,53,346\}$,	$\{0,56,345\}$,	$\{0,57,347\}$,
$\{0,76,178\}$,	$\{0,77,193\}$,	$\{0,84,187\}$,	$\{0,85,189\}$,	$\{0,86,371\}$,
$\{0,87,283\}$,	$\{0,41,302\}$,	$\{0,42,305\}$,	$\{0,43,303\}$,	$\{0,44,330\}$,
$\{0,55,379\}$,	$\{0,78,369\}$,	$\{0,88,382\}$,	$\{0,100,365\}$,	$\{0,115,377\}$,
$\{0,117,383\}$,	$\{0,173,190\}$,	$\{0,364,373\}$,	$\{0,366,372\}$,	$\{0,370,374\}$.

Now we deal with the case of $i \geq 5$. Start from a 24-regular 1-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^2)$ and a 48-regular 2-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48 \times 2^2)$ by Example 3.6. Take a 2-regular $(2^{i-2}, 4; 1)$-ICDM from Lemma 3.2. Then apply Construction 3.7 with $g = 24$, $m = 2^{i-2}$, $h = 2$ and $v = 24 \times 2^2$ to obtain a $24 \times 2^{i-2}$-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i)$. Combine with a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^3)$ and a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^4)$ from above, we apply Construction 3.3 inductively on i to get a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i)$ for any integer $i \geq 5$. □

Lemma 4.5 *There exists a 24-regular* CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72 \times 2^i)$ *for any integer* $i \geq 1$.

Proof For $i = 1$, a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72 \times 2)$ exists by Example 3.6. For $i = 2$, the 33 base blocks of a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72 \times 2^2)$ are listed as follows.

$\{0,1,115,164\}$,	$\{0,2,112,167\}$,	$\{0,3,119,162\}$,	$\{0,4,122,185\}$,	$\{0,5,136,187\}$,
$\{0,6,139,189\}$,	$\{0,7,141,186\}$,	$\{0,8,143,196\}$,	$\{0,9,146,193\}$,	$\{0,10,148,205\}$,
$\{0,11,209,232\}$,	$\{0,13,130\}$,	$\{0,14,127\}$,	$\{0,15,215\}$,	$\{0,16,217\}$,
$\{0,17,128\}$,	$\{0,18,220\}$,	$\{0,19,222\}$,	$\{0,20,227\}$,	$\{0,21,234\}$,
$\{0,22,236\}$,	$\{0,25,94\}$,	$\{0,26,91\}$,	$\{0,27,253\}$,	$\{0,28,98\}$,
$\{0,29,258\}$,	$\{0,31,230\}$,	$\{0,32,242\}$,	$\{0,33,97\}$,	$\{0,34,246\}$,
$\{0,37,248\}$,	$\{0,39,247\}$,	$\{0,38,244\}$.		

There exist a 8-regular 1-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 32)$, a 16-regular 2-perfect CP$(\{3,4\}, 1, \{2/3, 1/3\}; 64)$ and a 2-regular $(18, 4; 1)$-ICDM from Example 3.6 and Lemma 3.2. Then apply Construction 3.7 with $g = 8$, $m = 18$, $h = 2$ and $v = 32$ to obtain a 72×2-regular CP$(\{3,4\},1,\{2/3,1/3\};72\times 2^3)$. Combine with the existence of a 24-regular CP$(\{3,4\},1,\{2/3, 1/3\};72\times 2)$, then a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};72\times 2^3)$ follows from Construction 3.3.

For $i \geq 4$, start from a 24-regular 1-perfect CP$(\{3,4\},1,\{2/3,1/3\};24\times 6)$ and a 48-regular 2-perfect CP$(\{3,4\},1,\{2/3,1/3\};48\times 6)$, which exist by Example 3.6. Then take a 2-regular$(2^{i-1},4;1)$-ICDM from Lemma 3.2, and apply Construction 3.7 with $g = 24$, $m = 2^{i-1}$, $h = 2$, $v = 24\times 6$, to obtain a $24\times 2^{i-1}$-regular CP$(\{3,4\},1,\{2/3,1/3\};72\times 2^i)$. By Lemma 4.4 there is a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^{i-1})$. Then apply Construction 3.3 to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};72\times 2^i)$. $\square$

Lemma 4.6 *There exists a 24-regular CP*$(\{3,4\},1,\{2/3,1/3\};24\times 2^i3^j)$ *for any integer* $i,j\geq 2$.

Proof When $i \geq 2$ and $j = 2$. Start from a 3-regular CP$(\{3,4\},1,\{2/3,1/3\};9\times 3)$ and a 6-regular CP$(\{3,4\},1,\{2/3,1/3\};18\times 3)$, which exist by Example 1.3. Take a 2-regular $(2^{i+3},4;1)$-ICDM from Lemma 3.2. Then apply Construction 3.5 with $g = 3$, $m = 2^{i+3}$, $h = 2$, $v = 9\times 3$, to obtain a 24×2^i-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^i3^2)$. There exists a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^i)$ by Lemma 4.4. We apply Construction 3.3 to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^i3^2)$.

When $i\geq 2$ and $j\geq 3$. Start from a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^i)$, a $(3^j,4;1)$-CDM, and a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 3^j)$, which exist by Lemmas 4.4, 3.1 and 4.1, respectively. Then apply Construction 3.4 with $v = 24\times 2^i$, $g = 24$, and $m = 3^j$, to obtain a 24-regular CP$(\{3,4\},1,\{2/3,1/3\};24\times 2^i3^j)$. $\square$

Lemma 4.7 *There exists a g-regular CP*$(\{3,4\},1,\{2/3,1/3\};g\times 5^i)$ *for* $g = 48,72$ *and* $i\geq 1$.

Proof For $(g,i) = (48,1)$, the 24 base blocks of a 48-regular CP$(\{3,4\},1,\{2/3,1/3\};48\times 5)$ are

$\{0,1,77,128\}$,	$\{0,2,81,133\}$,	$\{0,3,86,129\}$,	$\{0,4,78,136\}$,	$\{0,6,88,144\}$,
$\{0,7,91,148\}$,	$\{0,8,97,214\}$,	$\{0,9,103,201\}$,	$\{0,11,224\}$,	$\{0,12,118\}$,
$\{0,13,181\}$,	$\{0,14,186\}$,	$\{0,17,193\}$,	$\{0,18,87\}$,	$\{0,21,194\}$,
$\{0,22,221\}$,	$\{0,23,124\}$,	$\{0,29,198\}$,	$\{0,32,209\}$,	$\{0,33,212\}$,
$\{0,37,204\}$,	$\{0,38,216\}$,	$\{0,44,93\}$,	$\{0,53,119\}$.	

For $(g,i) = (72,1)$, the 36 base blocks of a 72-regular CP$(\{3,4\},1,\{2/3,1/3\};72\times 5)$ are

$\{0,31,122,183\}$,	$\{0,32,121,188\}$,	$\{0,33,126,184\}$,	$\{0,34,131,182\}$,	$\{0,36,123,189\}$,
$\{0,37,129,181\}$,	$\{0,38,124,187\}$,	$\{0,39,147,201\}$,	$\{0,41,139,332\}$,	$\{0,43,137,186\}$,
$\{0,44,127,243\}$,	$\{0,46,128,264\}$,	$\{0,62,133\}$,	$\{0,64,132\}$,	$\{0,72,146\}$,
$\{0,73,154\}$,	$\{0,76,164\}$,	$\{0,77,134\}$,	$\{0,78,157\}$,	$\{0,84,333\}$,
$\{0,99,358\}$,	$\{0,102,158\}$,	$\{0,103,222\}$,	$\{0,104,351\}$,	$\{0,106,352\}$,
$\{0,109,301\}$,	$\{0,141,253\}$,	$\{0,118,166\}$,	$\{0,163,169\}$,	$\{0,307,318\}$,
$\{0,313,331\}$,	$\{0,334,346\}$,	$\{0,336,343\}$,	$\{0,337,341\}$,	$\{0,338,339\}$,
$\{0,344,347\}$.				

For $i > 1$, start from a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; g\times 5)$, then apply Construction 3.4 inductively with a $(5, 4; 1)$-CDM from Lemma 3.1 to obtain a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; g\times 5^i)$. □

Lemma 4.8 *If* $u > 1$ *is an integer such that* $\gcd(6, u) = 1$*, then there exists a* g*-regular CP*$(\{3,4\}, 1, \{2/3, 1/3\}; gu)$ *for* $g = 48, 72$.

Proof For $u > 1$ such that $\gcd(6, u) = 1$, write $u = 5^i u'$ where $i \geq 0$ and $5 \nmid u'$. If $u' = 1$, then $i \geq 1$, the conclusion follows from Lemma 4.7. If $u' > 1$, then $\gcd(u', 30) = 1$. When $i = 0$ and $u' > 1$, there exists a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; gu')$ by Lemmas 2.2 and 2.3. When $i \geq 1$ and $u' > 1$, take a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; gu')$, a $(5^i, 4; 1)$-CDM from Lemma 3.1, and a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; g \times 5^i)$ from Lemma 4.7, then apply Construction 3.4 to obtain a g-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; gu)$. □

Lemma 4.9 *There exists an optimal CP*$(\{3,4\}, 1, \{2/3, 1/3\}; 24u)$ *for any positive integer* $u > 1$.

Proof Let $u = 2^i 3^j u'$, where $\gcd(6, u') = 1$. We have the following two cases.

Case 1 When $u' = 1$, then $24u = 24 \times 2^i 3^j$.

If $i = 0$, then $j \geq 1$. When $j = 1$, there is an optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72)$ by Example 1.2. When $j \geq 2$, there is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^j)$ by Lemma 4.1. If $i = 1$, then $j \geq 0$. When $j = 0$, there is an optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48)$ by Example 1.2. When $j \geq 1$, there is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48 \times 3^j)$ by Lemma 4.2. If $i \geq 2$, then $j \geq 0$, there exists a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i 3^j)$ by Lemmas 4.4–4.6.

Case 2 When $u' > 1$, then $24u = 24 \times 2^i 3^j u'$.

Case 2.1 If $i = 0$, then $j \geq 0$. When $j = 0$, there is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24u)$ by Lemma 1.7. When $j = 1$, start from a 72-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72u')$ from Lemma 4.8, then apply Construction 3.3 to get an optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72u')$, where the needed optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 72)$ comes from Example 1.2. When $j \geq 2$, there is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 3^j u')$ by Lemma 4.3.

Case 2.2 If $i = 1$, then $j \geq 0$. When $j = 0$, there exist a 48-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48u')$, and an optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48)$, which come from Lemma 4.8 and Example 1.2, respectively. Then apply Construction 3.3 to obtain an optimal CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48u')$. When $j \geq 1$, there is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 48 \times 3^j)$ by Lemma 4.3.

Case 2.3 If $i \geq 2$, then $j \geq 0$. Start from a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i 3^j)$, which exists by Lemmas 4.4–4.6. Take a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24u')$ from Lemma 1.7. Then apply Construction 3.4 with a $(u', 4; 1)$-CDM from Lemma 3.1, to obtain a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24 \times 2^i 3^j u')$. That is a 24-regular CP$(\{3,4\}, 1, \{2/3, 1/3\}; 24u)$.

By Lemma 1.5, the resulting 24-regular CPs from above are also optimal. □

Combine the results of Lemmas 1.4 and 4.9, we complete the proof of Theorem 1.8.

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References

[1] J. A. SALEHI. *Code division multiple access techniques in optical fiber networks-Part I fundamental principles*. IEEE Trans. Commum., 1989, **37**(8): 824–833.

[2] J. A. SALEHI, C. A. BRACKETT. *Code division multiple access techniques in optical fiber networks-Part II systems performance analysis*. IEEE Trans. Commum., 1989, **37**(8): 834–842.

[3] F. R. K. CHUNG, J. A. SALEHI, V. K. WEI. *Optical orthogonal codes: design, analysis, and applications*. IEEE Trans. Inform. Theory, 1989, **35**(3): 595–604.

[4] S. W. GOLOMB. *Digital Communication with Space Application*. Peninsula, Los Alots, CA, 1982.

[5] J. L. MASSEY, P. MATHYS. *The collision channel without feedback*. IEEE Trans. Inform. Theory, 1985, **31**(2): 192–204.

[6] J. A. SALEHI. *Emerging optical code-division multiple-access communications systems*. IEEE Network, 1989, **3**(2): 31–39.

[7] M. P. VECCHI, J. A. SALEHI. *Neuromorphic networks based on sparse optical orthogonal codes*. Neural Information Processing Systems-Natural and Synthetic, Amer. Inst. Phys., 1988, 814–823.

[8] R. J. R. ABEL, M. BURATTI. *Some progress on $(v, 4, 1)$ difference families and optical orthogonal codes*. J. Combin. Theory, 2004, **106**(1): 59–75.

[9] S. BITAN, T. ETZION. *Constructions for optimal constant weight cyclically permutable codes and diffeence families*. IEEE Trans. Inform. Theory, 1995, **41**(1): 77–87.

[10] M. BURATTI. *Cyclic designs with block size 4 and related optimal optical orthogonal codes*. Des. Codes Cryptogr., 2002, **26**(1-3): 111–125.

[11] Yanxun CHANG, R. FUJI-HARA, Ying MIAO. *Combinatorial constructions of optimal optical orthogonal codes with weight 4*. IEEE Trans. Inform. Theory, 2003, **49**(5): 1283–1292.

[12] Yanxun CHANG, Lijun JI. *Optimal $(4up, 5, 1)$ optical orthogonal codes*. J. Combin. Des., 2004, **12**(5): 346–361.

[13] Yanxun CHANG, Ying MIAO. *Constructions for optimal optical orthogonal codes*. Discrete Math., 2003, **261**(1-3): 127–139.

[14] Kejun CHEN, Gennian GE, Lie ZHU. *Starters and related codes*. J. Statist. Plann. Inference, 2000, **86**(2): 379–395.

[15] Wensong CHU, C. J. COLBOURN. *Recursive constructions for optimal $(n, 4, 2)$-OOCs*. J. Combin. Des., 2004, **12**(5): 333–345.

[16] Wensong CHU, S. W. GOLOMB. *A new recursive construction for optical orthogonal codes*. IEEE Trans. Inform. Theory, 2003, **49**(11): 3072–3076.

[17] H. CHUNG, P. V. KUMAR. *Optical orthogonal codes new bounds and an optimal construction*. IEEE Trans. Inform. Theory, 1990, **36**(4): 866–873.

[18] Tao FENG, Yanxun CHANG, Lijun JI. *Constructions for strictly cyclic 3-designs and applications to optimal OOCs with $\lambda = 2$*. J. Combin. Theory Ser. A, 2008, **115**(8): 1527–1551.

[19] Tao FENG, Yanxun CHANG, Lijun JI. *Constructions for rotational Steiner quadruple systems*. J. Combin. Des., 2009, **17**(5): 353–368.

[20] R. FUJI-HARA, Ying MIAO. *Optical orthogonal codes: Their bounds and new optimal constructions*. IEEE Trans. Inform. Theory, 2000, **46**(7): 2396–2406.

[21] R. FUJI-HARA, Ying MIAO, Jianxing YIN. *Optimal $(9v, 4, 1)$ optical orthogonal codes*. SIAM J. Discrete Math., 2001, **14**(2): 256–266.

[22] Gennian GE, Jianxing YIN. *Constructions for optimal $(v, 4, 1)$ optical orthogonal codes*. IEEE Trans. Inform. Theory, 2001, **47**(7): 2998–3004.

[23] Gennian GE, Ying MIAO, Xianwei SUN. *Perfect difference families, perfect difference matrices, and related combinatorial structures*. J. Combin. Des., 2010, **18**(6): 415–449.

[24] Shikui MA, Yanxun CHANG. *A new class of optimal optical orthogonal codes with weight five*. IEEE Trans. Inform. Theory, 2004, **50**(8): 1848–1850.

[25] Shikui MA, Yanxun CHANG. *Constructions of optimal optical orthogonal codes with weight five*. J. Combin. Des., 2005, **13**(1): 54–69.

[26] Xiaomiao WANG, Yanxun CHANG. *Further results on $(v, 4, 1)$-perfect difference families*. Discrete Math., 2010, **310**(13-14): 1995–2006.

[27] Jianxing YIN. *Some combinatorial constructions for optical orthogonal codes*. Discrete Math., 1998, **185**(1-3): 201–219.

[28] F. R. GU, Jingshown WU. *Construction and performance analysis of variable-weight optical orthogonal codes for asynchronous optical CDMA systems*. J. Lightw. Technol., 2005, **23**(2): 740–748.

[29] G. C. YANG. *Variable-weight optical orthogonal codes for CDMA networks with multiple performance requirements*. IEEE Trans. Commun., 1996, **44**(1): 47–55.

[30] M. BURATTI, Yueer WEI, Dianhua WU, et al. *Relative difference families with variable block sizes and their related OOCs*. IEEE Trans. Inform. Theory, 2011, **57**(11): 7489–7497.

[31] Jing JIANG, Dianhua WU, Pingzhi FAN. *General constructions of optimal variable-weight optical orthogonal codes*. IEEE Trans. Inform. Theory, 2011, **57**(7): 4488–4496.

[32] Jing JIANG, Dianhua WU, M. H. LEE. *Some infinite classes of optimal* $(v, \{3,4\}, 1, Q)$*-OOCs with* $Q \in \{(\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3})\}$. Graphs and Combinatorics, 2013, **29**(6): 1795–1811.

[33] Yan LIU, Dianhua WU. *Constructions of optimal variable-weight OOCs via quadratic residues*. Front. Math. China, 2013, **8**(4): 869–890.

[34] Dianhua WU, Pingzhi FAN, Hengchao LI, et al. *Optimal variable-weight optical orthogonal codes via cyclic difference families*. Proc. 2009 IEEE Int. Symp. Information Theory (ISIT' 09), Jun. 28-Jul. 3, 448–452.

[35] Dianhua WU, Pingzhi FAN, Xun WANG, et al. *A new class of optimal variable-weight OOCs based on cyclic difference families*. Proc. 4th Int. Workshop on Signal Design and its Application in Communications (IWSDA' 2009), Oct. 19-23, 16–19.

[36] Dianhua WU, Hengming ZHAO, Pingzhi FAN, et al. *Optimal variable-weight optical orthogonal codes via difference packing*. IEEE Trans. Inform. Theory, 2010, **56**(8): 4053–4060.

[37] G. C. YANG. *Variable weight optical orthogonal codes for CDMA networks with multiple performance requirements*. Proc. IEEE GLOBECOM' 93, 1993, **1**(1): 488–492.

[38] Hengming ZHAO, Dianhua WU, Pingzhi FAN. *Constructions of optimal variable-weight optical orthogonal codes*. J. Combin. Des., 2010, **18**(4): 274–291.

[39] Hengming ZHAO, Dianhua WU, Zhengfang MO. *Further results on optimal* $(v, \{3,k\}, 1, \{1/2, 1/2\})$*-OOCs for* $k = 4, 5$. Discrete Math., 2011, **311**(1): 16–23.

[40] Hengming ZHAO. *On balanced optimal* $(18u, \{3,4\}, 1)$ *optical orthogonal codes*. J. Combin. Des., 2012, **20**(6): 290–303.

[41] Gennian GE. *On* $(g, 4; 1)$*-difference matrices*. Discrete Math., 2005, **301**(2-3): 164–174.