Journal of Mathematical Research with Applications
Jul., 2016, Vol. 36, No. 4, pp. 394–406
DOI:10.3770/j.issn:2095-2651.2016.04.002
Http://jmre.dlut.edu.cn

Lattice of Interval-Valued $(\in, \in\vee q)$-Fuzzy LI-Ideals in Lattice Implication Algebras

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Abstract In the present paper, the interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal theory in lattice implication algebras is further studied. Some new properties of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals are given. Representation theorem of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal which is generated by an interval-valued fuzzy set is established. It is proved that the set consisting of all interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals in a lattice implication algebra, under the partial order $\sqsubseteq$, forms a complete distributive lattice.

Keywords lattice-valued logic; lattice implication algebra; interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal; complete distributive lattice

MR(2010) Subject Classification 03B50; 03G25; 03E72

1. Introduction

In the field of many-valued logic, lattice-valued logic [1] plays an important role for two aspects: Firstly, it extends the chain-type truth-value field of some well-known presented logics (such as two-valued logic, three-valued logic, n-valued logic, the Łukasiewicz logic with truth values in the interval $[0, 1]$ and Zadeh's infinite-valued logic, and so on) to some relatively general lattices. Secondly, the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. In order to establish a logic system with truth value in a relatively general lattice, in 1990, Xu [2] firstly proposed the concept of lattice implication algebra by combining lattice and implication algebra, and researched many useful properties. It provided the foundation to establish the corresponding logic system from the algebraic viewpoint. Since then, this kind of logical algebra has been extensively investigated by many authors [3–9]. For the general development of lattice implication algebras, the ideal theory plays an important role. Jun introduced the notions of LI-ideals and prime LI-ideals in lattice implication algebras and investigated their properties in [10,11]. Liu etc. [12] studied several properties of prime LI-ideals and ILI-ideals in lattice implication algebras. The concept of fuzzy sets was presented firstly by Zadeh [13] in 1965. At present, fuzzy sets have

Received September 13, 2015; Accepted March 8, 2016
Supported by the National Natural Science Foundation of China (Grant No. 60774073) and the Higher School Research Foundation of Inner Mongolia (Grant No. NJSY14283).
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been applied to the field of algebraic structures, and the study of fuzzy algebras has achieved great success. Many wonderful and valuable results have been obtained by some mathematical researchers, such as Rosenfeld [14], Mordeson and Malik [15], Hedayati [16], Zhan and Dudek [17] and Liu [18]. As a generalization of the concept of fuzzy sets, Zadeh [19] introduced the notion of interval-valued fuzzy set and constructed a method of approximate inference using this notion. Recently, based on the study of interval-valued fuzzy sets, more and more researchers have devoted themselves to applying some results of interval-valued fuzzy sets to algebraic structures [20–25]. Among them, Liu [25] introduced the notions of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals and interval-valued $(\in, \in\vee q)$-fuzzy lattice ideals in lattice implication algebras. However, more properties of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals, especially, from the point of lattice theory, are less frequent.

In this paper, we will further research the properties of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals in lattice implication algebras. The rest of this article is organized as follows. In Section 2, we review related basic knowledge of lattice implication algebras and interval-valued fuzzy sets. In Section 3, we give several new properties of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals. In Section 4, we firstly introduce the concept of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal which is generated by an interval-valued fuzzy set and establish its representation theorem. And then, we investigate the lattice structural feature of the set containing all of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals in a lattice implication algebra. Finally, we conclude this paper in Section 5.

2. Preliminaries

In this section, we review related basic knowledge of lattice implication algebras [1,2] and interval-valued fuzzy sets [19].

Definition 2.1 ([2]) *Let $(L, \vee, \wedge, ', \rightarrow, O, I)$ be a bounded lattice with an order-reversing involution $'$, where I and O are the greatest and the smallest elements of L, respectively, $\rightarrow: L\times L \rightarrow L$ is a mapping. Then $(L, \vee, \wedge, ', \rightarrow, O, I)$ is called a lattice implication algebra if the following conditions hold for all $x, y, z \in L$:*

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(I_2) $x \rightarrow x = I$;

(I_3) $x \rightarrow y = y' \rightarrow x'$;

(I_4) $x \rightarrow y = y \rightarrow x = I$ *implies* $x = y$;

(I_5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;

(l_1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;

(l_2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

In the sequel, a lattice implication algebra $(L, \vee, \wedge, ', \rightarrow, O, I)$ will be denoted by L in short.

Lemma 2.2 ([1]) *Let L be a lattice implication algebra. Then for all $x, y, z \in L$,*

(i) $O \rightarrow x = I$, $x \rightarrow I = I$, $I \rightarrow x = x$ *and* $x' = x \rightarrow O$;

(ii) $x \leqslant y$ *if and only if* $x \rightarrow y = I$;

(iii) $x \to y \leqslant (y \to z) \to (x \to z)$ *and* $x \vee y = (x \to y) \to y$*;*

(iv) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$ *and* $z \to x \leqslant z \to y$*;*

(v) $x \to (y \vee z) = (x \to y) \vee (x \to z)$ *and* $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$*;*

(vi) $x \oplus y = y \oplus x$ *and* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$*;*

(vii) $O \oplus x = x$, $I \oplus x = I$ *and* $x \oplus x' = I$*;*

(viii) $x \vee y \leqslant x \oplus y$ *and* $x \leqslant (x \to y)' \oplus y$*;*

(ix) $x \leqslant y$ *implies* $x \oplus z \leqslant y \oplus z$*,*

where, $x \oplus y = x' \to y$ *for all* $x, y \in L$*.*

In the unit interval $[0,1]$ equipped with the natural order, we denote $\vee = \max$ and $\wedge = \min$. By an interval number $\overline{a}$, we mean an interval $[a^-, a^+]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Denote the set of all interval numbers by $D[0,1]$. The interval $[a,a]$ can be identified by the number $a \in [0,1]$. For the interval numbers $\{\overline{a}_i\}_{i=1}^n = \{[a_i^-, a_i^+]\}_{i=1}^n \in D[0,1]$, $n \in \mathbb{Z}_+$, we define:

(i) $\overline{a}_1 \sqcap \overline{a}_2 \sqcap \cdots \sqcap \overline{a}_n = [a_1^- \wedge a_2^- \wedge \cdots \wedge a_n^-, a_1^+ \wedge a_2^+ \wedge \cdots \wedge a_n^+]$;

(ii) $\overline{a}_1 \sqcup \overline{a}_2 \sqcup \cdots \sqcup \overline{a}_n = [a_1^- \vee a_2^- \vee \cdots \vee a_n^-, a_1^+ \vee a_2^+ \vee \cdots \vee a_n^+]$.

And for the interval numbers $\{\overline{a}_\lambda\}_{\lambda \in \Lambda} = \{[a_\lambda^-, a_\lambda^+]\}_{\lambda \in \Lambda} \in D[0,1]$, we define:

(iii) $\bigsqcap_{\lambda \in \Lambda} \overline{a}_\lambda = [\bigwedge_{\lambda \in \Lambda} a_\lambda^-, \bigwedge_{\lambda \in \Lambda} a_\lambda^+]$;

(iv) $\bigsqcup_{\lambda \in \Lambda} \overline{a}_\lambda = [\bigvee_{\lambda \in \Lambda} a_\lambda^-, \bigvee_{\lambda \in \Lambda} a_\lambda^+]$.

For any interval numbers $\overline{a}_1, \overline{a}_2 \in D[0,1]$, we define an order relation $\leqslant$ on $D[0,1]$ by:

(i) $\overline{a}_1 \leqslant \overline{a}_2$ if and only if $a_1^- \leqslant a_2^-$ and $a_1^+ \leqslant a_2^+$;

(ii) $\overline{a}_1 = \overline{a}_2$ if and only if $a_1^- = a_2^-$ and $a_1^+ = a_2^+$;

(iii) $\overline{a}_1 < \overline{a}_2$ if and only if $\overline{a}_1 \leqslant \overline{a}_2$ and $\overline{a}_1 \neq \overline{a}_2$;

(iv) $k\overline{a} = [ka^-, ka^+]$, whenever $0 \leqslant k \leqslant 1$.

We easily observe that $(D[0,1], \leqslant, \sqcap, \sqcup)$ forms a complete lattice with $0 = [0,0]$ as its least element and $1 = [1,1]$ as its greatest element.

Definition 2.3 ([19,21]) *Let a set* X *be the fixed domain. An interval-valued fuzzy set on* X *is a mapping* $\overline{A}: X \to D[0,1]$*, we denote for each* $x \in X$, $\overline{A}(x) = [A^-(x), A^+(x)] \in D[0,1]$*, where* A^- *and* A^+ *are fuzzy sets on* X *such that* $A^-(x) \leqslant A^+(x)$*, for all* $x \in X$*.*

For the sake of simplicity, in the sequel, we shall use the symbol $\overline{A} = [A^-, A^+]$ to denote an interval-valued fuzzy set on X. Let $\overline{A} = [A^-, A^+]$ and $\overline{B} = [B^-, B^+]$ be two interval-valued fuzzy sets on X. We define $\overline{A} \Cap \overline{B}$ and $\overline{A} \Cup \overline{B}$ as follows: for all $x \in X$,

(i) $(\overline{A} \Cap \overline{B})(x) = \overline{A}(x) \sqcap \overline{B}(x) = [A^-(x) \wedge B^-(x), A^+(x) \wedge B^+(x)]$;

(ii) $(\overline{A} \Cup \overline{B})(x) = \overline{A}(x) \sqcup \overline{B}(x) = [A^-(x) \vee B^-(x), A^+(x) \vee B^+(x)]$.

3. On interval-valued $(\in, \in\vee\, q)$-fuzzy LI-ideals

An interval-valued fuzzy set $\overline{A} = [A^-, A^+]$ on a lattice implication algebra L of the form

$$\overline{A}(y) = \begin{cases} \overline{\alpha}(\neq [0,0]), & \text{if } y = x \\ [0,0], & \text{if } y \neq x \end{cases} \tag{1}$$

is said to be an interval-valued fuzzy point with support x and interval value $\overline{\alpha}$, denoted by $x_{\overline{\alpha}}$. We now call $x_{\overline{\alpha}}$ belongs to (or resp., is quasi-coincident with) an interval-valued fuzzy set $\overline{A}$, written by $x_{\overline{\alpha}} \in \overline{A}$ (resp., $x_{\overline{\alpha}}q\overline{A}$, if $\overline{A}(x) \geqslant \overline{\alpha}$ (resp., $\overline{A}(x)+\overline{\alpha} = [A^-(x)+\alpha^-, A^+(x)+\alpha^+] > [1,1]$). If $x_{\overline{\alpha}} \in \overline{A}$ or $x_{\overline{\alpha}}q\overline{A}$, then we write $x_{\overline{\alpha}} \in\vee\, q\overline{A}$.

In the sequel, we emphasize that every interval-valued fuzzy set $\overline{A} = [A^-, A^+]$ on L must satisfy the comparable condition and the following properties:

$$[A^-(x), A^+(x)] < [0.5, 0.5] \text{ or } [0.5, 0.5] \leqslant [A^-(x), A^+(x)], \text{ for all } x \in L. \tag{2}$$

Definition 3.1 ([25]) *Let L be a lattice implication algebra. An interval-valued fuzzy set $\overline{A} = [A^-, A^+]$ on L is called an interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal of L if it satisfies the following conditions for all $\overline{\alpha}, \overline{\beta} \in D[0,1]$ and $x, y \in L$,*

(IVFI1) $x_{\overline{\alpha}} \in \overline{A}$ *implies* $O_{\overline{\alpha}} \in\vee\, q\overline{A}$;

(IVFI2) $(x \to y)'_{\overline{\alpha}} \in \overline{A}$ *and* $y_{\overline{\beta}} \in \overline{A}$ *imply* $x_{\overline{\alpha}} \in\vee\, q\overline{A}$.

The set of all interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals of L is denoted by $\mathrm{IVFLI}(L)$.

Theorem 3.2 ([25]) *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$ an interval-valued fuzzy set on L. Then $\overline{A} \in \mathrm{IVFLI}(L)$ if and only if it satisfies the following conditions:*

(IVFI3) $\overline{A}(O) \geqslant \overline{A}(x) \sqcap [0,5,0,5]$, *for all* $x \in L$;

(IVFI4) $\overline{A}(x) \geqslant \overline{A}((x \to y)') \sqcap \overline{A}(y) \sqcap [0.5, 0.5]$, *for all* $x, y \in L$.

Theorem 3.3 ([25]) *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$ an interval-valued fuzzy set on L. Then $\overline{A} \in$ **IVFLI**(L) if and only if it satisfies the following conditions:*

(IVFI5) $x \leqslant y$ *implies* $\overline{A}(x) \geqslant \overline{A}(y) \sqcap [0.5, 0.5]$, *for all* $x, y \in L$;

(IVFI6) $\overline{A}(x \oplus y) \geqslant \overline{A}(x) \sqcap \overline{A}(y) \sqcap [0.5, 0.5]$, *for all* $x, y \in L$.

Definition 3.4 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$ an interval-valued fuzzy set on L. An interval-valued fuzzy set $\overline{A}^{\overline{\alpha}}$ on L is defined as follows:*

$$\overline{A}^{\overline{\alpha}}(x) = \begin{cases} \overline{A}(x), & x \neq O \\ \overline{A}(O) \sqcup \overline{\alpha}, & x = O \end{cases} \tag{3}$$

for all $x \in L$, where $\overline{\alpha} \in D[0,1]$.

Example 3.5 Let $L = \{O, a, I\}$, $O < a < I$, $\wedge = \min$, $\vee = \max$, $O' = I, a' = a, I' = O$, and $O \to O = O \to a = O \to I = a \to a = a \to I = I \to I = I, a \to O = I \to a = a, I \to O = O$. Then $(L, \vee, \wedge, ', \to, O, I)$ is a lattice implication algebra by Example 4 in [2]. Define an interval-valued fuzzy set $\overline{A} = [A^-, A^+]$ on L by $\overline{A}(O) = [0.3, 0.6], \overline{A}(a) = \overline{A}(I) = [0.2, 0.4]$. Putting $\alpha = [0.2, 0.8]$, we have $\overline{A}^{\overline{\alpha}}(O) = [0.3, 0.8], \overline{A}^{\overline{\alpha}}(a) = \overline{A}^{\overline{\alpha}}(I) = [0.2, 0.4]$ by Definition 3.4.

Theorem 3.6 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+] \in \mathrm{IVFLI}(L)$. Then for all $\overline{\alpha} \in D[0,1]$, $\overline{A}^{\overline{\alpha}} \in \mathrm{IVFLI}(L)$.*

Proof Firstly, for all $x, y \in L$, let $x \leqslant y$. We consider the following two cases:

(i) Assume that $x = O$. If $y = O$, we have that

$$\overline{A}^{\overline{\alpha}}(x) = \overline{A}(O) \sqcup \overline{\alpha} = \overline{A}^{\overline{\alpha}}(y) \geqslant \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5].$$

If $y \neq O$, by using $\overline{A} \in \mathrm{IVFLI}(L)$ and (IVFI3), we have that

$$\overline{A}^{\overline{\alpha}}(x) = \overline{A}(O) \sqcup \overline{\alpha} \geqslant \overline{A}(O) \geqslant \overline{A}(y) \sqcap [0.5, 0.5] = \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5].$$

(ii) Assume that $x \neq O$, then $y \neq O$. It follows that

$$\overline{A}^{\overline{\alpha}}(x) = \overline{A}(x) \geqslant \overline{A}(y) \sqcap [0.5, 0.5] = \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5]$$

from $\overline{A} \in \mathrm{IVFLI}(L)$ and (IVFI5).

Summarize these two cases, we conclude that $x \leqslant y$ implies $\overline{A}^{\overline{\alpha}}(x) \geqslant \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5]$, for all $x, y \in L$. i.e., $\overline{A}^{\overline{\alpha}}$ satisfies (IVFI5).

Secondly, for all $x, y \in L$, we consider the following two cases:

(i) Assume that $x \oplus y = O$. If $x = y = O$, it is obvious that

$$\overline{A}^{\overline{\alpha}}(x \oplus y) = \overline{A}(O) \sqcup \overline{\alpha} = \overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \geqslant \overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5].$$

If $x = O, y \neq O$ or $x \neq O, y = O$, then $x \oplus y \neq O$, it is a contradiction.

If $x \neq O$ and $y \neq O$, it follows that

$$\begin{aligned}\overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5] =& \overline{A}(x) \sqcap \overline{A}(y) \sqcap [0.5, 0.5] \\ \leqslant& \overline{A}(x \oplus y) = \overline{A}(O) \leqslant \overline{A}(O) \sqcup \overline{\alpha} = \overline{A}^{\overline{\alpha}}(x \oplus y)\end{aligned}$$

from $\overline{A} \in \mathrm{IVFLI}(L)$, (IVFI6) and (3).

(ii) Assume that $x \oplus y \neq O$. If $x = y = O$, it is obviously a contradiction.

If $x = O, y \neq O$ or $x \neq O, y = O$, assume $x = O, y \neq O$, then $x \oplus y = O' \to y = y$, and so $\overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5] \leqslant \overline{A}^{\overline{\alpha}}(y) = \overline{A}(y) = \overline{A}(x \oplus y) = \overline{A}^{\overline{\alpha}}(x \oplus y)$.

If $x \neq O$ and $y \neq O$, it follows that

$$\overline{A}^{\overline{\alpha}}(x \oplus y) = \overline{A}(x \oplus y) \geqslant \overline{A}(x) \sqcap \overline{A}(y) \sqcap [0.5, 0.5] = \overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5]$$

from $\overline{A} \in \mathrm{IVFLI}(L)$ and (IVFI6).

Summarize these two cases, we conclude that $\overline{A}^{\overline{\alpha}}(x \oplus y) \geqslant \overline{A}^{\overline{\alpha}}(x) \sqcap \overline{A}^{\overline{\alpha}}(y) \sqcap [0.5, 0.5]$, for all $x, y \in L$. i.e., $\overline{A}^{\overline{\alpha}}$ satisfies (IVFI6).

Thus it follows that $\overline{A}^{\overline{\alpha}} \in \mathrm{IVFLI}(L)$ from Theorem 3.3. □

Definition 3.7 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$, $\overline{B} = [B^-, B^+]$ interval-valued fuzzy sets on L. Interval-valued fuzzy sets $\overline{A}^{\overline{B}}$ and $\overline{B}^{\overline{A}}$ on L are defined as follows: for all $x \in L$,*

$$\overline{A}^{\overline{B}}(x) = \begin{cases} \overline{A}(x), & x \neq O \\ \overline{A}(O) \sqcup \overline{B}(O), & x = O \end{cases} \quad \text{and} \quad \overline{B}^{\overline{A}}(x) = \begin{cases} \overline{B}(x), & x \neq O \\ \overline{B}(O) \sqcup \overline{A}(O), & x = O \end{cases}. \tag{4}$$

Example 3.8 Let $(L, \vee, \wedge, ', \to, O, I)$ is the lattice implication algebra given in Example 3.5. Define interval-valued fuzzy sets $\overline{A} = [A^-, A^+]$ and $\overline{B} = [B^-, B^+]$ on L by $\overline{A}(O) = [0.6, 0.9]$, $\overline{A}(a) = \overline{A}(I) = [0.3, 0.4]$ and $\overline{B}(O) = \overline{B}(a) = \overline{B}(I) = [0.7, 0.8]$. Then $\overline{A}^{\overline{B}}(O) = [0.7, 0.9]$, $\overline{A}^{\overline{B}}(a) = \overline{A}^{\overline{B}}(I) = [0.3, 0.4]$ and $\overline{B}^{\overline{A}}(O) = [0.7, 0.9]$, $\overline{B}^{\overline{A}}(a) = \overline{B}^{\overline{A}}(I) = [0.7, 0.8]$ by Definition 3.7.

Corollary 3.9 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$, $\overline{B} = [B^-, B^+] \in$*

IVFLI(L). *Then* $\overline{A}^{\overline{B}}, \overline{B}^{\overline{A}} \in$ IVFLI(L).

Definition 3.10 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+]$ interval-valued fuzzy sets on L. An interval-valued fuzzy set $\overline{A} \uplus \overline{B}$ on L is defined as follows: for all $x, a, b \in L$,*

$$(\overline{A} \uplus \overline{B})(x) = \bigsqcup_{x \leqslant a \oplus b} \{\overline{A}(a) \sqcap \overline{B}(b) \sqcap [0.5, 0.5]\}. \tag{5}$$

Example 3.11 Let $(L, \vee, \wedge, ', \rightarrow, O, I)$ be the lattice implication algebra given in Example 3.5. Define interval-valued fuzzy sets $\overline{A} = [A^-, A^+]$ and $\overline{B} = [B^-, B^+]$ on L by $\overline{A}(O) = [0.25, 0.45], \overline{A}(a) = \overline{A}(I) = [0.1, 0.2]$ and $\overline{B}(O) = [0.3, 0.4], \overline{B}(a) = \overline{B}(I) = [0.2, 0.3]$. Then $\left(\overline{A} \uplus \overline{B}\right)(O) = [0.25, 0.4]$, $\left(\overline{A} \uplus \overline{B}\right)(a) = [0.2, 0.3]$, $\left(\overline{A} \uplus \overline{B}\right)(I) = [0.1, 0.2]$ by Definition 3.10.

Theorem 3.12 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+] \in$ IVFLI(L). Then $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}} \in$ IVFLI(L).*

Proof Firstly, for all $x, y \in L$, let $x \leqslant y$. Then $\{a \oplus b | x \leqslant a \oplus b\} \supseteq \{a \oplus b | y \leqslant a \oplus b\}$, and so

$$\begin{aligned}\left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x) = \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} &\geqslant \bigsqcup_{y \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\ &= [0.5, 0.5] \sqcap \bigsqcup_{y \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\ &= \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(y) \sqcap [0.5, 0.5].\end{aligned}$$

Hence $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$ satisfies (IVFI5). Secondly, for all $x, y \in L$, we have that

$$\begin{aligned}\left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x \oplus y) &= \bigsqcup_{x \oplus y \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\ &\geqslant \bigsqcup_{x \leqslant a_1 \oplus a_2 \text{ and } y \leqslant b_1 \oplus b_2} \left\{\overline{A}^{\overline{B}}(a_1 \oplus b_1) \sqcap \overline{B}^{\overline{A}}(a_2 \oplus b_2) \sqcap [0.5, 0.5]\right\} \\ &\geqslant \bigsqcup_{x \leqslant a_1 \oplus a_2 \text{ and } y \leqslant b_1 \oplus b_2} \left\{\overline{A}^{\overline{B}}(a_1) \sqcap \overline{A}^{\overline{B}}(b_1) \sqcap \overline{B}^{\overline{A}}(a_2) \sqcap \overline{B}^{\overline{A}}(b_2) \sqcap [0.5, 0.5]\right\} \\ &= \bigsqcup_{x \leqslant a_1 \oplus a_2} \left\{\overline{A}^{\overline{B}}(a_1) \sqcap \overline{B}^{\overline{A}}(a_2) \sqcap [0.5, 0.5]\right\} \sqcap \\ &\quad \bigsqcup_{y \leqslant b_1 \oplus b_2} \left\{\overline{A}^{\overline{B}}(b_1) \sqcap \overline{B}^{\overline{A}}(b_2) \sqcap [0.5, 0.5]\right\} \\ &= [0.5, 0.5] \sqcap \bigsqcup_{x \leqslant a_1 \oplus a_2} \left\{\overline{A}^{\overline{B}}(a_1) \sqcap \overline{B}^{\overline{A}}(a_2) \sqcap [0.5, 0.5]\right\} \sqcap \\ &\quad [0.5, 0.5] \sqcap \bigsqcup_{y \leqslant b_1 \oplus b_2} \left\{\overline{A}^{\overline{B}}(b_1) \sqcap \overline{B}^{\overline{A}}(b_2) \sqcap [0.5, 0.5]\right\} \\ &= \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x) \sqcap \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(y) \sqcap [0.5, 0.5],\end{aligned}$$

and so $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$ also satisfies (IVFI6). Hence $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}} \in$ IVFLI(L) by Theorem 3.3. $\square$

4. The lattice of interval-valued $(\in, \in\vee\, q)$-fuzzy LI-ideals

In this section, we investigate the lattice structural feature of the set IVFLI(L).

Definition 4.1 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+]$ interval-valued fuzzy sets on L. The operation $\sqsubseteq$ is defined as follows:*

$$\overline{A} \sqsubseteq \overline{B} \Longleftrightarrow \overline{A}(x) \sqcap [0.5, 0.5] \leqslant \overline{B}(x), \text{ for all } x \in L, \tag{6}$$

and

$$\overline{A} = \overline{B} \Longleftrightarrow \overline{A} \sqsubseteq \overline{B} \text{ and } \overline{B} \sqsubseteq \overline{A}. \tag{7}$$

Remark 4.2 Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+], \overline{C} = [C^-, C^+]$ interval-valued fuzzy sets on L. Then according to Definition 4.1, we have that

$$\overline{A} \sqsubseteq \overline{B} \text{ implies } \overline{A}(x) \sqcap [0.5, 0.5] \leqslant \overline{B}(x) \sqcap [0.5, 0.5], \text{ for all } x \in L. \tag{8}$$

At the same time, it is easy to verify that the following assertions hold:

(i) By the definition of $\sqsubseteq$, $\overline{A} \sqsubseteq \overline{A}$;

(ii) By the definition of $\sqsubseteq$, $\overline{A} \sqsubseteq \overline{B}$ and $\overline{B} \sqsubseteq \overline{A}$ imply $\overline{A} = \overline{B}$;

(iii) $\overline{A} \sqsubseteq \overline{B}$ and $\overline{B} \sqsubseteq \overline{C}$ imply $\overline{A} \sqsubseteq \overline{C}$.

Hence, we can see that $\sqsubseteq$ is a partial order on the set of all interval-valued fuzzy sets on L.

Definition 4.3 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$ an interval-valued fuzzy set on L. The intersection of all interval-valued $(\in, \in\vee\, q)$-fuzzy LI-ideals of L containing $\overline{A}$ is called the interval-valued $(\in, \in\vee\, q)$-fuzzy LI-ideal generated by $\overline{A}$, and denoted by $\langle \overline{A} \rangle$.*

Theorem 4.4 *Let L be a lattice implication algebra and $\overline{A} = [A^-, A^+]$ an interval-valued fuzzy set on L. An interval-valued fuzzy set $\overline{B} = [B^-, B^+]$ on L is defined as follows:*

$$\overline{B}(x) = \bigsqcup \big\{ \overline{A}(a_1) \sqcap \cdots \overline{A}(a_n) \sqcap [0.5, 0.5] | a_1, a_2, \ldots, a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \big\} \tag{9}$$

for all $x \in L$. Then $\overline{B} = \langle \overline{A} \rangle$.

Proof Firstly, we prove that $\overline{B} \in$ IVFLI(L). For all $x, y \in L$, let $x \leqslant y$. Then

$$\begin{aligned}
&\overline{B}(y) \sqcap [0.5, 0.5] \\
&\quad = [0.5, 0.5] \sqcap \bigsqcup \big\{ \overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5, 0.5] | a_1, a_2, \ldots, a_n \in L \text{ and } y \leqslant a_1 \oplus \cdots \oplus a_n \big\} \\
&\quad = \bigsqcup \big\{ \overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5, 0.5] | a_1, a_2, \ldots, a_n \in L \text{ and } y \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \big\} \\
&\quad \leqslant \bigsqcup \big\{ \overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5, 0.5] | a_1, a_2, \ldots, a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \big\} \\
&\quad = \overline{B}(x).
\end{aligned}$$

Thus $\overline{B}$ satisfies (IVFI5). Assume that there are $a_1, a_2, \ldots, a_n \in L$ and $b_1, b_2, \ldots, b_m \in L$ such that $x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n$ and $y \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_m$, we have that $x \oplus y \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus b_1 \oplus b_2 \oplus \cdots \oplus b_m$ by Lemma 2.2(ix). Thus, we can obtain that

$$\overline{B}(x) \sqcap \overline{B}(y) \sqcap [0.5, 0.5]$$

$$= [0.5,0.5] \sqcap \bigsqcup\big\{\overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus \cdots \oplus a_n\big\} \sqcap$$
$$\bigsqcup\big\{\overline{A}(b_1) \sqcap \cdots \sqcap \overline{A}(b_m) \sqcap [0.5,0.5] | b_1,b_2,\ldots,b_m \in L \text{ and } y \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_m\big\}$$
$$= \bigsqcup\big\{\overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n\big\} \sqcap$$
$$\bigsqcup\big\{\overline{A}(b_1) \sqcap \cdots \sqcap \overline{A}(b_m) \sqcap [0.5,0.5] | b_1,b_2,\ldots,b_m \in L \text{ and } y \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_m\big\}$$
$$= \bigsqcup\{\overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap \overline{A}(b_1) \sqcap \cdots \sqcap \overline{A}(b_m) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n,b_1,\ldots,b_m \in L$$
$$\text{such that } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \text{ and } y \leqslant b_1 \oplus b_2 \oplus \cdots \oplus b_m\}$$
$$\leqslant \bigsqcup\{\overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap \overline{A}(b_1) \sqcap \cdots \sqcap \overline{A}(b_m) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n,b_1,\ldots,b_m \in L$$
$$\text{such that } x \oplus y \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n \oplus b_1 \oplus b_2 \oplus \cdots \oplus b_m\}$$
$$\leqslant \bigsqcup\big\{\overline{A}(c_1) \sqcap \cdots \overline{A}(c_k) \sqcap [0.5,0.5] | c_1,c_2,\ldots,c_k \in L \text{ and } x \oplus y \leqslant c_1 \oplus \cdots \oplus c_k\big\} = \overline{B}(x \oplus y).$$

Hence $\overline{B}$ also satisfies (IVFI6). It follows from Theorem 3.3 that $\overline{B} \in \text{IVFLI}(L)$.

Secondly, for any $x \in L$, it follows from $x \leqslant x$ and the definition of $\overline{B}$ that $\overline{A}(x) \sqcap [0.5,0.5] \leqslant \overline{B}(x)$. This means that $\overline{A} \sqsubseteq \overline{B}$.

Finally, assume that $\overline{C} \in \text{IVFLI}(L)$ with $\overline{A} \sqsubseteq \overline{C}$. Then for any $x \in L$, we have

$$\overline{B}(x) \sqcap [0.5,0.5]$$
$$= [0.5,0.5] \sqcap \bigsqcup\big\{\overline{A}(a_1) \sqcap \cdots \sqcap \overline{A}(a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus \cdots \oplus a_n\big\}$$
$$\leqslant [0.5,0.5] \sqcap \bigsqcup\big\{\overline{C}(a_1) \sqcap \cdots \sqcap \overline{C}(a_n) | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n\big\}$$
$$= \bigsqcup\big\{\overline{C}(a_1) \sqcap \cdots \sqcap \overline{C}(a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n\big\}$$
$$= [0.5,0.5] \sqcap \bigsqcup\big\{\overline{C}(a_1) \sqcap \cdots \sqcap \overline{C}(a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus \cdots \oplus a_n\big\}$$
$$\leqslant [0.5,0.5] \sqcap \bigsqcup\big\{\overline{C}(a_1 \oplus \cdots \oplus a_n) | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n\big\}$$
$$= \bigsqcup\big\{\overline{C}(a_1 \oplus \cdots \oplus a_n) \sqcap [0.5,0.5] | a_1,a_2,\ldots,a_n \in L \text{ and } x \leqslant a_1 \oplus a_2 \oplus \cdots \oplus a_n\big\}$$
$$\leqslant \bigsqcup\big\{\overline{C}(x)\big\} = \overline{C}(x).$$

Hence $\overline{B} \sqsubseteq \overline{C}$ holds. To sum up, we have that $\overline{B} = \langle \overline{A} \rangle$. □

Example 4.5 Let $L = \{O,a,b,c,d,I\}$, $O' = I, a' = c, b' = d, c' = a, d' = b, I' = O$, the Hasse diagram of L be defined as Figure 1, and the operator $\to$ of L be defined as in Table 1.

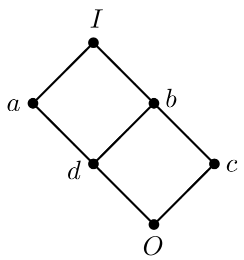

Figure 1 Hasse Diagram of L

$\to$	O	a	b	c	d	I
O	I	I	I	I	I	I
a	c	I	b	c	b	I
b	d	a	I	b	a	I
c	a	a	I	I	a	I
d	b	I	I	b	I	I
I	O	a	b	c	d	I

Table 1 Operator $\to$ of L

Then $(L, \vee, \wedge, ', \to, O, I)$ is a lattice implication algebra. Define an interval-valued fuzzy set $\overline{A} = [A^-, A^+]$ on L by $\overline{A}(O) = \overline{A}(a) = [0.7,0.8], \overline{A}(b) = \overline{A}(c) = \overline{A}(d) = \overline{A}(I) = [0.2,0.3]$. Since

$d \leqslant a$ but $\overline{A}(d) \sqcap [0.5, 0.5] = [0.2, 0.3] \ngeqslant [0.7, 0.8] = \overline{A}(a)$, we know that $\overline{A} \notin \mathrm{IVFLI}(L)$. It is easy to verify that $\langle\overline{A}\rangle \in \mathrm{IVFLI}(L)$ from Theorem 4.4, where $\langle\overline{A}\rangle(O) = \langle\overline{A}\rangle(a) = [0.5, 0.5], \langle\overline{A}\rangle(b) = \langle\overline{A}\rangle(c) = \langle\overline{A}\rangle(d) = \langle\overline{A}\rangle(I) = [0.2, 0.3]$.

Theorem 4.6 *Let L be a lattice implication algebra. Then* $(\mathrm{IVFLI}(L), \sqsubseteq)$ *is a complete lattice.*

Proof For any $\{\overline{A}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathrm{IVFLI}(L)$, where Λ is an indexed set. It is easy to verify that $\Cap_{\lambda\in\Lambda}\overline{A}_\lambda$ is infimum of $\{\overline{A}_\lambda\}_{\lambda\in\Lambda}$, where $t(\Cap_{\lambda\in\Lambda}\overline{A}_\lambda)(x) = \bigsqcap_{\lambda\in\Lambda}\overline{A}_\lambda(x)$ for all $x \in L$. i.e., $\bigwedge_{\lambda\in\Lambda}\overline{A}_\lambda = \Cap_{\lambda\in\Lambda}\overline{A}_\lambda$. Define $\Cup_{\lambda\in\Lambda}\overline{A}_\lambda$ as follows: $(\Cup_{\lambda\in\Lambda}\overline{A}_\lambda)(x) = \bigsqcup_{\lambda\in\Lambda}\overline{A}_\lambda(x)$ for all $x \in L$. Then $\langle\Cup_{\lambda\in\Lambda}\overline{A}_\lambda\rangle$ is supremum of $\{\overline{A}_\lambda\}_{\lambda\in\Lambda}$, where $\langle\Cup_{\lambda\in\Lambda}\overline{A}_\lambda\rangle$ is the interval-valued $(\in, \in\vee\, q)$-fuzzy LI-ideal generated by $\Cup_{\lambda\in\Lambda}\overline{A}_\lambda$ of L. i.e., $\bigvee_{\lambda\in\Lambda}\overline{A}_\lambda = \langle\Cup_{\lambda\in\Lambda}\overline{A}_\lambda\rangle$. Therefore, $(\mathrm{IVFLI}(L), \sqsubseteq)$ is a complete lattice. The proof is completed. $\square$

Remark 4.7 Let L be a lattice implication algebra. For all $\overline{A}, \overline{B} \in \mathrm{IVFLI}(L)$, by Theorem 4.6 we know that $\overline{A} \wedge \overline{B} = \overline{A} \Cap \overline{B}$ and $\overline{A} \vee \overline{B} = \langle\overline{A} \Cup \overline{B}\rangle$. The following example shows that $\overline{A} \vee \overline{B} \neq \overline{A} \Cup \overline{B}$ in general.

Example 4.8 Let $L = \{O, a, b, I\}$, $O' = I, a' = b, b' = a, I' = O$, the Hasse diagram of L be defined as Figure 2, and the operator $\rightarrow$ of L be defined as in Table 2.

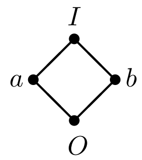

Figure 2 Hasse Diagram of L

$\rightarrow$	O	a	b	I
O	I	I	I	I
a	b	I	b	I
b	a	a	I	I
I	O	a	b	I

Table 2 Operator $\rightarrow$ of L

Then $(L, \vee, \wedge, ', \rightarrow, O, I)$ is a lattice implication algebra. Define interval-valued fuzzy sets $\overline{A} = [A^-, A^+]$ and $\overline{B} = [B^-, B^+]$ on L by $\overline{A}(O) = \overline{A}(b) = [0.8, 0.9], \overline{A}(a) = \overline{A}(I) = [0.2, 0.3]$, and $\overline{B}(O) = \overline{B}(a) = [0.8, 0.9], \overline{B}(b) = \overline{B}(I) = [0.2, 0.3]$, then $\overline{A}, \overline{B} \in \mathrm{IVFLI}(L)$. It is easily to verify that $\overline{C} = \overline{A} \Cup \overline{B} \notin \mathrm{IVFLI}(L)$, where $\overline{C}(O) = \overline{C}(a) = \overline{C}(b) = [0.8, 0.9], \overline{C}(I) = [0.2, 0.3]$. In fact, $\overline{C}(I) = [0.2, 0.3] \ngeqslant [0.5, 0.5] = \overline{C}((I \rightarrow a)') \sqcap \overline{C}(a) \sqcap [0.5, 0.5]$.

Theorem 4.9 *Let L be a lattice implication algebra. Then For all $\overline{A}, \overline{B} \in \mathrm{IVFLI}(L)$, $\overline{A} \vee \overline{B} = \langle\overline{A} \Cup \overline{B}\rangle = \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$ in the complete lattice* $(\mathrm{IVFLI}(L), \sqsubseteq)$.

Proof For all $\overline{A}, \overline{B} \in \mathrm{IVFLI}(L)$, it is obvious that $\overline{A} \sqsubseteq \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$ and $\overline{B} \sqsubseteq \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$, that is, $\overline{A}(x) \sqcap [0.5, 0.5] \leqslant (\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}})(x)$ and $\overline{B}(x) \sqcap [0.5, 0.5] \leqslant (\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}})(x)$ for all $x \in L$. Thus

$$\begin{aligned}(\overline{A} \Cup \overline{B})(x) \sqcap [0.5, 0.5] &= \left[\overline{A}(x) \sqcup \overline{B}(x)\right] \sqcap [0.5, 0.5] \\ &= \left(\overline{A}(x) \sqcap [0.5, 0.5]\right) \sqcup \left(\overline{B}(x) \sqcap [0.5, 0.5]\right) \leqslant \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x),\end{aligned}$$

that is, $\overline{A} \Cup \overline{B} \sqsubseteq \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$, and thus $\langle\overline{A} \Cup \overline{B}\rangle \sqsubseteq \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}} \in \mathrm{IVFLI}(L)$ by Theorem 3.9. Let

$\overline{C} \in \mathrm{IVFLI}(L)$ such that $\overline{A} \uplus \overline{B} \sqsubseteq \overline{C}$, for all $x \in L$, we consider the following two cases:

(i) If $x = O$, then

$$\begin{aligned}
&\left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(O) \sqcap [0.5, 0.5] \\
&= [0.5, 0.5] \sqcap \bigsqcup_{O \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} = \overline{A}^{\overline{B}}(O) \sqcap \overline{B}^{\overline{A}}(O) \sqcap [0.5, 0.5] \\
&= \left[\overline{A}(O) \sqcup \overline{B}(O)\right] \sqcap [0.5, 0.5] = \left(\overline{A} \uplus \overline{B}\right)(O) \sqcap [0.5, 0.5] \leqslant \overline{C}(O),
\end{aligned}$$

thus $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}} \sqsubseteq \overline{C}$ for this case.

(ii) If $x > O$, then we have

$$\begin{aligned}
&\left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x) \sqcap [0.5, 0.5] \\
&= [0.5, 0.5] \sqcap \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant a} \left\{\overline{A}(a) \sqcap \left[\overline{A}(O) \sqcup \overline{B}(O)\right] \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant b} \left\{\left[\overline{A}(O) \sqcup \overline{B}(O)\right] \sqcap \overline{B}(b) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant a} \left\{\overline{A}(a) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant b} \left\{\overline{B}(b) \sqcap [0.5, 0.5]\right\} \\
&\leqslant \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\overline{C}(a) \sqcap \overline{C}(b) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant a} \left\{\overline{C}(a) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant b} \left\{\overline{C}(b) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{C}(a) \sqcap \overline{C}(b) \sqcap [0.5, 0.5]\right\} \\
&= [0.5, 0.5] \sqcap \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{C}(a) \sqcap \overline{C}(b) \sqcap [0.5, 0.5]\right\} \\
&\leqslant \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{C}(a \oplus b) \sqcap [0.5, 0.5]\right\} \leqslant \overline{C}(x),
\end{aligned}$$

thus $\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}} \sqsubseteq \overline{C}$ for this case too.

By Definition 4.3 and Theorem 4.4 we have that $\overline{A} \vee \overline{B} = \langle \overline{A} \uplus \overline{B} \rangle = \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}$. $\square$

Finally, we investigate the distributivity of lattice $(\mathrm{IVFLI}(L), \sqsubseteq)$.

Theorem 4.10 *Let L be a lattice implication algebra. Then $(\mathrm{IVFLI}(L), \sqsubseteq)$ is a distributive lattice, where, $\overline{A} \wedge \overline{B} = \overline{A} \Cap \overline{B}$ and $\overline{A} \vee \overline{B} = \langle \overline{A} \uplus \overline{B} \rangle$, for all $\overline{A}, \overline{B} \in \mathrm{IVFLI}(L)$.*

Proof To finish the proof, it suffices to show that $\overline{C} \wedge (\overline{A} \vee \overline{B}) = (\overline{C} \wedge \overline{A}) \vee (\overline{C} \wedge \overline{B})$, for all $\overline{A}, \overline{B}, \overline{C} \in \mathrm{IVFLI}(L)$. Since the inequality $(\overline{C} \wedge \overline{A}) \vee (\overline{C} \wedge \overline{B}) \sqsubseteq \overline{C} \wedge (\overline{A} \vee \overline{B})$ holds automatically in a lattice, we need only to show the inequality $\overline{C} \wedge (\overline{A} \vee \overline{B}) \sqsubseteq (\overline{C} \wedge \overline{A}) \vee (\overline{C} \wedge \overline{B})$. i.e., we need only to show that $(\overline{C} \Cap \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}})(x) \sqcap [0.5, 0.5] \leqslant (\overline{C} \Cap \overline{A}^{\overline{C} \Cap \overline{B}} \uplus \overline{C} \Cap \overline{B}^{\overline{C} \Cap \overline{A}})(x)$, for all $x \in L$. For these, we consider the following two cases:

(i) If $x = O$, we have

$$\begin{aligned}
&\left(\overline{C} \Cap \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(O) \sqcap [0.5, 0.5] \\
&= \overline{C}(O) \sqcap \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(O) \sqcap [0.5, 0.5] = \overline{C}(O) \sqcap \bigsqcup_{O \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\
&= \overline{C}(O) \sqcap \left[\overline{A}^{\overline{B}}(O) \sqcap \overline{B}^{\overline{A}}(O) \sqcap [0.5, 0.5]\right] = \overline{C}(O) \sqcap \left[\overline{A}(O) \sqcup \overline{B}(O)\right] \sqcap [0.5, 0.5] \\
&= \left\{\left[\overline{C}(O) \sqcap \overline{A}(O)\right] \sqcup \left[\overline{C}(O) \sqcap \overline{B}(O)\right]\right\} \sqcap [0.5, 0.5] = \left[\left(\overline{C} \Cap \overline{A}\right)(O) \sqcup \left(\overline{C} \Cap \overline{B}\right)(O)\right] \sqcap [0.5, 0.5] \\
&= \left(\overline{C} \Cap \overline{A}^{\overline{C} \Cap \overline{B}}\right)(O) \sqcap \left(\overline{C} \Cap \overline{B}^{\overline{C} \Cap \overline{A}}\right)(O) \sqcap [0.5, 0.5] \\
&= \bigsqcup_{O \leqslant a \oplus b} \left\{\left(\overline{C} \Cap \overline{A}^{\overline{C} \Cap \overline{B}}\right)(a) \sqcap \left(\overline{C} \Cap \overline{B}^{\overline{C} \Cap \overline{A}}\right)(b) \sqcap [0.5, 0.5]\right\} \\
&= \left(\overline{C} \Cap \overline{A}^{\overline{C} \Cap \overline{B}} \uplus \overline{C} \Cap \overline{B}^{\overline{C} \Cap \overline{A}}\right)(O).
\end{aligned}$$

(ii) If $x > O$, we have

$$\begin{aligned}
&\left(\overline{C} \Cap \overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x) \sqcap [0.5, 0.5] = \overline{C}(x) \sqcap \left(\overline{A}^{\overline{B}} \uplus \overline{B}^{\overline{A}}\right)(x) \sqcap [0.5, 0.5] \\
&= \overline{C}(x) \sqcap \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b} \left\{\overline{C}(x) \sqcap \overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\overline{C}(x) \sqcap \overline{A}^{\overline{B}}(a) \sqcap \overline{B}^{\overline{A}}(b) \sqcap [0.5, 0.5]\right\} \sqcup \\
&\bigsqcup_{x \leqslant b} \left\{\overline{C}(x) \sqcap \overline{A}^{\overline{B}}(O) \sqcap \overline{B}(b) \sqcap [0.5, 0.5]\right\} \sqcup \bigsqcup_{x \leqslant a} \left\{\overline{C}(x) \sqcap \overline{A}(a) \sqcap \overline{B}^{\overline{A}}(O) \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\left[\overline{C}(x) \sqcap \overline{A}(a)\right] \sqcap \left[\overline{C}(x) \sqcap \overline{B}(b)\right] \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant b} \left\{\left[\overline{C}(x) \sqcap [0.5, 0.5] \sqcap \overline{A}^{\overline{B}}(O)\right] \sqcap \left[\overline{C}(x) \sqcap \overline{B}(b)\right] \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant a} \left\{\left[\overline{C}(x) \sqcap \overline{A}(a)\right] \sqcap \left[\overline{C}(x) \sqcap [0.5, 0.5] \sqcap \overline{B}^{\overline{A}}(O)\right] \sqcap [0.5, 0.5]\right\} \\
&\leqslant \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\left[\overline{C}(x \wedge a) \sqcap \overline{A}(x \wedge a)\right] \sqcap \left[\overline{C}(x \wedge b) \sqcap \overline{B}(x \wedge b)\right] \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant b} \left\{\left[\overline{C}(O) \sqcap \left(\overline{A}(O) \sqcup \overline{B}(O)\right)\right] \sqcap \left[\overline{C}(x \wedge b) \sqcap \overline{B}(x \wedge b)\right] \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant a} \left\{\left[\overline{C}(x \wedge a) \sqcap \overline{A}(x \wedge a)\right] \sqcap \left[\overline{C}(O) \sqcap \left(\overline{B}(O) \sqcup \overline{A}(O)\right)\right] \sqcap [0.5, 0.5]\right\} \\
&= \bigsqcup_{x \leqslant a \oplus b, a \neq O, b \neq O} \left\{\left(\overline{C} \Cap \overline{A}\right)(x \wedge a) \sqcap \left(\overline{C} \Cap \overline{B}\right)(x \wedge b) \sqcap [0.5, 0.5]\right\} \sqcup \\
&\quad \bigsqcup_{x \leqslant b} \left\{\left[\left(\overline{C} \Cap \overline{A}\right)(O) \sqcup \left(\overline{C} \Cap \overline{B}\right)(O)\right] \sqcap \left(\overline{C} \Cap \overline{B}\right)(x \wedge b) \sqcap [0.5, 0.5]\right\} \sqcup
\end{aligned}$$

$$\bigsqcup_{x\leqslant a}\left\{(\overline{C}\Cap\overline{A})(x\wedge a)\sqcap\left[(\overline{C}\Cap\overline{B})(O)\sqcup(\overline{C}\Cap\overline{A})(O)\right]\sqcap[0.5,0.5]\right\}$$

$$=\bigsqcup_{x\leqslant a\oplus b, a\neq O, b\neq O}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(x\wedge a)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(x\wedge b)\sqcap[0.5,0.5]\right\}\sqcup$$

$$\bigsqcup_{x\leqslant b}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(O)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(x\wedge b)\sqcap[0.5,0.5]\right\}\sqcup$$

$$\bigsqcup_{x\leqslant a}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(x\wedge a)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(O)\sqcap[0.5,0.5]\right\}$$

$$=\bigsqcup_{x\leqslant a\oplus b}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(x\wedge a)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(x\wedge b)\sqcap[0.5,0.5]\right\}.$$

Let $a\wedge x=c$ and $b\wedge x=d$. Since $x\leqslant a\oplus b$, using Lemma 2.2, we get that $c\oplus d=(a\wedge x)\oplus(b\wedge x)=((a\wedge x)\oplus b)\wedge((a\wedge x)\oplus x)=(a\oplus b)\wedge(x\oplus b)\wedge(a\oplus x)\wedge(x\oplus x)\geqslant(a\oplus b)\wedge x\wedge x\wedge x=(a\oplus b)\wedge x\geqslant x\wedge x=x$. Hence we can conclude that

$$\left(\overline{C}\Cap\overline{A}^{\overline{B}}\uplus\overline{B}^{\overline{A}}\right)(x)\sqcap[0.5,0.5]$$

$$\leqslant\bigsqcup_{x\leqslant a\oplus b}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(x\wedge a)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(x\wedge b)\sqcap[0.5,0.5]\right\}$$

$$\leqslant\bigsqcup_{x\leqslant c\oplus d}\left\{\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\right)(c)\sqcap\left(\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(d)\sqcap[0.5,0.5]\right\}$$

$$=\left(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\uplus\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}}\right)(x).$$

To sum up, we have that $(\overline{C}\Cap\overline{A}^{\overline{B}}\uplus\overline{B}^{\overline{A}})(x)\sqcap[0.5,0.5]\leqslant(\overline{C}\Cap\overline{A}^{\overline{C}\Cap\overline{B}}\uplus\overline{C}\Cap\overline{B}^{\overline{C}\Cap\overline{A}})(x)$, for all $x\in L$. The proof is completed. □

5. Concluding remarks

As well known, LI-ideal is an important concept for studying the structural features of lattice implication algebras. In this paper, the interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal theory in lattice implication algebras is further studied. Some new properties of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals are given. Representation theorem of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal which is generated by an interval-valued fuzzy set is established. It is proved that the set consisting of all interval-valued $(\in, \in\vee q)$-fuzzy LI-ideals in a lattice implication algebra, under the partial order $\sqsubseteq$, forms a complete distributive lattice. Results obtained in this paper not only enrich the content of interval-valued $(\in, \in\vee q)$-fuzzy LI-ideal theory in lattice implication algebras, but also show interactions of algebraic technique and interval-valued fuzzifying method in the studying logic problems. We hope that more links of interval-valued fuzzy sets and logics emerge by the stipulating of this work.

Acknowledgements We thank the referees for their time and comments.

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