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The Null and Column Spaces of Combinations of Two Projectors

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Abstract This paper establishes some new equalities and inequalities for the null and column spaces of combinations of two projectors P and Q. Some new necessary and sufficient conditions for $P \pm Q$ to be invertible are given by the structure of null and column space of some combinations of P and Q. In addition, the inclusion relation of $\mathcal{N}(PQ+QP)$ and $\mathcal{N}(PQ-QP)$ is discussed and necessary and sufficient conditions for them to be equal are also studied.

Keywords projector; orthogonal projector; null space; column space

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1. Introduction

Throughout this paper $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ matrices over the complex field $\mathbb{C}$, I_n stands for the identity matrix of order n. We use $\mathbb{C}^n$ to represent the linear space of all column vectors of dimension n over $\mathbb{C}$. For $A\in\mathbb{C}^{n\times n}$, denote by $r(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$, A^*, A^- and $|A|$ the rank, the null space, the column space, the conjugate transpose, a generalized inverse (the matrix A^- satisfies $AA^-A=A$) and the determinant of A, respectively. For a matrix $A\in\mathbb{C}^{n\times n}$, we say that A is group invertible if there exists a matrix $X\in\mathbb{C}^{n\times n}$ such that

$$AXA=A,\quad XAX=X,\quad AX=XA \tag{1.1}$$

hold. If such an X exists, then it is unique, and it is called the group inverse of A. It is well-known that A is group invertible if and only if $r(A)=r(A^2)$ (see [1]). We use $V\le\mathbb{C}^n$ to say that V is a subspace of $\mathbb{C}^n$, and use $V^\perp$ to represent the orthogonal complement of V in $\mathbb{C}^n$. If $V\le\mathbb{C}^n$, $T\in\mathbb{C}^{n\times n}$, denote $TV=\{T\alpha|\alpha\in V\}$, then $TV\le\mathbb{C}^n$. If T is an invertible matrix of order n, then $TV\cong V$ (meaning that TV is isomorphic to V). If $V\le\mathbb{C}^n$, denote by $\dim V$ the dimension of V. A matrix $A\in\mathbb{C}^{n\times n}$ is a projector if $A^2=A$; it is an orthogonal projector if, in

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addition, $A^*=A$. In what follows, $\mathbb{C}_n^P$ will mean the set of all projectors in $\mathbb{C}^{n\times n}$, i.e.,

$$\mathbb{C}_n^P=\left\{A\in\mathbb{C}^{n\times n}\,|\,A^2=A\right\},$$

whereas the symbol $\mathbb{C}_n^{OP}$ will denote a subset of the set of $\mathbb{C}_n^P$ consisting of orthogonal projectors, i.e., $\mathbb{C}_n^{OP}=\{A\in\mathbb{C}^{n\times n}\,|\,A^2=A=A^*\}$.

As one of the fundamental building blocks in matrix theory, idempotent matrices are very useful in many contexts and have been extensively studied in the literature [2–21]. Recently, to investigate the invertibility of $P+Q$ and $P-Q$ of $P,Q\in\mathbb{C}_n^P$, is of great interest in matrix theory, as it is closely connected with the problem of when the space $\mathbb{C}^n$ is the direct sum of its two subspaces and the existence of idempotent transformations satisfying some systems of equations. For instance, Groß and Trenkler in [2] considered the nonsingularity of $P-Q$ by employing the relations for the ranks of matrices developed by Marsaglia and Styan [3]; Koliha, Rakočević and Straškraba [4] obtained some new characterizations of the nonsingularity of $P\pm Q$ in terms of the nonsingularity of $P+Q$ or $P-Q$ by considering the kernel of a matrix to establish its nonsingularity; Tian and Styan [5] presented many interesting equalities for the ranks of combinations of projectors and applied them to the invertibility of $P-Q$ and $P+Q$; Baksalary and Trenkler reinvestigated the results of [5] from the point of view of the question: which relationships given in [5] remain valid when ranks are replaced with column spaces? Their work shed additional light on the links between subspaces attributed to various functions of $P,Q\in\mathbb{C}_n^{OP}$; Koliha, Rakočević in [6], Zuo and Xie in [7,8] found new relations between the nonsingularity of $P\pm Q$ and combinations of P and Q; Liu, Wu and Yu in [9] investigated the group inverse of the combinations of two projectors; Koliha, Rakočević in [10,11], Buckholtz in [12,13], Deng in [14,15], Rakočević and Wei in [16] discussed the invertibility in other settings, such as rings, Hilbert space and C*-algebras.

In this note, we follow the line of Baksalary and Trenkler's idea and find several new and interesting identities concerning the null and column spaces of $P\pm Q$, $(P-Q)^2$, $PQ\pm QP$, $Q-PQ$, $I-PQ$, $aP+bQ+cPQ$ $(a,b,c\in\mathbb{C},\ ab\neq 0)$ with $P,Q\in\mathbb{C}_n^P$ or $\mathbb{C}_n^{OP}$. Through these identities, we derive a variety of new characterizations for the invertibility of $P\pm Q$. Simultaneously, we also discuss inclusion relation between the null spaces of $PQ+QP$ and $PQ-QP$ and get some interesting rank equalities and inequalities.

To prove the main results, we shall begin with some lemmas.

Lemma 1.1 *Let $A,B\in\mathbb{C}^{n\times n}$ and T be an invertible matrix in $\mathbb{C}^{n\times n}$. Then*

(a) $(T\mathcal{N}(A))^{\perp}=(T^*)^{-1}\mathcal{R}(A^*)$, $\mathcal{N}(A)^{\perp}=\mathcal{R}(A^*)$;

(b) $\mathcal{R}(A)\cap\mathcal{R}(B)=\{0\}\Leftrightarrow\mathcal{N}(A^*)+\mathcal{N}(B^*)=\mathbb{C}^n$;

(c) $\mathcal{N}(A)\cap\mathcal{N}(B)=\{0\}\Leftrightarrow\mathcal{R}(A^*)+\mathcal{R}(B^*)=\mathbb{C}^n$.

Proof (a) Let $(T^*)^{-1}A^*\beta\in(T^*)^{-1}\mathcal{R}(A^*)$ with $\beta\in\mathbb{C}^n$ and $T\alpha\in T\mathcal{N}(A)$ with $\alpha\in\mathcal{N}(A)$. Since $[(T^*)^{-1}A^*\beta]^*\,T\alpha=\beta^*A\alpha=0$, we have $(T^*)^{-1}\mathcal{R}(A^*)\leq(T\mathcal{N}(A))^{\perp}$. Note that $\dim(T^*)^{-1}\mathcal{R}(A^*)$ $=\dim\mathcal{R}(A^*)=r(A^*)=r(A)=\dim\mathcal{N}(A)^{\perp}=\dim(T\mathcal{N}(A))^{\perp}$, then $(T\mathcal{N}(A))^{\perp}=(T^*)^{-1}\mathcal{R}(A^*)$. The second identity follows by setting $T=I_n$ in the first identity of (a).

(b) and (c) If M and N are two subspaces of $\mathbb{C}^n$, then from the following two identities

$$(M\cap N)^{\perp}=M^{\perp}+N^{\perp},\ \ (M+N)^{\perp}=M^{\perp}\cap N^{\perp}, \tag{1.2}$$

the statements (b) and (c) can be derived. □

Lemma 1.2 ([5]) *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $r(P-Q)=r\binom{P}{Q}+r(P,Q)-r(P)-r(Q)$;

(b) $r(P+Q)=r\begin{pmatrix}P&Q\\Q&0\end{pmatrix}-r(Q)=r\begin{pmatrix}Q&P\\P&0\end{pmatrix}-r(P)$;

(c) $r(P+Q)+r(PQ-QP)=r(P-Q)+r(PQ+QP)$.

Lemma 1.3 ([1]) *Let $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times k}$ and $C\in\mathbb{C}^{l\times n}$. Then*

(a) $(A,B)^-=\begin{pmatrix}A^-\{I_m-B[(I_m-AA^-)B]^-(I_m-AA^-)\}\\ [(I_m-AA^-)B]^-(I_m-AA^-)\end{pmatrix}$;

(b) $\begin{pmatrix}A\\C\end{pmatrix}^-=(\{I_n-(I_n-A^-A)[C(I_n-A^-A)]^-C\}A^-,(I_n-A^-A)[C(I_n-A^-A)]^-)$.

Lemma 1.4 ([3]) *Let $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times k}$ and $C\in\mathbb{C}^{l\times n}$. Then*

(a) $r(A,B)=r(A)+r(B-AA^-B)=r(B)+r(A-BB^-A)$;

(b) $r\begin{pmatrix}A\\C\end{pmatrix}=r(A)+r(C-CA^-A)=r(C)+r(A-AC^-C)$;

(c) $r\begin{pmatrix}A&B\\C&0\end{pmatrix}=r(B)+r(C)+r[(I_m-BB^-)A(I_n-C^-C)]$.

Lemma 1.5 ([17]) *Let $P,Q\in\mathbb{C}_n^{OP}$. Then there exists a unitary matrix $U\in\mathbb{C}^{n\times n}$ such that*

$$P=U\begin{pmatrix}I&0&&&&\\0&0&&&&\\&&I&&&\\&&&I&&\\&&&&0&\\&&&&&0\end{pmatrix}U^*,\ Q=U\begin{pmatrix}C^2&CS&&&&\\CS&S^2&&&&\\&&I&&&\\&&&0&&\\&&&&I&\\&&&&&0\end{pmatrix}U^*, \tag{1.3}$$

where C,S are positive diagonal real matrices such that $C^2+S^2=I$, the symbol I denotes identity matrices of various sizes, and the corresponding blocks in the two projection matrices are of the same size.

2. The null and column spaces of combinations of two projectors

In this section, we will present some identities concerning the null and column spaces of $(P-Q)^2$, $PQ\pm QP$, $Q-PQ$, $I-PQ$, $aP+bQ+cPQ$, $aP+bQ-cPQ-dQP-ePQP$; $a_1P+b_1Q+a_2PQ+b_2QP+a_3PQP+b_3QPQ+\cdots+a_{2n-1}(PQ)^{n-1}P+b_{2n-1}(QP)^{n-1}Q+a_{2n}(PQ)^n$. We also use these identities to derive some new characterizations for the invertibility of $P\pm Q$.

Theorem 2.1 *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $\mathcal{N}(P-Q)=(\mathcal{R}(P)\cap\mathcal{R}(Q))\oplus(\mathcal{N}(P)\cap\mathcal{N}(Q))$;

(b) $\mathcal{R}(P-Q)=(\mathcal{R}(P)+\mathcal{R}(Q))\cap(\mathcal{N}(P)+\mathcal{N}(Q))$.

Proof (a) It is clear that $(\mathcal{R}(P)\cap\mathcal{R}(Q))+(\mathcal{N}(P)\cap\mathcal{N}(Q))\leq\mathcal{N}(P-Q)$ and $(\mathcal{R}(P)\cap$

$\mathcal{R}(Q))\bigcap(\mathcal{N}(P)\cap\mathcal{N}(Q))=\{0\}$. For any $\alpha\in\mathcal{N}(P-Q)$, we have $P\alpha=Q\alpha=PQ\alpha$ and $\alpha=Q\alpha+(\alpha-Q\alpha)\in(\mathcal{R}(P)\cap\mathcal{R}(Q))+(\mathcal{N}(P)\cap\mathcal{N}(Q))$. Hence, $\mathcal{N}(P-Q)=(\mathcal{R}(P)\cap\mathcal{R}(Q))\oplus(\mathcal{N}(P)\cap\mathcal{N}(Q))$.

(b) Note that $P^*,Q^*\in\mathbb{C}_n^P$, then by (a) we have

$$\mathcal{N}(P^*-Q^*)=(\mathcal{R}(P^*)\cap\mathcal{R}(Q^*))\oplus(\mathcal{N}(P^*)\cap\mathcal{N}(Q^*)). \tag{2.1}$$

The statement (b) follows by taking orthogonal complement to both sides of Eq. (2.1) and by applying the results of the Lemma 1.1. □

Note that a matrix $A\in\mathbb{C}^{n\times n}$ is invertible if and only if $\mathcal{N}(A)=\{0\}$, then some necessary and sufficient conditions for which $P-Q$ is invertible is characterized by the null and column spaces of P and Q as follows.

Corollary 2.2 ([4]) *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $P-Q$ is invertible $\Leftrightarrow\mathcal{R}(P)\cap\mathcal{R}(Q)=\mathcal{N}(P)\cap\mathcal{N}(Q)=\{0\}$;

(b) $P-Q$ is invertible $\Leftrightarrow\mathcal{R}(P)+\mathcal{R}(Q)=\mathcal{N}(P)+\mathcal{N}(Q)=\mathbb{C}^n$.

By Theorem 2.1, the rank of $P-Q$ has the following representation.

Corollary 2.3 ([5]) *Let $P,Q\in\mathbb{C}_n^P$. Then*

$$r(P-Q)=r\begin{pmatrix}P\\Q\end{pmatrix}+r(P,Q)-r(P)-r(Q).$$

Proof Note that $\dim\mathcal{N}(P-Q)=n-r(P-Q)$, we have $\dim(\mathcal{R}(P)\cap\mathcal{R}(Q))=r(P)+r(Q)-r(P,Q)$ and $\dim(\mathcal{N}(P)\cap\mathcal{N}(Q))=\dim(\mathcal{N}\begin{pmatrix}P\\Q\end{pmatrix})=n-r\begin{pmatrix}P\\Q\end{pmatrix}$. The desired rank identity follows by substituting the three equalities into (a) of Theorem 2.1. □

The null and column spaces of $P+Q$ and $P-Q$ are closely related with those of $PQ+QP$, $PQ-QP$ and $I-P-Q$, which is given below.

Theorem 2.4 *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $\mathcal{N}(PQ-QP)=\mathcal{N}(P-Q)\oplus\mathcal{N}(I-P-Q)$;

(b) $\mathcal{N}(PQ+QP)=\mathcal{N}(P+Q)\oplus\mathcal{N}(I-P-Q)$;

(c) $\mathcal{R}(PQ-QP)=\mathcal{R}(P-Q)\cap\mathcal{R}(I-P-Q)$ and $\mathcal{R}(P-Q)+\mathcal{R}(I-P-Q)=\mathbb{C}^n$;

(d) $\mathcal{R}(PQ+QP)=\mathcal{R}(P+Q)\cap\mathcal{R}(I-P-Q)$ and $\mathcal{R}(P+Q)+\mathcal{R}(I-P-Q)=\mathbb{C}^n$.

Proof (a) Note that $(P-Q)(I-P-Q)=-(I-P-Q)(P-Q)=QP-PQ$, then we have $\mathcal{N}(P-Q)+\mathcal{N}(I-P-Q)\leq\mathcal{N}(PQ-QP)$. For any $\alpha\in\mathcal{N}(P-Q)\cap\mathcal{N}(I-P-Q)$, we have $P\alpha=Q\alpha$, $\alpha=P\alpha+Q\alpha$. Thus, $PQ\alpha=QP\alpha=0=P\alpha=Q\alpha$. Therefore, $\alpha=0$. Hence $\mathcal{N}(P-Q)\oplus\mathcal{N}(I-P-Q)\leq\mathcal{N}(PQ-QP)$. Next, we claim that the equality $\dim\mathcal{N}(P-Q)+\dim\mathcal{N}(I-P-Q)=\dim\mathcal{N}(PQ-QP)$ holds. To prove it, it suffices to verify $r(PQ-QP)+n=r(P-Q)+r(I-P-Q)$. On the one hand,

$$\begin{aligned}r\begin{pmatrix}I_n & I_n-P-Q\\ P-Q & 0\end{pmatrix}&=r\begin{pmatrix}I_n-QP-PQ & I_n-P-Q\\ P-Q & 0\end{pmatrix}\\&=r\begin{pmatrix}I_n-P-Q & I_n-P-Q\\ P-Q & 0\end{pmatrix}=r\left(P-Q\right)+r\left(I_n-P-Q\right).\end{aligned} \tag{2.2}$$

On the other hand,

$$r\begin{pmatrix} I_n & I_n-P-Q \\ P-Q & 0 \end{pmatrix} = r\begin{pmatrix} I_n & I_n-P-Q \\ 0 & QP-PQ \end{pmatrix} = r\,(PQ-QP)+n. \tag{2.3}$$

By Eqs. (2.2) and (2.3), we have $r(PQ-QP)+n = r(P-Q)+r(I-P-Q)$. Consequently, the desired equality $\mathcal{N}(PQ-QP) = \mathcal{N}(P-Q)\oplus\mathcal{N}(I-P-Q)$ holds.

(b) Note that $(P+Q)(I-P-Q) = -PQ-QP = (I-P-Q)(P+Q)$, then $\mathcal{N}(P+Q)+\mathcal{N}(I-P-Q) \le \mathcal{N}(PQ+QP)$. It is easy to see that $\mathcal{N}(P+Q)\cap\mathcal{N}(I-P-Q) = \{0\}$, thus $\mathcal{N}(P+Q)\oplus\mathcal{N}(I-P-Q) \le \mathcal{N}(PQ+QP)$. By applying the method similar to the proof of (a), we can also obtain $r(PQ+QP)+n = r(P+Q)+r(I-P-Q)$. Consequently, we have $\mathcal{N}(PQ+QP) = \mathcal{N}(P+Q)\oplus\mathcal{N}(I-P-Q)$.

(c) and (d) Note that $P^*, Q^* \in \mathbb{C}_n^P$, then (c) (resp., (d)) can be proved by using the results of Lemma 1.1 and (a) (resp., (b)). □

By Theorem 2.4, we get some more characterizations about the invertibility of $P-Q$, $P+Q$ and $I-P-Q$ by some identities of null spaces as follows.

Corollary 2.5 *Let $P, Q \in \mathbb{C}_n^P$. Then*

(a) $P-Q$ is invertible $\Leftrightarrow \mathcal{N}(PQ-QP) = \mathcal{N}(I-P-Q)$;

(b) $P+Q$ is invertible $\Leftrightarrow \mathcal{N}(PQ+QP) = \mathcal{N}(I-P-Q)$;

(c) $I-P-Q$ is invertible $\Leftrightarrow \mathcal{N}(PQ-QP) = \mathcal{N}(P-Q) \Leftrightarrow \mathcal{N}(PQ+QP) = \mathcal{N}(P+Q)$.

The null and column spaces of $(P-Q)^2$ can also be described by the null and column spaces of $P+Q, I-PQ, I-QP$ as follows.

Theorem 2.6 *Let $P, Q \in \mathbb{C}_n^P$. Then*

(a) $\mathcal{N}((P-Q)^2) = \mathcal{N}(P+Q)+\mathcal{N}(I-PQ)+\mathcal{N}(I-QP)$;

(b) $\mathcal{R}((P-Q)^2) = \mathcal{R}(P+Q)\cap\mathcal{R}(I-PQ)\cap\mathcal{R}(I-QP)$.

Proof (a) For $\alpha \in \mathcal{N}(P+Q)$, we have $P\alpha = -Q\alpha$. Therefore $(P-Q)^2\alpha = (P+Q-PQ-QP)\alpha = 0$, implying $\alpha \in \mathcal{N}((P-Q)^2)$. So

$$\mathcal{N}(P+Q) \le \mathcal{N}((P-Q)^2). \tag{2.4}$$

For any $\alpha \in \mathcal{N}(I-PQ)$, we have $\alpha = PQ\alpha = P\alpha$. Then $(P-Q)^2\alpha = (P+Q-PQ-QP)\alpha = 0$, therefore $\alpha \in \mathcal{N}((P-Q)^2)$, which implies

$$\mathcal{N}(I-PQ) \le \mathcal{N}((P-Q)^2). \tag{2.5}$$

Similarly, we can also prove

$$\mathcal{N}(I-QP) \le \mathcal{N}((P-Q)^2). \tag{2.6}$$

By Eqs. (2.4)–(2.6), we have

$$\mathcal{N}(P+Q)+\mathcal{N}(I-PQ)+\mathcal{N}(I-QP) \le \mathcal{N}((P-Q)^2). \tag{2.7}$$

On the other hand, for any $\alpha \in \mathcal{N}((P-Q)^2)$, then $(P+Q-PQ-QP)\alpha = 0$. Therefore, $P\alpha = PQP\alpha$, $Q\alpha = QPQ\alpha$, $QP\alpha = (QP)^2\alpha$, $PQ\alpha = (PQ)^2\alpha$. Consequently, $(I-PQ)PQ\alpha = 0$, $(I-QP)QP\alpha = 0$. Hence, $PQ\alpha \in \mathcal{N}(I-PQ)$, $QP\alpha \in \mathcal{N}(I-QP)$. Moreover, the identity

$(P+Q)(2\alpha - PQ\alpha - QP\alpha) = 2P\alpha + 2Q\alpha - PQ\alpha - QPQ\alpha - PQP\alpha - QP\alpha = P\alpha + Q\alpha - PQ\alpha - QP\alpha = (P-Q)^2\alpha = 0$ yields that $2\alpha - PQ\alpha - QP\alpha \in \mathcal{N}(P+Q)$. Therefore $\alpha = \frac{1}{2}(2\alpha - PQ\alpha - QP\alpha) + \frac{1}{2}PQ\alpha + \frac{1}{2}QP\alpha \in \mathcal{N}(P+Q) + \mathcal{N}(I-PQ) + \mathcal{N}(I-QP)$, which together with Eq. (2.7) implies that $\mathcal{N}((P-Q)^2) = \mathcal{N}(P+Q) + \mathcal{N}(I-PQ) + \mathcal{N}(I-QP)$.

(b) Since $P, Q \in \mathbb{C}_n^P$, we have $P^*, Q^* \in \mathbb{C}_n^P$. By applying (a) and Lemma 1.1, the statement (b) can be obtained. □

Another characterization of the invertibility of $P-Q$ can be given by Theorem 2.6.

Corollary 2.7 ([4]) *Let $P, Q \in \mathbb{C}_n^P$. Then $P-Q$ is invertible $\Leftrightarrow$ both $P+Q$ and $I-PQ$ are invertible $\Leftrightarrow$ both $P+Q$ and $I-QP$ are invertible.*

Proof Note that $\dim\mathcal{N}(I-PQ) = \dim\mathcal{N}(I-QP)$. Then $P-Q$ is invertible $\Leftrightarrow (P-Q)^2$ is invertible $\Leftrightarrow \mathcal{N}((P-Q)^2) = \{0\} \Leftrightarrow \mathcal{N}(P+Q) = \mathcal{N}(I-PQ) = \mathcal{N}(I-QP) = \{0\} \Leftrightarrow \mathcal{N}(P+Q) = \mathcal{N}(I-PQ) = \{0\} \Leftrightarrow P+Q$ and $I-PQ$ are all invertible $\Leftrightarrow P+Q$ and $I-QP$ are all invertible. □

By the proof of Corollary 2.7, we observe that $\dim\mathcal{N}(I-PQ) = \dim\mathcal{N}(I-QP)$. Therefore, $\mathcal{N}(I-PQ)$ and $\mathcal{N}(I-QP)$ are isomorphic as linear space. But the two spaces may not always be the same. There is an example to illustrate it.

Example 2.8 Let

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P, Q \in \mathbb{C}_n^P$ and

$$I - PQ = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ I - QP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear $\mathcal{N}(I-PQ) \neq \mathcal{N}(I-QP)$. But if P and Q are orthogonal projectors, then we have the following results.

Theorem 2.9 *Let $P, Q \in \mathbb{C}_n^{OP}$. Then*

(a) $\mathcal{N}(I-PQ) = \mathcal{N}(I-QP)$;

(b) $\mathcal{N}((P-Q)^2) = \mathcal{N}(P+Q) \oplus \mathcal{N}(I-PQ)$;

(c) $\mathcal{R}((P-Q)^2) = \mathcal{R}(P+Q) \cap \mathcal{R}(I-PQ)$.

Proof (a) By Lemma 1.5, for $P, Q \in \mathbb{C}_n^{OP}$, there exists a unitary matrix $U \in \mathbb{C}^n$ such that

$$P = U\begin{pmatrix} I & 0 & & & & \\ 0 & 0 & & & & \\ & & I & & & \\ & & & I & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}U^*, \quad Q = U\begin{pmatrix} C^2 & CS & & & & \\ CS & S^2 & & & & \\ & & I & & & \\ & & & 0 & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix}U^*,$$

where C, S are defined as those in Lemma 1.5. Direct calculations show that

$$I–PQ = U\begin{pmatrix} I-C^2 & -CS & & & & \\ 0 & I & & & & \\ & & 0 & & & \\ & & & I & & \\ & & & & I & \\ & & & & & I \end{pmatrix}U^*,\ I–QP = U\begin{pmatrix} I-C^2 & 0 & & & & \\ -CS & I & & & & \\ & & 0 & & & \\ & & & I & & \\ & & & & I & \\ & & & & & I \end{pmatrix}U^*.$$

Note that $|I - C^2| \neq 0$, we have $\mathcal{N}(I - PQ) = \mathcal{N}(I - QP)$.

(b) By the statement (a) and Theorem 2.6, we have $\mathcal{N}((P-Q)^2) = \mathcal{N}(P+Q)+\mathcal{N}(I-PQ)$. In addition, for any $\beta \in \mathcal{N}(P + Q) \cap \mathcal{N}(I - PQ)$, we have $(P + Q)\beta = 0$ and $\beta = PQ\beta = P\beta$, thus $2\beta = P(P + Q)\beta = 0$, so we conclude that $\mathcal{N}(P + Q) \cap \mathcal{N}(I - PQ) = \{0\}$. Hence $\mathcal{N}((P - Q)^2) = \mathcal{N}(P + Q) \oplus \mathcal{N}(I - PQ)$.

(c) Note that $P, Q \in \mathbb{C}_n^{OP}$, then $P^* = P$ and $Q^* = Q$. By the results of (b) and Lemma 1.1, the identity of (c) can be obtained. □

The null space of $Q - PQ$ and the column space of $Q - QP$ can be described as follows.

Theorem 2.10 *Let* $P, Q \in \mathbb{C}_n^P$. *Then*

(a) $\mathcal{N}(Q - PQ) = \mathcal{N}(Q) \oplus (\mathcal{R}(P) \cap \mathcal{R}(Q))$;

(b) $\mathcal{R}(Q - QP) = \mathcal{R}(Q) \cap (\mathcal{N}(P) + \mathcal{N}(Q))$.

Proof (a) It is clear that $\mathcal{N}(Q) + (\mathcal{R}(P) \cap \mathcal{R}(Q)) \leq \mathcal{N}(Q - PQ)$, and the sum of spaces $\mathcal{N}(Q) + (\mathcal{R}(P) \cap \mathcal{R}(Q))$ is direct sum. For any $\alpha \in \mathcal{N}(Q - PQ)$, we have $Q\alpha = PQ\alpha$. Therefore $\alpha = (\alpha - Q\alpha) + Q\alpha \in \mathcal{N}(Q) + (\mathcal{R}(P) \cap \mathcal{R}(Q))$. Hence $\mathcal{N}(Q - PQ) \leq \mathcal{N}(Q) + (\mathcal{R}(P) \cap \mathcal{R}(Q))$. Then (a) follows.

(b) Since $P^*, Q^* \in \mathbb{C}_n^P$, then (b) can be obtained from (a) and Lemma 1.1. □

Similarly, we can obtain the following results concerning the null and column spaces of $I - PQ$, for which the proof is similar to that of Theorem 2.9 and is omitted.

Theorem 2.11 *Let* $P, Q \in \mathbb{C}_n^P$. *Then*

(a) $\mathcal{N}(I - PQ) = \mathcal{R}(P) \cap \mathcal{N}(P - PQ)$;

(b) $\mathcal{R}(I - PQ) = \mathcal{N}(Q) + \mathcal{R}(Q - PQ)$.

Theorem 2.12 *Let* $P, Q \in \mathbb{C}_n^P$. *Then*

(a) $\mathcal{N}(P + Q - PQ) = \mathcal{N}(Q - PQ) \cap \mathcal{N}(P)$;

(b) $\mathcal{R}(P + Q - PQ) = \mathcal{R}(P - PQ) + \mathcal{R}(Q)$.

Proof For any $\alpha \in \mathcal{N}(Q - PQ) \cap \mathcal{N}(P)$, we have $Q\alpha = PQ\alpha$ and $P\alpha = 0$, which imply $(P + Q - PQ)\alpha = 0$. Therefore $\mathcal{N}(Q - PQ) \cap \mathcal{N}(P) \leq \mathcal{N}(P + Q - PQ)$. In addition, for any $\alpha \in \mathcal{N}(P + Q - PQ)$, then $(P + Q - PQ)\alpha = 0$. Thus, $P\alpha = 0$ and $(Q - PQ)\alpha = 0$, leading to $\mathcal{N}(P + Q - PQ) \leq \mathcal{N}(Q - PQ) \cap \mathcal{N}(P)$. Hence $\mathcal{N}(P + Q - PQ) = \mathcal{N}(Q - PQ) \cap \mathcal{N}(P)$.

(b) By the fact that $P^*, Q^* \in \mathbb{C}_n^P$ and the statement (a), we have $\mathcal{N}(P^* + Q^* - Q^*P^*) = \mathcal{N}(P^* - Q^*P^*) \cap \mathcal{N}(Q^*)$. Taking orthogonal complement on both sides of the equality, we have

$\mathcal{R}(P+Q-PQ)=\mathcal{R}(P-PQ)+\mathcal{R}(Q)$. $\square$

In general, the null and column spaces of $aP+bQ-cPQ$ $(a,b,c\in\mathbb{C}\ ab\neq 0)$ are described in the following.

Theorem 2.13 *Let $P,Q\in\mathbb{C}_n^P$ and $a,b,c\in\mathbb{C}$ $(ab\neq 0)$. Then*

(a) If $c=a+b$, then $\mathcal{N}(aP+bQ-cPQ)=\mathcal{N}(P-Q)$;

(b) If $c\neq a+b$, then $\mathcal{N}(aP+bQ-cPQ)=T\mathcal{N}(P+Q)\cong\mathcal{N}(P+Q)$, where $T=I+\frac{a+c-b}{a+b-c}Q$;

(c) If $c\neq a+b$, then $\mathcal{N}(aP+bQ-cPQ)=\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)$;

(d) If $c=a+b$, then $\mathcal{R}(aP+bQ-cPQ)=\mathcal{R}(P-Q)$;

(e) If $c\neq a+b$, then $\mathcal{R}(aP+bQ-cPQ)=K\mathcal{R}(P+Q)\cong\mathcal{R}(P+Q)$, where $K=I+\frac{b+c-a}{a+b-c}P$;

(f) If $c\neq a+b$, then $\mathcal{R}(aP+bQ-cPQ)=\mathcal{R}(P-PQ)+\mathcal{R}(Q+\frac{a-c}{b}P)$.

Proof (a) If $c=a+b$, then $(I-\frac{c}{a}P)(aP+bQ-cPQ)=b(Q-P)$. Note that $b\neq 0$ and $I-\frac{c}{a}P$ is invertible, we have $\mathcal{N}(aP+bQ-cPQ)=\mathcal{N}(P-Q)$.

(b) If $c\neq a+b$, then

$$(I+\frac{b+c-a}{a+b-c}P)(aP+bQ-cPQ)(I+\frac{a+c-b}{a+b-c}Q)=\frac{2ab}{a+b-c}(P+Q). \tag{2.8}$$

Note that $\frac{2ab}{a+b-c}\neq 0$ and both $I+\frac{b+c-a}{a+b-c}P$ and $I+\frac{a+c-b}{a+b-c}Q$ are invertible, we have $\mathcal{N}(aP+bQ-cPQ)=T\mathcal{N}(P+Q)\cong\mathcal{N}(P+Q)$.

(c) If $c\neq a+b$, then for any $\alpha\in\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)$, we have $(Q-PQ)\alpha=(P+\frac{b-c}{a}Q)\alpha=0$, which imply $PQ\alpha=Q\alpha$ and $P\alpha=\frac{c-b}{a}Q\alpha$. Thus $(aP+bQ-cPQ)\alpha=0$. Hence $\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)\leq\mathcal{N}(aP+bQ-cPQ)$. In addition, for any $\alpha\in\mathcal{N}(aP+bQ-cPQ)$, then $(aP+bQ-cPQ)\alpha=0$. Thus, $P\alpha=\frac{c-b}{a}PQ\alpha$ and $Q\alpha=PQ\alpha$, which imply $\alpha\in\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)$. Therefore $\mathcal{N}(aP+bQ-cPQ)\leq\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)$. Consequently, we have $\mathcal{N}(aP+bQ-cPQ)=\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)$.

(d) The statement (a) and Lemma 1.1 can be applied to obtain the desired statement.

(e) By using the equality of (2.1), the statement (e) can be obtained.

(f) Since $c\neq a+b$, we have $\bar{c}\neq\bar{a}+\bar{b}$ (where $\bar{a}$ is the conjugate of a). Note that $P^*,Q^*\in\mathbb{C}_n^P$, then by the statement (c) we have $\mathcal{N}(\bar{b}Q^*+\bar{a}P^*-\bar{c}Q^*P^*)=\mathcal{N}(P^*-Q^*P^*)\cap\mathcal{N}(Q^*+\frac{\bar{a}-\bar{c}}{\bar{b}}P^*)$. Taking orthogonal complement to both sides of the equation and applying Lemma 1.1, we have $\mathcal{R}(aP+bQ-cPQ)=\mathcal{R}(P-PQ)+\mathcal{R}(Q+\frac{a-c}{b}P)$. $\square$

From Theorem 2.12, the rank and the invertibility of $aP+bQ-cPQ$ are described as follows.

Corollary 2.14 *Let $P,Q\in\mathbb{C}_n^P$ and $a,b,c\in\mathbb{C}$ $(ab\neq 0)$. Then*

(a) $r(aP+bQ-cPQ)=\begin{cases} r(P-Q), & \text{if } c=a+b,\\ r(P+Q), & \text{if } c\neq a+b;\end{cases}$

(b) If $c\neq a+b$, then $aP+bQ-cPQ$ is invertible $\Leftrightarrow\mathcal{N}(Q-PQ)\cap\mathcal{N}(P+\frac{b-c}{a}Q)=\{0\}\Leftrightarrow\mathcal{R}(P-PQ)+\mathcal{R}(Q+\frac{a-c}{b}P)=\mathbb{C}^n$;

(c) If $c=a+b$, then $aP+bQ-cPQ$ is invertible $\Leftrightarrow P-Q$ is invertible.

Remark 2.15 It is worth pointing out that the statement (c) of Corollary 2.14 cannot be generalized to the case $aP+bQ-cPQ-dQP$. There is an example to illustrate it. Let

$$P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad Q=\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}.$$

It is clear that $P,Q\in\mathbb{C}_2^P$. If $a=12, b=-5, c=10, d=-3$, we can conclude that $a+b=c+d$ and $aP+bQ-cPQ-dQP$ is not invertible. If $a=b=1$ and $c=d=-1$, then the condition $a+b=c+d$ still holds, but $aP+bQ-cPQ-dQP$ is invertible. This implies that the ivertibility of $aP+bQ-cPQ-dQP$ does not ramain constant under the assumption $a+b=c+d$ with $a,b,c,d\in\mathbb{C}$ and $ab\neq 0$.

The null space and the invertibility of $P+Q$ can also be derived from Theorem 2.13.

Corollary 2.16 *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $\mathcal{N}(P+Q)\cong(\mathcal{N}(Q)\oplus(\mathcal{R}(P)\cap\mathcal{R}(Q)))\cap\mathcal{N}(P)$;

(b) $P+Q$ *is invertible* $\Leftrightarrow(\mathcal{N}(Q)\oplus(\mathcal{R}(P)\cap\mathcal{R}(Q)))\cap\mathcal{N}(P)=\{0\}$.

Proof (a) Substituting $a=b=c=1$ in the statement (b) of Theorem 2.13, we have $\mathcal{N}(P+Q)\cong\mathcal{N}(P+Q-PQ)$. By Theorems 2.10 and 2.12, we have $\mathcal{N}(P+Q-PQ)=(\mathcal{N}(Q)\oplus(\mathcal{R}(P)\cap\mathcal{R}(Q)))\cap\mathcal{N}(P)$. Hence $\mathcal{N}(P+Q)\cong(\mathcal{N}(Q)\oplus(\mathcal{R}(P)\cap\mathcal{R}(Q)))\cap\mathcal{N}(P)$.

(b) The statement (b) follows directly from that of (a). □

If $P,Q\in\mathbb{C}_n^{OP}$, then the null space of $ap+bQ-cPQ-dQP-ePQP$ can be described by the null spaces of $P+Q$ and $P-Q$ in the following.

Theorem 2.17 *Let $P,Q\in\mathbb{C}_n^{OP}$, $a,b,c,d,e\in\mathbb{C}$, $ab\neq 0$ and $|abI-(be+cd)C^2|\neq 0$, where C is the matrix in the CS decomposition of P and Q. Then*

(a) If $a+b=c+d+e$, then $\mathcal{N}(aP+bQ-cPQ-dQP-ePQP)=\mathcal{N}(P-Q)$;

(b) If $a+b\neq c+d+e$, then $\mathcal{N}(aP+bQ-cPQ-dQP-ePQP)=\mathcal{N}(P+Q)$.

Proof Consider the CS decomposition of P and Q, there exists a unitary matrix U such that P,Q can be presented as those in Lemma 1.5. Then we have

$$P-Q=U\begin{pmatrix} I-C^2 & -CS & & & & \\ -CS & -S^2 & & & & \\ & & 0 & & & \\ & & & I & & \\ & & & & -I & \\ & & & & & 0 \end{pmatrix}U^*,\quad P+Q=U\begin{pmatrix} I+C^2 & CS & & & & \\ CS & S^2 & & & & \\ & & 2I & & & \\ & & & I & & \\ & & & & I & \\ & & & & & 0 \end{pmatrix}U^*,$$

and

$$aP+bQ-cPQ-dQP-ePQP$$

$$=U\begin{pmatrix} aI+(b-c-d-e)C^2 & (b-c)CS & & & & \\ (b-d)CS & bS^2 & & & & \\ & & (a+b-c-d-e)I & & & \\ & & & aI & & \\ & & & & bI & \\ & & & & & 0 \end{pmatrix}U^*.$$

Since $\begin{vmatrix} I-C^2 & -CS \\ -CS & -S^2 \end{vmatrix} = |-S^2| \neq 0$, $\begin{vmatrix} I+C^2 & CS \\ CS & S^2 \end{vmatrix} = |S^2| \neq 0$, and

$$\begin{vmatrix} aI+(b-c-d-e)C^2 & (b-c)CS \\ (b-d)CS & bS^2 \end{vmatrix} = |abI-(be+cd)C^2||S^2| \neq 0,$$

the statements (a) and (b) can be derived. $\square$

For any positive integer n, we consider the null space of the combinations of $P,Q \in \mathbb{C}_n^P$ as $T = a_1P + b_1Q + a_2PQ + b_2QP + a_3PQP + b_3QPQ + \cdots + a_{2n-1}(PQ)^{n-1}P + b_{2n-1}(QP)^{n-1}Q + a_{2n}(PQ)^n$, where $a_i, b_j \in \mathbb{C}$ $(1 \leq i \leq 2n, 1 \leq j \leq 2n-1)$ and $a_1b_1 \neq 0$.

Theorem 2.18 *Let $P,Q \in \mathbb{C}_n^P$ such that $(PQ)^n = (QP)^n$.*

(a) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \neq 0$, then $\mathcal{N}(T) = \mathcal{N}(P) \bigcap \mathcal{N}(Q)$.

(b) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j = 0$, then $\mathcal{N}(T) = \mathcal{N}(P-Q)$.

Proof (a) It is clear that $\mathcal{N}(P) \bigcup \mathcal{N}(Q) \subseteq \mathcal{N}(T)$. On the other hand, for any $\alpha \in \mathcal{N}(T)$, we have

$$(a_1P + b_1Q + a_2PQ + b_2QP + a_3PQP + b_3QPQ + \cdots + a_{2n}(PQ)^n + b_{2n}(QP)^n)\,\alpha = 0. \quad (2.9)$$

Multiplying $(PQ)^n$ left to Eq. (2.9) and using the condition $(PQ)^n = (QP)^n$, we have

$$\Big(\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j\Big)(PQ)^n\alpha = 0.$$

The above identity and the condition $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \neq 0$ imply

$$(PQ)^n\alpha = 0. \quad (2.10)$$

Similarly, multiplying $(QP)^n$ left to Eq. (2.10), we have $(QP)^n\alpha = 0$. Multiplying $(PQ)^{n-1}P$ left to Eq. (2.9) and observing Eq. (2.10), we have $a_1(PQ)^{n-1}P\alpha = 0$. Since $a_1 \neq 0$, we have

$$(PQ)^{n-1}P\alpha = 0. \quad (2.11)$$

Multiplying $(QP)^{n-1}Q$ left to Eq. (2.9) and using $(QP)^n\alpha = 0$, we can deduce $(QP)^{n-1}Q\alpha = 0$. Multiplying

$$(PQ)^{n-1}, (QP)^{n-1}, (PQ)^{n-2}P, (QP)^{n-2}Q, \ldots, PQ, QP, P, Q$$

left to Eq. (2.9), respectively and taking similar deductions as above, we can obtain

$$(PQ)^{n-1}\alpha = (PQ)^{n-2}P\alpha = \cdots = PQP\alpha = PQ\alpha = P\alpha = 0$$

and

$$(QP)^{n-1}\alpha = (QP)^{n-2}Q\alpha = \cdots = QPQ\alpha = QP\alpha = Q\alpha = 0.$$

These imply $\alpha \in \mathcal{N}(P) \bigcup \mathcal{N}(Q)$. Hence $\mathcal{N}(T) \subseteq \mathcal{N}(P) \bigcup \mathcal{N}(Q)$ and the statement (a) follows.

(b) For any $\alpha \in \mathcal{N}(T)$, Eq. (2.9) still holds. Multiplying $(PQ)^{n-1}P$ left to Eq. (2.9) and note that $(PQ)^n = (QP)^n$, we have

$$a_1(PQ)^{n-1}P\alpha + \Big(\sum_{i=2}^{2n} a_i + \sum_{j=1}^{2n-1} b_j\Big)(PQ)^n\alpha = 0.$$

Since $a_1 + (\sum_{i=2}^{2n} a_i + \sum_{j=1}^{2n-1} b_j) = 0$ and $a_1 \neq 0$, we have

$$(PQ)^{n-1}P\alpha = (PQ)^n\alpha = 0. \tag{2.12}$$

Multiplying $(PQ)^{n-1}$ left to Eq. (2.9) and using Eq. (2.12), we get

$$b_1(PQ)^{n-1}\alpha + \Big(\sum_{i=1}^{2n} a_i + \sum_{j=2}^{2n-1} b_j\Big)(PQ)^n\alpha = 0.$$

The condition $b_1 + (\sum_{i=1}^{2n} a_i + \sum_{j=2}^{2n} b_j) = 0$ and $b_1 \neq 0$ imply

$$(PQ)^{n-1}\alpha = (PQ)^n\alpha. \tag{2.13}$$

Multiplying $(PQ)^{n-2}P, (PQ)^{n-2}, (PQ)^{n-3}P, \ldots, PQP, PQ, P$ left to Eq. (2.9), respectively, we have $(PQ)^{n-2}P\alpha = (PQ)^{n-2}\alpha = \cdots = PQP\alpha = PQ\alpha = P\alpha$. Consequently,

$$(PQ)^n\alpha = (PQ)^{n-1}P\alpha = \cdots = PQP\alpha = PQ\alpha = P\alpha. \tag{2.14}$$

Multiplying $(QP)^{n-1}Q, (QP)^{n-1}, \ldots, QPQ, QP, Q$ left to Eq. (2.9), respectively, we have

$$(QP)^n\alpha = (QP)^{n-1}Q\alpha = \cdots = QPQ\alpha = QP\alpha = Q\alpha. \tag{2.15}$$

Since $(PQ)^n = (QP)^n$, and by Eqs. (2.14) and (2.15), we have $P\alpha = Q\alpha$. Therefore $\alpha \in \mathcal{N}(P - Q)$. Hence $\mathcal{N}(T) \subseteq \mathcal{N}(P-Q)$. On the other hand, for any $\alpha \in \mathcal{N}(P-Q)$, we have $P\alpha = Q\alpha$. It is easy to see that Eqs. (2.14) and (2.15) hold. Consequently, $T\alpha = (\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j)P\alpha = 0$. That is, $\alpha \in \mathcal{N}(T)$. Therefore, $\mathcal{N}(P - Q) \subseteq \mathcal{N}(T)$. The statement (b) then follows. □

Corollary 2.19 *Let $P, Q \in \mathbb{C}_n^P$ such that $(PQ)^n = (QP)^n$.*

(a) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \neq 0$, then $r(T) = r(P + Q)$ and therefore T is invertible if and only if $P + Q$ is invertible.

(b) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j = 0$, then $r(T) = r(P - Q)$ and therefore T is invertible if and only if $P - Q$ is invertible.

(c) T is group invertible.

Proof (a) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \neq 0$, then by Theorem 2.18 (a), we have $r(T) = n - \dim\mathcal{N}(T) = n - \dim(\mathcal{N}(P) \bigcup \mathcal{N}(Q))$, which is a constant independent of the choices of the coefficients $a_i, b_j, (1 \leq i \leq 2n, 1 \leq j \leq 2n-1)$ such that $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \neq 0$ and $a_1b_1 \neq 0$. In particular, take $a_1 = b_1 = 1$ and $a_i = b_j = 0$ for $2 \leq i \leq 2n, 2 \leq j \leq 2n-1$, then $r(T) = r(P + Q)$. As a consequence, T is invertible if and only if $P + Q$ is invertible.

(b) If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j = 0$, then by Theorem 2.18 (b) we have $r(T) = n - \dim\mathcal{N}(T) = n - \dim(\mathcal{N}(P - Q))$, which is a constant independent of the choices of the coefficients $a_i, b_j, (1 \leq i \leq 2n, 1 \leq j \leq 2n-1)$ such that $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j = 0$ and $a_1b_1 \neq 0$. In particular, take

$a_1 = -b_1 = 1$ and $a_i = b_j = 0$ for $2 \le i \le 2n, 2 \le j \le 2n-1$, then $r(T) = r(P-Q)$. As a consequence, T is invertible if and only if $P-Q$ is invertible.

(c) Note that T^2 can be represented as a linear combinations of

$$P, Q, PQ, QP, \ldots, (PQ)^{n-1}P, (QP)^{n-1}P, (PQ)^n$$

and the sum of the coefficients of T^2 is $(\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j)^2$. If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j \ne 0$, then by the statement (a), we have $r(T^2) = r(P+Q) = r(T)$, which implies that T is group invertible. If $\sum_{i=1}^{2n} a_i + \sum_{j=1}^{2n-1} b_j = 0$, then by the statement (b), $r(T^2) = r(P-Q) = r(T)$ and T is group invertible. $\square$

Corollary 2.20 ([10]) *Let $P, Q \in \mathbb{C}_n^P$ such that $(PQ)^2 = (QP)^2$. Then*

$$aP + bQ + cPQ + dQP + ePQP + fQPQ + gPQPQ$$

is group invertible for any $a, b, c, d, e, f, g \in \mathbb{C}$ and $ab \ne 0$.

3. Inclusion relations of the null spaces of $PQ + QP$ and $PQ - QP$

In this section, we will discuss the inclusion relations for the null spaces of $PQ + QP$ and $PQ - QP$ with $P, Q \in \mathbb{C}_n^P$ and $P, Q \in \mathbb{C}_n^{OP}$, respectively. Some necessary and sufficient conditions for the two spaces to be equal are established. Consequently, more characterizations for the rank and invertibility of $P+Q$ and $P-Q$ are also obtained.

It is noted that the invertibility of $P-Q$ is sufficient, but not necessary, for the invertibility of $P+Q$, as demonstrated in [4]. Moreover, we can recall the statements (a) and (b) of Theorem 2.4. The above observations motivate us to consider the inclusion relations for $\mathcal{N}(PQ+QP)$ and $\mathcal{N}(PQ-QP)$. In general, one direction of the inclusion is not always right, with an example given below:

Example 3.1 Let

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P, Q \in \mathbb{C}_n^P$. Simple calculations show that

$$PQ + QP = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad PQ - QP = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that the two equalities $\mathcal{N}(PQ+QP) = \{(-2x, x, 0)' \mid x \in \mathbb{C}\}$ and $\mathcal{N}(PQ-QP) = \{(0, x, y)' \mid x, y \in \mathbb{C}\}$ hold. Then, $\mathcal{N}(PQ+QP) \le \mathcal{N}(PQ-QP)$ is not right. But we have the following rank inequality:

Theorem 3.2 *Let $P, Q \in \mathbb{C}_n^P$. Then P and Q satisfy*

$$r(PQ-QP) \le r(PQ+QP).$$

Proof By Lemma 1.2, we have $r(P+Q)+r(PQ-QP)=r(P-Q)+r(PQ+QP)$. To prove the statement, it suffices to prove $r(P-Q)\le r(P+Q)$. By Lemmas 1.2 and 1.4, we have

$$r(P+Q)=r\begin{pmatrix}P & Q\\ Q & 0\end{pmatrix}-r(Q)=r\begin{pmatrix}P\\ Q\end{pmatrix}+r\left(\left[I_{2n}-\begin{pmatrix}P\\ Q\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}^{-}\right]\begin{pmatrix}Q\\ 0\end{pmatrix}\right)-r(Q), \qquad (3.1)$$

and

$$r(P-Q)=r\begin{pmatrix}P\\ Q\end{pmatrix}+r(P,Q)-r(P)-r(Q)=r\begin{pmatrix}P\\ Q\end{pmatrix}+r(Q-PQ)-r(Q). \qquad (3.2)$$

By Lemma 1.3, we have

$$\begin{pmatrix}P\\ Q\end{pmatrix}^{-}=\left(P-(I_n-P)[Q(I_n-P)]^{-}QP,\ \ (I_n-P)[Q(I_n-P)]^{-}\right).$$

Therefore

$$\left[I_{2n}-\begin{pmatrix}P\\ Q\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}^{-}\right]\begin{pmatrix}Q\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}I_n-P & 0\\ -QP+Q(I_n-P)[Q(I_n-P)]^{-}QP & I_n-Q(I_n-P)[Q(I_n-P)]^{-}\end{pmatrix}\begin{pmatrix}Q\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}Q-PQ\\ -Q+Q(I_n-P)[Q(I_n-P)]^{-}Q\end{pmatrix},$$

which implies

$$r\left(\left(I_{2n}-\begin{pmatrix}P\\ Q\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}^{-}\right)\begin{pmatrix}Q\\ 0\end{pmatrix}\right)\ge r(Q-PQ). \qquad (3.3)$$

By Eqs. (3.1)–(3.3), we have $r(P+Q)\ge r(P-Q)$. $\square$

By the proof of Theorem 3.2 and the statement (a) of Corollary 2.14, we can now give the upper and lower bounds of the rank of $aP+bQ-cPQ$ with $a,b,c\in\mathbb{C}$ and $ab\ne 0$.

Corollary 3.3 *Let $P,Q\in\mathbb{C}_n^P$. Then we have*

$$\max_{a,b,c\in\mathbb{C},ab\ne 0} r(aP+bQ-cPQ)=r(P+Q)$$

and

$$\min_{a,b,c\in\mathbb{C},ab\ne 0} r(aP+bQ-cPQ)=r(P-Q).$$

A necessary and sufficient condition for the null (resp., column) space of $PQ+QP$ and $PQ-QP$ to be equal is established in the following.

Theorem 3.4 *Let $P,Q\in\mathbb{C}_n^P$. Then*

(a) $\mathcal{N}(PQ+QP)=\mathcal{N}(PQ-QP)\Leftrightarrow\mathcal{R}(P)\cap\mathcal{R}(Q)=\{0\}$;

(b) $\mathcal{R}(PQ+QP)=\mathcal{R}(PQ-QP)\Leftrightarrow\mathcal{N}(P)+\mathcal{N}(Q)=\mathbb{C}^n$.

Proof (a) "$\Leftarrow$". If $\mathcal{R}(P)\cap\mathcal{R}(Q)=\{0\}$, then for any $\alpha\in\mathcal{N}(PQ-QP)$, we have $PQ\alpha=QP\alpha\in\mathcal{R}(P)\cap\mathcal{R}(Q)=\{0\}$. Thus, $PQ\alpha=QP\alpha=0$, which implies $(PQ+QP)\alpha=0$. Therefore $\mathcal{N}(PQ-QP)\le\mathcal{N}(PQ+QP)$. Similarly, we can also prove $\mathcal{N}(PQ+QP)\le\mathcal{N}(PQ-QP)$. Hence $\mathcal{N}(PQ+QP)=\mathcal{N}(PQ-QP)$.

" $\Rightarrow$ ". If $\alpha \in \mathcal{R}(P)\cap\mathcal{R}(Q)$, then $\alpha = P\alpha = Q\alpha$, which implies $(PQ-QP)\alpha = 0$. Therefore $(PQ+QP)\alpha = 0 \Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$. Hence $\mathcal{R}(P)\cap\mathcal{R}(Q) = \{0\}$.

(b) Note that $P^*, Q^* \in \mathbb{C}_n^P$, then the statement (b) is derived by Lemma 1.1 and (a). □

According to Theorem 3.4 and Corollary 2.2, a new characterization for $P-Q$ to be invertible by the properties of the null spaces of $PQ+QP$ and $PQ-QP$ is given below.

Corollary 3.5 *Let $P, Q \in \mathbb{C}_n^P$. Then $P-Q$ is invertible $\Leftrightarrow \mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$ and $\mathcal{N}(P-PQ)\cap\mathcal{N}(Q-QP) = \{0\}$.*

Proof " $\Rightarrow$ " If $P-Q$ is invertible, then by Corollary 2.2 we have $\mathcal{R}(P)\cap\mathcal{R}(Q) = \{0\}$. By Theorem 3.3 we have $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$. For any $\alpha \in \mathcal{N}(P-PQ)\cap\mathcal{N}(Q-QP)$, then $P\alpha = PQ\alpha, Q\alpha = QP\alpha$. Thus, $(P-Q)^2\alpha = 0$, which implies $\alpha = 0$. Hence, $\mathcal{N}(P-PQ)\cap\mathcal{N}(Q-QP) = \{0\}$.

" $\Leftarrow$ ". By $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$ and Theorem 3.3, we have $\mathcal{R}(P)\cap\mathcal{R}(Q) = \{0\}$. Since $\mathcal{N}(P-PQ)\cap\mathcal{N}(Q-QP) = \{0\}$, we have $\mathcal{N}(P)\cap\mathcal{N}(Q) = \{0\}$. Consequently, by Corollary 2.2, $P-Q$ is invertible. □

A sufficient condition for which the identity $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$ holds is found below.

Theorem 3.6 *Let $P, Q \in \mathbb{C}_n^P$. Then $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$ if $I-PQ$ is invertible.*

Proof On the one hand, for any $\alpha \in \mathcal{N}(PQ-QP)$, then $PQ\alpha = QP\alpha$. Thus, $PQ\alpha = QP\alpha = PQP\alpha$. Consequently, $(I-PQ)(PQ+QP)\alpha = PQ\alpha + QP\alpha - PQPQ\alpha - PQP\alpha = PQ\alpha + QP\alpha - PQQP\alpha - PQP\alpha = 0$. The invertibility of $I-PQ$ implies $(PQ+QP)\alpha = 0$. Hence, $\mathcal{N}(PQ-QP) \le \mathcal{N}(PQ+QP)$. On the other hand, for any $\alpha \in \mathcal{N}(PQ+QP)$, we have $PQ\alpha = -QP\alpha = -PQP\alpha$. Thus, $(I-PQ)(PQ-QP)\alpha = PQ\alpha - QP\alpha - PQPQ\alpha + PQP\alpha = PQ\alpha - QP\alpha + PQP\alpha + PQP\alpha = 0$. The invertibility of $I-PQ$ implies $(PQ-QP)\alpha = 0$. Therefore, $\mathcal{N}(PQ+QP) \le \mathcal{N}(PQ-QP)$. Hence, $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$. □

Subsequently, by specializing the condition $P, Q \in \mathbb{C}_n^P$ into $P, Q \in \mathbb{C}_n^{OP}$, we continue to study the relation of $\mathcal{N}(PQ+QP)$ and $\mathcal{N}(PQ-QP)$.

Theorem 3.7 *Let $P, Q \in \mathbb{C}_n^{OP}$. Then $\mathcal{N}(PQ+QP) \le \mathcal{N}(PQ-QP)$.*

Proof Consider the CS decomposition of P and Q. By Lemma 1.5, there exists a unitary matrix U such that P, Q can be presented as Eq. (1.3). Direct calculations show that

$$PQ+QP = U\begin{pmatrix} 2C^2 & CS & & & & \\ CS & 0 & & & & \\ & & 2I & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}U^*,\ PQ-QP = U\begin{pmatrix} 0 & CS & & & & \\ -CS & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}U^*,$$

where $\begin{pmatrix} 2C^2 & CS \\ CS & 0 \end{pmatrix} \in \mathbb{C}^{2p\times 2p}$, $I \in \mathbb{C}^{t\times t}$.

Note that both $\begin{pmatrix} 2C^2 & CS \\ CS & 0 \end{pmatrix} \in \mathbb{C}^{2p\times 2p}$ and $\begin{pmatrix} 0 & CS \\ -CS & 0 \end{pmatrix} \in \mathbb{C}^{2p\times 2p}$ are invertible, then

$$\mathcal{N}(PQ+QP) = \{U\,(0_{1\times 2p}, 0_{1\times t}, X)'\,U^* \mid X \in \mathbb{C}^{1\times(n-2p-t)}\}$$

and

$$\mathcal{N}(PQ-QP) = \{U\,(0_{1\times 2p}, Y_{1\times t}, X)'\,U^* \mid X \in \mathbb{C}^{1\times(n-2p-t)}\}.$$

Therefore $\mathcal{N}(PQ+QP) \leq \mathcal{N}(PQ-QP)$. $\square$

If $P, Q \in \mathbb{C}_n^{OP}$, a necessary and sufficient condition for which the identity $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$ holds is given below.

Theorem 3.8 *Let $P, Q \in \mathbb{C}_n^{OP}$. Then $I - PQ$ is invertible $\Leftrightarrow \mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$.*

Proof The proof of necessity can be derived from Theorem 3.6. Consider the CS decomposition of P and Q and by the identity $\mathcal{N}(PQ+QP) = \mathcal{N}(PQ-QP)$, P, Q can be represented as

$$P = U\begin{pmatrix} I & 0 & & & \\ 0 & 0 & & & \\ & & I & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}U^*, \quad Q = U\begin{pmatrix} C^2 & CS & & & \\ CS & S^2 & & & \\ & & 0 & & \\ & & & I & \\ & & & & 0 \end{pmatrix}U^*.$$

It follows that

$$I - PQ = U\begin{pmatrix} I - C^2 & -CS & & & \\ 0 & I & & & \\ & & I & & \\ & & & I & \\ & & & & I \end{pmatrix}U^*$$

is invertible. This proves the sufficiency. $\square$

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