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On π-Semicommutative Rings

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Abstract A ring R is said to be π-semicommutative if $a, b \in R$ satisfy $ab = 0$ then there exists a positive integer n such that $a^n Rb^n = 0$. We study the properties of π-semicommutative rings and the relationship between such rings and other related rings. In particular, we answer a question on left GWZI rings negatively.

Keywords semicommutative rings; left GWZI rings; π-semicommutative rings

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1. Introduction

Throughout this note a ring is associative with identity unless otherwise stated. For a ring R, we use $N(R)$ to denote the set of nilpotent elements in R, $Z(R)$ its center, $N_*(R)$ its prime radical, and $J(R)$ its Jacobson radical. The symbol $T_n(R)$ stands for the ring of $n \times n$ upper triangular matrices over R, $S_n(R)$ its subring in which each matrix has the identical principal diagonal elements, I_n the $n \times n$ identity matrix, and E_{ij} $(i, j = 1, 2, \ldots, n)$ the $n \times n$ matrix units. For a nonempty subset X of R, we use $l(X)$ and $r(X)$ to denote the left and right annihilators of X in R, respectively. The ring of integers modulo a positive integer n is denoted by $\mathbb{Z}_n$.

A ring is reduced if it has no nonzero nilpotent elements, a ring is abelian if all idempotents are central, and a ring is 2-primal if its prime radical coincides with the set of nilpotent elements in it. Due to Bell [1], a ring R is called to satisfy the Insertion-of-Factors-Property (simply, an IFP ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Shin [2] used the term SI for the IFP, while Narbonne [3] used semicommutative in place of the IFP, and Habeb [4] used the term zero insertive (simply, ZI) for the IFP. In this paper, we choose a semicommutative ring in the above names, so as to cohere with other related references. It is known by [2, Lemma 1.2] that a ring R is semicommutative if and only if for any $a \in R$, $l(a)$ (resp., $r(a)$) is an ideal in R. There are many authors to study semicommutative rings and their generalizations. Liang et al. [5] called a ring R weakly semicommutaive if $ab = 0$ implies $aRb \subseteq N(R)$ for $a, b \in R$. Agayev et al. [6] defined a ring R to be central semicommutative if $ab = 0$ implies $aRb \subseteq Z(R)$ for $a, b \in R$, and they proved that a central semicommutative ring is a 2-primal ring.

According to Zhou [7], a left ideal L of R is called a generalized weak ideal (simply, a GW-ideal) if for any $a \in L$, there exists a positive integer n such that $a^n R \subseteq L$. Based on this notion,

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Du et al. [8] called a ring R to be left generalized weak zero insertive (simply, left GWZI), if $l(a)$ is a GW-ideal for any $a \in R$. Similarly, a right ideal M of R is called a GW-ideal in case for any $a \in M$, $Ra^n \subseteq M$ for some positive integer n, and a ring R is called right GWZI if $r(a)$ is a GW-ideal of R for any $a \in R$. If a ring R is a left and right GWZI ring, then it is said to be a GWZI ring. Some properties of left GWZI rings were investigated in [8]. However, it is a question whether a left GWZI ring is right GWZI. The main motivation of this note is to answer the question in the negative. Moreover we define a ring R to be π-semicommutative if $a, b \in R$ satisfy $ab = 0$ then there exists a positive integer n such that $a^n R b^n = 0$. It is proved that there exists a π-semicommutative ring which is neither left nor right GWZI, and that a ring R is π-semicommutative if and only if $S_n(R)$ is π-semicommutative for any positive integer $n \geq 2$.

2. Π-semicommutative rings

We start this section with the following observation.

Proposition 2.1 *A ring R is left GWZI if and only if $ab = 0$ implies that there exists a positive integer n such that $a^n R b = 0$ for $a, b \in R$.*

Proof It is clear. □

Symmetrically, a ring R is right GWZI if and only if $ab = 0$ implies that there exists a positive integer n such that $aRb^n = 0$ for $a, b \in R$.

Definition 2.2 *A ring R is called π-semicommutative if $a, b \in R$ satisfy $ab = 0$ then there exists a positive integer n such that $a^n R b^n = 0$.*

Clearly, the class of π-semicommutative rings is closed under subrings and finite direct sums.

Proposition 2.3 *Every central semicommutative ring is π-semicommutative.*

Proof Let R be a central semicommutative ring. If $a, b \in R$ satisfy $ab = 0$, then we have $arb \in Z(R)$ for any $r \in R$. This means that $a^2 r b^2 = aarbb = abarb = 0$. Thus R is a π-semicommutative ring by Definition 2.2. □

The converse of Proposition 2.3 is not true in general.

Example 2.4 A π-semicommutative ring need not be central semicommutative.

Proof It is known by [9, Corollary 13] that for every countable field F, there exists a nil algebra S over F such that $S[x]$ is not nil. Let $R = F + S$. Then R is a local ring with $J(R) = S$. We claim that R is a GWZI ring. Let $a, b \in R$ with $ab = 0$. If $a \in J(R)$, then a is nilpotent. There exists a positive integer n such that $a^n R b = 0$. If $a \notin J(R)$, then a is a unit. This implies that $b = 0$, and so $a^n R b = 0$. It follows that R is a left GWZI ring. Similarly, R is a right GWZI ring. Thus R is a π-semicommutative ring. Now we prove that R is not a 2-primal ring. Assume on the contrary, then one has $N_*(R) = N(R) = S$. This means that $N_*(R[x]) = N_*(R)[x] = S[x]$ is nil, a contradiction. Since a central semicommutative ring is 2-primal by [6, Theorem 2.4], R is not central semicommutative. □

The above proof shows that a GWZI ring need not be central semicommutative.

Clearly, a left (resp., right) GWZI ring is π-semicommutative, but the converse is not true by help of next example.

Example 2.5 There exists a right GWZI ring which is not left GWZI.

Proof Let F be a field and $A = F[a, b, c]$ the free algebra of polynomials with zero constant terms in noncommutative indeterminates a, b, c over F. Clearly, A is a ring without identity. Consider an ideal I of $F + A$ generated by cc, ac and crc for all $r \in A$. Let $R = (F + A)/I$. We prove that R is a right GWZI ring but R is not a left GWZI ring. Let I_1 be the linear space over F, of the monomials in A with exactly one c and $I_2 = F[a, b]$, the free algebra of polynomials with zero constant terms in noncommuting indeterminates a, b over F. Certainly we have $A = I + I_1 + I_2$. Let $A[x], I[x], I_1[x]$, and $I_2[x]$ denote the polynomial rings without identity over A, I, I_1 and I_2 respectively, where x is the indeterminate over R. For simplicity, in what follows we will use the claim which appears in the proof of [10, Example 14].

Claim. If $f(x), g(x) \in A[x]$ satisfy $f(x)g(x) \in I[x]$, then $f(x) \in I_1[x] + I[x]$ and $g(x) \in I_1[x] + I[x]$ (when $f(x) \notin I[x]$), or $f(x) \in I_1[x] + I[x] + I_2[x]a$ (when $g(x) \notin I[x]$) and $g(x) \in cI_2[x] + I[x]$. In particular, if $s, t \in A$ satisfy $st \in I$, then $s, t \in I_1 + I$ or $s \in I_1 + I + I_2a$ and $t \in cI_2 + I$ by taking into account the fact $sx, tx \in A[x]$ with $sxtx = stx^2 \in I[x]$.

We will prove that if $s, t \in F + A$ satisfy $st \in I$, then either $s \in I$ or $t^2 \in I$. First we show that the conclusion is true for $s, t \in A$. In this situation, we have $s, t \in I_1 + I$ or $s \in I_1 + I + I_2a$ and $t \in cI_2 + I$ by the claim. Thus $t^2 \in I$ holds in both cases by the definitions of I_1, I and I_2. Generally, let $s, t \in (F + A)$ with $st \in I$. We may write $s = k_1 + s_1, t = k_2 + t_2$ where $k_1, k_2 \in F$ and $s_1, t_2 \in A$. Thus we have $st = k_1k_2 + k_1t_2 + k_2s_1 + s_1t_2 \in I$. This means that $k_1k_2 = 0$ by the definition of I. If $k_1 = k_2 = 0$, then we have $t^2 \in I$ by the above argument. Assume that $k_1 = 0$ and $k_2 \neq 0$. Then we have $st = k_2s_1 + s_1t_2 \in I$. We claim that $s = s_1 \in I$. In fact, let $\hat{I}_1$ be the subspace in I_1, of the monomials in A with exactly one c but no ac as a factor, for example, $k_1ac, k_2bac, k_3bacb \notin \hat{I}_1$ where $k_1, k_2, k_3 \in F\backslash\{0\}$. Since $A = I + I_1 + I_2$, we have $A = I \oplus \hat{I}_1 \oplus I_2$ (as linear spaces). Let $s_1 = i + i_1 + i_2$, and $t_2 = i' + i'_1 + i'_2$ where $i, i' \in I, i_1, i'_1 \in \hat{I}_1, i_2, i'_2 \in I_2$. Then $st = k_2s_1 + s_1t_2 \in I$ implies that $k_2i_1 + i_1i'_2 + i_2i'_1 + k_2i_2 + i_2i'_2 \in I$. It is easy to see that $k_2i_1 + i_1i'_2 + i_2i'_1 \in \hat{I}_1$, and $k_2i_2 + i_2i'_2 \in I_2$. Thus we have $k_2i_1 + i_1i'_2 + i_2i'_1 = k_2i_2 + i_2i'_2 = 0$. Since $k_2 \neq 0$, we get $i_2 = 0$. Now $k_2i_1 + i_1i'_2 + i_2i'_1 = 0$ gives that $i_1 = 0$. Similarly, if $k_1 \neq 0$ and $k_2 = 0$ then we have $t \in I$. From the above discussion, we conclude that for any $s, t \in F + A$ with $st \in I$, then either $s \in I$ or $t^2 \in I$. Thus R is a right GWZI ring. On the other hand, since $ac \in I$ and $a^nbc \notin I$ for any positive integer n, we have $a^nRc \notin I$. This means that R is not a left GWZI ring. □

Example 2.5 gives a negative answer to the question of [8, p.255] whether the property of GWZI is left-right symmetric. Moreover, let R_1 be a right GWZI ring which is not left GWZI, and $R_2 = R_1^{op}$ be the opposite ring of R_1. Then it is easy to see that the ring direct sum $R = R_1 \oplus R_2$ is a π-semicommutative ring which is neither a left nor a right GWZI ring.

Proposition 2.6 *A π-semicommutative ring R is weakly semicommutative.*

Proof Let $a, b \in R$ with $ab = 0$. Then we have $(ba)^2 = 0$, and so $(rba)(bar) = 0$ for any $r \in R$. There exists a positive integer n such that $(rba)^n r(bar)^n = 0$ by the π-semicommutativity of R. Observing that $(bar)^n = ba(rba)^{n-1}r$, we have $0 = (rba)^n r(bar)^n = (rba)^n rba(rba)^{n-1}r = (rba)^{2n}r$. It follows that $(rba)^{2n+1} = 0$, i.e., $rba \in N(R)$, equivalently, $arb \in N(R)$ for any $r \in R$. □

Proposition 2.7 *A π-semicommutative ring R is abelian.*

Proof For any $e^2 = e$ and any $r \in R$, we have $e(1-e) = (1-e)e = 0$. The π-semicommutativity of R implies that there exists a positive integer n such that $e^n r(1-e)^n = (1-e)^n re^n = 0$, i.e., $er(1-e) = (1-e)re = 0$. This means that R is abelian. □

The converse of Proposition 2.7 is not true as the next example shows.

Example 2.8 ([6, Example 2.7]) Let $\mathbb{Z}$ be the ring of integers, and consider the ring $R =$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv d (\operatorname{mod} 2), b \equiv c (\operatorname{mod} 2), a, b, c, d \in \mathbb{Z} \right\}.$$

Then R is an abelian ring by the proof of [6, Example 2.7]. Let n be any positive integer, and

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}$$

in R. Clearly we have $AB = 0$, but $A^{2n+1}CB^{2n+1} = 2^{4n+2} \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \neq 0$. This means that R is not π-semicommutative. □

A ring R is called locally finite if every finite subset of R generates a finite multiplicative semigroup. For example, an algebraic closure of a finite field is locally finite but not finite.

Proposition 2.9 *A locally finite abelian ring R is a GWZI ring.*

Proof Let $a, b \in R$ with $ab = 0$. Since R is locally finite, there exist positive integers m, k such that $a^m = a^{m+k}$. This means that $a^m = a^m a^k = a^m a^{2k} = \cdots = a^m a^{mk} = a^m a^{(k-1)m} a^m$, and so $a^m a^{(k-1)m} = a^{km}$ is an idempotent. Thus we have $a^{km} rb = ra^{km} b = 0$ for any $r \in R$ since R is abelian, and hence R is a left GWZI ring. Similarly, R is a right GWZI ring. □

It is an open question in [8] whether $S_n(R)$ is left GWZI for any left GWZI ring R and any positive integer $n \geq 4$.

Corollary 2.10 *Let R be a left GWZI ring. If R is a finite ring, then $S_n(R)$ is a left GWZI ring for any positive integer n.*

Proof The hypothesis and Proposition 2.7 imply that R is abelian, and so is $S_n(R)$. Since R is a finite ring, $S_n(R)$ is a finite ring. Thus $S_n(R)$ is a left GWZI ring by Proposition 2.9. □

A ring R is called π-regular if for any $a \in R$, there exist a positive integer n and $b \in R$ such that $a^n = a^n ba^n$, and R is called regular in case $n = 1$ (see [11]).

Theorem 2.11 *If R is an abelian π-regular ring, then $S_n(R)$ is a GWZI ring for any positive* *integer n.*

Proof First we show that R is a GWZI ring. Let $a, b \in R$ with $ab = 0$. Since R is abelian π-regular, there exist a positive integer n and $c \in R$ such that $a^n = a^n c a^n$. Let $e = ca^n$. Then e is a central idempotent with $a^n = a^n e$. It follows that $a^n rb = a^n erb = a^n reb = a^n rca^n b = 0$ for any $r \in R$. Thus R is a left GWZI ring. Similarly, R is a right GWZI ring. To complete the proof, it suffices to show that $S_n(R)$ is abelian π-regular. It was proved in [11, Theorem 3] that an abelian ring R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. Now it is easily checked that R is abelian if and only if $S_n(R)$ is abelian, and that $N(R)$ is an ideal of R if and only if $N(S_n(R))$ is an ideal of $S_n(R)$. Moreover the ring isomorphism $S_n(R)/N(S_n(R)) \cong R/N(R)$ implies that $R/N(R)$ is regular if and only if $S_n(R)/N(S_n(R))$ is regular. Thus R is abelian π-regular if and only if $S_n(R)$ is abelian π-regular. The proof is completed from the above argument. □

Given a ring R and an (R, R)-bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. It is easily checked that this ring is isomorphic to the formal matrix ring

$$S = \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \mid a, \in R, m \in M \right\}.$$

In what follows we will identify $T(R, R)$ with the ring $S_2(R)$ canonically.

Theorem 2.12 *A ring R is a π-semicommutative ring if and only if the trivial extension $T(R, R)$ of R by R is a π-semicommutative ring.*

Proof Assume that R is π-semicommutative. Let $A, B \in T(R, R)$ with $AB = 0$. We may write $A = \begin{pmatrix} a & u \\ 0 & a \end{pmatrix}$, $B = \begin{pmatrix} b & v \\ 0 & b \end{pmatrix}$ where $a, u, b, v \in R$. Since $AB = 0$, we have $ab = 0$. There exists a positive integer n such that $a^n R b^n = 0$ by the π-semicommutativity of R. Let $\delta_k(r, s) = r^k s + r^{k-1} s r + \cdots + r s r^{k-1} + s r^k$ where $r, s \in R$ and k is a positive integer. Clearly, $\delta_{2n}(r, s)$ can be written as $\delta_{2n}(r, s) = r^n s_1 + s_2 r^n$ for some $s_1, s_2 \in R$. By a simple computation, we obtain that

$$A^{2n+1} = \begin{pmatrix} a^{2n+1} & \delta_{2n}(a, u) \\ 0 & a^{2n+1} \end{pmatrix}, \quad B^{2n+1} = \begin{pmatrix} b^{2n+1} & \delta_{2n}(b, v) \\ 0 & b^{2n+1} \end{pmatrix}.$$

For any $C \in T(R, R)$, there exist $c, w \in R$ such that $C = \begin{pmatrix} c & w \\ 0 & c \end{pmatrix}$. Thus we have

$$A^{2n+1} C B^{2n+1} = \begin{pmatrix} a^{2n+1} c b^{2n+1} & \delta \\ 0 & a^{2n+1} c b^{2n+1} \end{pmatrix}$$

where $\delta = a^{2n+1} c \delta_{2n}(b, v) + a^{2n+1} w b^{2n+1} + \delta_{2n}(a, u) c b^{2n+1}$. Noticing that $\delta_{2n}(a, u) = a^n c_1 + c_2 a^n$, and $\delta_{2n}(b, v) = b^n d_1 + d_2 b^n$ for some $c_1, c_2, d_1, d_2 \in R$, we get $a^{2n+1} c b^{2n+1} = \delta = 0$ by applying $a^n R b^n = 0$. Thus we have $A^{2n+1} C B^{2n+1} = 0$ for any $C \in T(R, R)$, and so $T(R, R)$ is π-semicommutative.

Conversely, suppose that $T(R,R)$ is π-semicommutative. Then its subring $RI_2 = \{aI_2 | a \in R\}$ is π-semicommutative, and thus $R \cong RI_2$ is π-semicommutative. □

For a ring R, we write $R^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in R\}$.

Lemma 2.13 *Let R be a π-semicommutative ring and $n \geq 2$ a positive integer. If $A, B \in S_n(R)$ satisfy $AB = 0$, then there exists a positive integer q such that $a^q \gamma_n B^q = 0$ for all $\gamma_n \in R^n$ where a is the principal diagonal element of A.*

Proof We proceed by induction on n. First, let $A, B \in S_2(R)$ with $AB = 0$. We may write

$$A = \begin{pmatrix} a & u \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} b & v \\ 0 & b \end{pmatrix}$$

where $a, u, b, v \in R$. Thus $AB = 0$ implies $ab = 0$, and there exists a positive integer m such that $a^m R b^m = 0$ by the π-semicommutativity of R. Using a simple computation, we obtain that

$$B^{2m+1} = \begin{pmatrix} b^{2m+1} & \delta_{2m} \\ 0 & b^{2m+1} \end{pmatrix}$$

where $\delta_{2m} = b^m r_1 + r_2 b^m$ for some $r_1, r_2 \in R$. Now for any $\gamma_2 = (c_1, c_2) \in R^2$, we may have

$$\begin{aligned} a^{2m+1}\gamma_2 B^{2m+1} &= (a^{2m+1}c_1, a^{2m+1}c_2) \begin{pmatrix} b^{2m+1} & \delta_{2m} \\ 0 & b^{2m+1} \end{pmatrix} \\ &= (a^{2m+1}c_1 b^{2m+1}, a^{2m+1}c_1\delta_{2m} + a^{2m+1}c_2 b^{2m+1}) = (0, a^{2m+1}c_1\delta_{2m}) \end{aligned}$$

by applying $a^m R b^m = 0$.

Observing that $a^{2m+1}c_1\delta_{2m} = a^{2m+1}c_1(b^m r_1 + r_2 b^m) = 0$, we have $a^{2m+1}\gamma_2 B^{2m+1} = 0$. Assume that the conclusion of the lemma is true for $n = k-1$. Let $A, B \in S_k(R)$ with $AB = 0$. We may write $A = \begin{pmatrix} a & \alpha \\ 0 & A_1 \end{pmatrix}$, $B = \begin{pmatrix} b & \beta \\ 0 & B_1 \end{pmatrix}$ where $a, b \in R$, $\alpha, \beta \in R^{k-1}$, and $A_1, B_1 \in S_{k-1}(R)$. Since $AB = 0$, we have $ab = 0$ and $A_1 B_1 = 0$. There exist positive integers p_1, p_2 such that $a^{p_1} R b^{p_1} = 0$ and $a^{p_2}\gamma_{k-1} B_1^{p_2} = 0$ for any $\gamma_{k-1} \in R^{k-1}$ by the π-semicommutativity of R and induction hypothesis. Let $p = \max\{p_1, p_2\}$. It follows that $a^p R b^p = 0$ and $a^p \gamma_{k-1} B_1^p = 0$. By a simple computation, we have

$$B^{2p+1} = \begin{pmatrix} b^{2p+1} & \Delta \\ 0 & B_1^{2p+1} \end{pmatrix}$$

where $\Delta = b^{2p}\beta + b^{2p-1}\beta B_1 + \cdots + b^l \beta B_1^{2p-l} + \cdots + \beta B_1^{2p}$. For any $\gamma_k = (c_1, c_2, \ldots, c_k) \in R^k$, we may write $\gamma_k = (c_1, \gamma_{k-1})$ with $\gamma_{k-1} \in R^{k-1}$. A direct computation yields that $a^{2p+1}\gamma_k B^{2p+1} = (a^{2p+1}c_1, a^{2p+1}\gamma_{k-1}) \begin{pmatrix} b^{2p+1} & \Delta \\ 0 & B_1^{2p+1} \end{pmatrix} = (a^{2p+1}c_1 b^{2p+1}, a^{2p+1}c_1\Delta + a^{2p+1}\gamma_{k-1}B_1^{2p+1})$. Using the facts $a^p R b^p = 0$ and $a^p \gamma_{k-1} B_1^p = 0$, we have $a^{2p+1}c_1 b^{2p+1} = 0$, $a^{2p+1}\gamma_{k-1}B_1^{2p+1} = 0$, and $a^{2p+1}c_1\Delta = a^{2p+1}c_1(b^{p-1}\beta B_1^p + \cdots + \beta B_1^{2p})$. Noticing that $c_1 b^{p-1}\beta, \ldots, c_1\beta$ are all in R^{k-1}, we get $a^{2p+1}c_1\Delta = 0$ by induction hypothesis. This means that $a^{2p+1}\gamma_k B^{2p+1} = 0$ for all $\gamma_k \in R^k$, and the proof is completed. □

Theorem 2.14 *Let R be a ring and $n \geq 2$ be a positive integer. Then R is π-semicommutative if and only if $S_n(R)$ is π-semicommutative.*

Proof Assume that R is a π-semicommutative ring. To prove $S_n(R)$ is π-semicommutative, we proceed by induction on n. The conclusion is true in the case $n = 2$ by Theorem 2.12. Assume that $S_{k-1}(R)$ is a π-semicommutative ring. Now let $A, B \in S_k(R)$ with $AB = 0$. We may write $A = \begin{pmatrix} a & \alpha \\ 0 & A_1 \end{pmatrix}$, $B = \begin{pmatrix} b & \beta \\ 0 & B_1 \end{pmatrix}$ where $a, b \in R$, $\alpha, \beta \in R^{k-1}$, and $A_1, B_1 \in S_{k-1}(R)$. Since $AB = 0$, we have $ab = 0$ and $A_1B_1 = 0$. There exist positive integers p_1, p_2 such that $a^{p_1}Rb^{p_1} = 0$ and $A_1^{p_2}S_{k-1}(R)B_1^{p_2} = 0$ by the π-semicommutativity of R and induction hypothesis. Applying Lemma 2.13, there exists a positive integer q such that $a^q\gamma_{k-1}B^q = 0$ for any $\gamma_{k-1} \in R^{k-1}$. Let $m = \max\{p_1, p_2, q\}$. We have $a^mRb^m = 0$, $A_1^mS_{k-1}(R)B_1^m = 0$, and $a^m\gamma_{k-1}B^m = 0$ for any $\gamma_{k-1} \in R^{k-1}$. By using a direct computation, we may obtain the following two equalities

$$A^{2m+1} = \begin{pmatrix} a^{2m+1} & \Delta(A) \\ 0 & A_1^{2m+1} \end{pmatrix}, \quad B^{2m+1} = \begin{pmatrix} b^{2m+1} & \Delta(B) \\ 0 & B_1^{2m+1} \end{pmatrix}$$

where $\Delta(A) = a^{2m}\alpha + a^{2m-1}\alpha A_1 + \cdots + \alpha A_1^{2m}$, $\Delta(B) = b^{2m}\beta + b^{2m-1}\beta B_1 + \cdots + \beta B_1^{2m}$. For any $C \in S_k(R)$, it can be written as $C = \begin{pmatrix} c & \gamma \\ 0 & C_1 \end{pmatrix}$ where $c \in R$, $C_1 \in S_{k-1}(R)$, and $\gamma \in R^{k-1}$. By a simple computation, we have the following equalities

$$A^{2m+1}CB^{2m+1} = \begin{pmatrix} a^{2m+1}c & a^{2m+1}\gamma + \Delta(A)C_1 \\ 0 & A_1^{2m+1}C_1 \end{pmatrix} \begin{pmatrix} b^{2m+1} & \Delta(B) \\ 0 & B_1^{2m+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{2m+1}cb^{2m+1} & a^{2m+1}c\Delta(B) + a^{2m+1}\gamma B_1^{2m+1} + \Delta(A)C_1B_1^{2m+1} \\ 0 & A_1^{2m+1}C_1B_1^{2m+1} \end{pmatrix}.$$

Applying the facts $a^mRb^m = 0$, $A_1^mS_{k-1}(R)B_1^m = 0$, and $a^m\gamma_{k-1}B^m = 0$, we get $A^{2m+1}CB^{2m+1} = \begin{pmatrix} 0 & \Delta_1 \\ 0 & 0 \end{pmatrix}$ where $\Delta_1 = a^{2m+1}c\Delta(B) + \Delta(A)C_1B_1^{2m+1}$. Since $cb^{m-1}\beta, \ldots, c\beta$ are in R^{k-1}, we have $a^{2m+1}c\Delta(B) = a^{2m+1}c(b^{m-1}\beta B_1^{m+1} + \cdots + \beta B_1^{2m}) = 0$. Similarly, since $\alpha C_1, \ldots, \alpha A_1^{m-1}C_1$ are in R^{k-1}, $A_1^mS_{k-1}(R)B_1^m = 0$, and $a^m\gamma_{k-1}B^m = 0$, we get $\Delta(A)C_1B_1^{2m+1} = (a^{2m}\alpha + a^{2m-1}\alpha A_1 + \cdots + a^{m+1}\alpha A_1^{m-1})C_1B_1^{2m+1} = 0$. It follows that $A^{2m+1}CB^{2m+1} = 0$ for any $C \in S_k(R)$. This completes the induction steps.

Conversely, if $S_n(R)$ is π-semicommutative, then R is π-semicommutative since $R \cong RI_n = \{rI_n | r \in R\}$ which is a subring of $S_n(R)$. □

Corollary 2.15 *A ring R is π-semicommutative if and only if $R[x]/(x^n)$ is π-semicommutative for any positive integer $n \geq 2$ where (x^n) is the ideal of $R[x]$ generated by x^n in $R[x]$.*

Proof Let $V = \sum_{i=1}^{n-1} E_{i,i+1}$, and $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ where $RV^k = \{rA | r \in R, A \in V^k\}$ for $1 \leq k \leq n-1$. Then we have $R[x]/(x^n) \cong V_n(R)$ in a natural way. If R is π-semicommutative, then $S_n(R)$ is π-semicommutative by Theorem 2.14, and so is $V_n(R)$ as a subring of $S_n(R)$. The validity of the converse of the corollary is rather obvious. □

A ring R in [12] is called linearly weak Armendariz (simply, LWA) if $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ satisfy $g(x)f(x) = 0$ then $a_ib_j \in N(R)$ for all i and j, equivalently, if $a, b \in R$ satisfy $a^2 = b^2 = 0$ then $a + b \in N(R)$ by [12, Proposition 2.2]. Thus a weakly semicommutative ring is LWA.

In the light of Theorem 2.14, it is natural to ask the question whether the subring

$$S(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots \\ 0 & a & a_{23} & \cdots \\ 0 & 0 & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

of the countable infinite upper triangular matrix ring is a π-semicommutative ring in case R is a π-semicommutative ring. The answer to this question is negative. In fact, $S(R)$ is not π-semicommutative for any ring R. Otherwise $S(R)$ is weakly semicommutative and so is LWA. Take $A = \sum_{i=1}^{\infty} E_{2i-1,2i}$ and $B = \sum_{i=1}^{\infty} E_{2i,2i+1}$ in $S(R)$, then clearly we have $A^2 = B^2 = 0$. But $A + B$ is not a nilpotent element, this shows that $S(R)$ is not LWA by [12, Proposition 2.2], a desried contradiction.

If R is a local ring with $J(R)$ nil, then R is π-semicommutative. In this case, $S(R)$ is a local ring since $S(R)/J(S(R)) \cong R/J(R)$. Thus $S(R)$ is an abelian ring, but $S(R)$ is not π-semicommutative from the above argument. This enables us to get more examples of anelian rings which are not π-semicommutative.

Example 2.16 There is a π-semicommutative ring R over which the polynomial ring $R[x]$ is not a π-semicommutative ring.

Proof By [12, Theorem 3.8], there exists a nil algebra S over some countable field F such that $S[x]$ is not LWA. Let $R = F + S$. Then R is a local ring with $J(R) = S$, and so R is π-semicommutative. We claim that $R[x]$ is not π-semicommutative. Otherwise $R[x]$ is weakly semicommutative, and so is LWA. This means that $S[x]$ is LWA as a subring of $R[x]$ without identity, a desired contradiction. □

As any ring is a factor of a polynomial domain containing sufficiently many noncommutative indeterminates, the homomorphic image of a π-semicommutative ring need not be π-semicommutative.

Proposition 2.17 *Let R be a ring and I an ideal of R. If I is reduced as a ring without identity and R/I is π-semicommutative, then R is π-semicommutative.*

Proof Write $\overline{R} = R/I$ and $\bar{r} = r + I$ for any $r \in R$. If $a, b \in R$ satisfy $ab = 0$, then $\bar{a}\bar{b} = \bar{0}$ in $\overline{R}$. There exists a positive integer n such that $\bar{a}^n\bar{r}\bar{b}^n = \bar{0}$, i.e., $a^nrb^n \in I$ for any $r \in R$ by the π-semicommutativity of $\overline{R}$. Since I is reduced and $(ba^nrb^na)^2 = 0$ in I, we get $ba^nrb^na = 0$. It follows that $(a^nrb^n)^3 = a^nrb^{n-1}(ba^nrb^na)a^{n-1}rb^n = 0$ in I. This gives $a^nrb^n = 0$ for any $r \in R$ by the reduceness of I. □

Proposition 2.18 *Let R be a left (resp., right) GWZI ring and I an ideal of R. Then $R/l(I)$*

(resp., $R/r(I)$) is a left (resp., right) GWZI ring.

Proof Write $\overline{R} = R/l(I)$ and $\bar{r} = r + l(I)$ where $r \in R$. For any $a, b \in R$, if $\bar{a}\bar{b} = \bar{0}$ in $\overline{R}$, then we have $ab \in l(I)$, and so $abv = 0$ for all $v \in I$. Since R is a left GWZI ring, there exists a positive integer n such that $a^n cbv = 0$ for any $c \in R$. This means that $a^n cb \in l(I)$, i.e., $\bar{a}^n \bar{c}\bar{b} = \bar{0}$ in $\overline{R}$ for any $\bar{c} \in \overline{R}$. The validity of the right version of the proposition is now clear. □

For any ring, reduced ⇒ semicommutative ⇒ GWZI ⇒ π-semicommutative ⇒ weakly semicommutative, and no converse implication holds. For example, $\mathbb{Z}_4$ is a semicommutative ring which is not reduced, $R = S_4(\mathbb{Z}_2)$ is a GWZI ring but not semicommutative, and $R = T_2(\mathbb{Z}_2)$ is a weakly semicommutative ring and not π-semicommutative [8]. Of course, there exists a π-semicommutative ring R which is neither left nor right GWZI by Example 2.5.

We conclude this note with the following proposition.

Proposition 2.19 *The following are equivalent for a ring R with $J(R) = 0$.*

(1) R is reduced;

(2) R is semicommutative;

(3) R is central semicommutative;

(4) R is left (right) GWZI;

(5) R is π-semicommutative;

(6) R is weakly semicommutative.

Proof It suffices to prove that (6) implies (1). Let $a \in R$ with $a^2 = 0$. Then we have $raa = 0$ for any $r \in R$, and so $rara \in N(R)$. Thus Ra is a nil left ideal in R. This means that $Ra \subseteq J(R) = 0$. Thus we have $a = 0$, and so R is reduced. □

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