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# Properties of Certain Nonlinear Integral Operator Associated with Janowski Starlike and Convex Functions

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**Abstract** In this paper, we consider a general nonlinear integral operator $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n; g_1,\ldots,g_n)(z)$. Some results including coefficient problems, univalency condition and radius of convexity for this integral operator are given. Furthermore, we discuss the mapping properties between $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ and subclasses of analytic functions with bounded boundary rotation. The same subjects for some corresponding classes are shown upon specializing the parameters in our main results.

**Keywords** analytic functions; Janowski functions; nonlinear integral operators; functions with bounded boundary rotation; subordination

**MR(2010) Subject Classification** 30C45; 30C50

## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$, which are analytic in the open unit disc $\Delta=\{z\in\mathbb{C}:|z|<1\}$ and are given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. By $\mathcal{S}$ we designate the subclass of $\mathcal{A}$ consisting of univalent functions in $\Delta$.

The function $f(z)\in\mathcal{A}$ is called subordinate to a function $g(z)\in\mathcal{A}$, written by $f\prec g$, if there exists a function $w(z)$, analytic in $\Delta$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=g(w(z))$. A function $p(z)\in\mathcal{P}[A,B]$ if $p(z)\in\mathcal{A}$ is subordinate to $\frac{1+Az}{1+Bz}$, where $p(z)$ is analytic in $\Delta$ with $p(0)=1$ and $-1\leqslant B<A\leqslant 1$. Janowski [1] introduced the class $\mathcal{P}[A,B]$. Furthermore, let $S^*[A,B]$ and $K[A,B]$ be subclasses of $\mathcal{S}$ consisting of starlike and convex Janowski functions, respectively defined by the following equalities:

$$S^*[A,B]=\big\{f(z)\in\mathcal{S}:\frac{zf'(z)}{f(z)}\in\mathcal{P}[A,B],\ z\in\Delta\big\},\tag{1.1}$$

$$K[A,B]=\big\{f(z)\in\mathcal{S}:1+\frac{zf''(z)}{f'(z)}\in\mathcal{P}[A,B],\ z\in\Delta\big\}.\tag{1.2}$$

In fact, the classes $K[A,B]$ and $S^*[A,B]$ have been extensively studied by many authors with different parameters $A$ and $B$ (see [2–9]). In particular, $K[1,-1]\equiv K$ and $S^*[1,-1]\equiv S^*$ are the class of convex functions and starlike functions, respectively. Moreover, $K[1-2\alpha,-1]\equiv K(\alpha)$

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and $S^*[1-2\alpha,-1] \equiv S^*(\alpha)$ $(0 < \alpha \leqslant 1)$ are the usual class of functions that are convex of order $\alpha$ and starlike of order $\alpha$ in $\Delta$, respectively. Noor et al. [10,11] defined the class $\mathcal{P}_k(\rho)$ as

$$\int_0^{2\pi} \Big|\frac{\Re(p(z))-\rho}{1-\rho}\Big| \mathrm{d}\theta \leqslant k\pi, \tag{1.3}$$

where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \rho < 1$. Pinchuk [12] studied the class $\mathcal{P}_k \equiv \mathcal{P}_k(0)$. Taking $b \in \mathbb{C} - \{0\}$, Noor et al. [10] also considered two important classes $V_k(\rho, b)$ and $R_k(\rho, b)$ related to $\mathcal{P}_k(\rho)$, where

$$V_k(\rho,b) = \Big\{f(z) \in \mathcal{S} : 1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \in \mathcal{P}_k(\rho)\Big\},$$

$$R_k(\rho,b) = \Big\{f(z) \in \mathcal{S} : 1 + \frac{1}{b}\Big(\frac{zf'(z)}{f(z)} + 1\Big) \in \mathcal{P}_k(\rho)\Big\}.$$

Notice that $V_k(0,1)$ and $R_k(0,1)$ are the well-known classes of analytic functions with bounded radius and bounded boundary rotations, respectively.

Now, let $\mathscr{H}_{\gamma,\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z) : \mathcal{A}^n \to \mathcal{A}$ be the nonlinear integral operator defined by

$$\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z) = \int_0^z \Big[\prod_{i=1}^n (f_i'(t))^{\alpha_i}\Big(\frac{g_i(t)}{t}\Big)^{\beta_i}\Big]\mathrm{d}t, \tag{1.4}$$

where $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ for all $i = 1, 2, \ldots, n$. Here, we need to note some special cases.

**Remark 1.1** (1) If $f_i(z) = g_i(z)$ for all $i = 1, 2, \ldots, n$, we obtain the integral operator introduced and studied by Frasin [13].

(2) If $\beta_i = 0$ for all $i = 1, 2, \ldots, n$, we obtain the integral operator introduced and studied by Breaz et al. [14].

(3) If $\alpha_i = 0$ for all $i = 1, 2, \ldots, n$, we obtain the integral operator introduced and studied by Breaz and Breaz [15].

(4) For $n = 1, \alpha_1 = \alpha, \beta_1 = \beta$ and $f_1 = f, g_1 = g$, we obtain the integral operator defined as

$$\mathscr{H}_{\alpha,\beta}(f,g) = \int_0^z (f'(t))^\alpha\Big(\frac{g(t)}{t}\Big)^\beta \mathrm{d}t.$$

(5) For $\alpha_1 = \alpha_2 = \cdots = \alpha$ and $\beta_1 = \beta_2 = \cdots = \beta$, we obtain the integral operator defined as

$$\mathscr{H}_{\alpha,\beta}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z) = \int_0^z \Big[\prod_{i=1}^n (f_i'(t))^{\alpha}\Big(\frac{g_i(t)}{t}\Big)^{\beta}\Big]\mathrm{d}t.$$

(6) For $n = 1$ and $\alpha_1 = 0, \beta_1 = \beta, g_1 = g$, we obtain the integral operator introduced and studied by Miller et al. [16].

(7) For $n = 1$ and $\alpha_1 = \alpha, \beta_1 = 0, f_1 = f$, we obtain the integral operator introduced and studied by Pascu and Pescar [17].

Also, kinds of different integral operators are studied by several authors (For more details, see [13,17–23]).

In the present paper, we study several properties of the operator $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$.

Throughout this paper we assume that in the sequel every many-valued function is taken with the principal branch.

## 2. Preliminary results

In the proofs of our main results we will make use of the following Lemmas:

**Lemma 2.1** ([5]) *Let $p(z) \in \mathcal{P}[A,B]$ and $z = re^{i\theta}$ $(0 \leqslant r \leqslant 1)$. Then*

$$\frac{1-Ar}{1-Br} \leqslant \Re(p(z)) \leqslant |p(z)| \leqslant \frac{1+Ar}{1+Br}.$$

In the above inequality, suppose that the function $\psi(r) = \frac{1+Ar}{1+Br}$ $(0 \leqslant r \leqslant 1)$, then

$$\psi'(r) = \frac{A-B}{(1+Br)^2} > 0,$$

which implies that the $\psi(r)$ is increasing function with respect to $r$. Thus, we have

$$\psi(r) = \frac{1+Ar}{1+Br} \leqslant \psi(1) = \frac{1+A}{1+B}, \quad B \neq -1.$$

**Lemma 2.2** ([24]) *Let $\gamma$ be complex number with $\Re(\gamma) > 0$. If $h(z) \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)} \Big| \frac{zh''(z)}{h'(z)} \Big| \leqslant 1,$$

*for all $z \in \Delta$, then the integral operator $F_\gamma(z) = \{\gamma \int_0^z t^{\gamma-1} h'(t) \mathrm{d}t\}^{\frac{1}{\gamma}}$ is in the class $\mathcal{S}$.*

**Lemma 2.3** ([25]) *Let the function $f(z) \in K$ with $z = re^{i\theta}$ $(0 \leqslant \theta \leqslant 2\pi)$. Then*

$$\frac{r}{1+r} \leqslant |f(z)| \leqslant \frac{r}{1-r}, \quad \frac{1}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2}.$$

*The results are sharp.*

## 3. Main results

**Theorem 3.1** *Let $\mathscr{L}(z) = \mathscr{H}_{\alpha,\beta}(f_1,\ldots,f_n; g_1,\ldots,g_n)(z)$ with $f_i(z), g_i(z) \in K$ $(i = 1,2,\ldots,n)$, $0 < \alpha < 1$, $0 < \beta < 1$, $z = re^{i\theta}$ $(0 < r < 1)$ and $\alpha + \beta = 1$. If $L(r, \mathscr{L}(z)) = \int_0^{2\pi} |z\mathscr{L}'(z)| \mathrm{d}\theta$, then*

$$L(r, \mathscr{L}(z)) \leqslant \frac{2\pi r}{(1-r)^{2n\alpha+n\beta}}.$$

**Proof** It is clear from (5) of Remark 1.1 that

$$\mathscr{H}_{\alpha,\beta}(f_1,\ldots,f_n; g_1,\ldots,g_n)(z) = \mathscr{L}(z) = \int_0^z \Big[\prod_{i=1}^n (f_i'(t))^\alpha \Big(\frac{g_i(t)}{t}\Big)^\beta\Big] \mathrm{d}t. \tag{3.1}$$

Differentiating both sides of (3.1), it follows that $\mathscr{L}'(z) = \prod_{i=1}^n (f_i'(z))^\alpha (\frac{g_i(z)}{z})^\beta$. Taking $z = re^{i\theta}$, we have

$$\begin{aligned} L(r, \mathscr{L}(z)) &= \int_0^{2\pi} |z\mathscr{L}'(z)| \mathrm{d}\theta = \int_0^{2\pi} \Big| z \prod_{i=1}^n (f_i'(z))^\alpha \Big(\frac{g_i(z)}{z}\Big)^\beta \Big| \mathrm{d}\theta \\ &= \int_0^{2\pi} \Big| z^{1-n\beta} \prod_{i=1}^n (f_i'(z))^\alpha (g_i(z))^\beta \Big| \mathrm{d}\theta \end{aligned}$$

$$=r^{1-n\beta}\int_0^{2\pi}\Big|\prod_{i=1}^{n}(f_i'(z))\Big|^{\alpha}\Big|\prod_{i=1}^{n}(g_i(z))\Big|^{\beta}\mathrm{d}\theta. \tag{3.2}$$

Using the well-known Holder's inequality in (3.2) with $0<\alpha,\beta<1$ and $\alpha+\beta=1$, then we can write

$$L(r,\mathscr{L}(z))\leqslant 2\pi r^{1-n\beta}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|\prod_{i=1}^{n}f_i'(z)\Big|\mathrm{d}\theta\Big)^{\alpha}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|\prod_{i=1}^{n}g_i(z)\Big|\mathrm{d}\theta\Big)^{\beta}. \tag{3.3}$$

Since $f_i(z)\in K$, $g_i(z)\in K$, from Lemma 2.3 and (3.3), we complete the proof of Theorem. □

**Theorem 3.2** *Let $\mathscr{L}(z)=\mathscr{H}_{\alpha,\beta}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=z+\sum_{k=2}^{\infty}\mathfrak{B}_k z^k$ with $f_i(z)\in K$, $g_i(z)\in K$ $(i=1,2,\ldots,n)$, $0<\alpha<1, 0<\beta<1$, $z=re^{i\theta}$ $(0<r<1)$ and $\alpha+\beta=1$. Then*

$$\mathfrak{B}_k\leqslant\frac{1}{k}\frac{1}{r^{k-1}}\frac{1}{(1-r)^{2n\alpha+n\beta}}.$$

**Proof** By Cauchy's formula, we have

$$\mathfrak{B}_k=\frac{1}{2\pi i}\int_{|z|=r}\frac{\mathscr{L}(z)}{z^{n+1}}\mathrm{d}z,\quad 0<r<1.$$

With $z=re^{i\theta}$, namely,

$$|\mathfrak{B}_k|\leqslant\frac{1}{2\pi r^k}\int_0^{2\pi}|\mathscr{L}'(re^{i\theta})|\mathrm{d}\theta. \tag{3.4}$$

From the Theorem 3.1 and (3.4) it follows that

$$k\mathfrak{B}_k\leqslant\frac{1}{2\pi r^k}\int_0^{2\pi}|z\mathscr{L}'(re^{i\theta})|\mathrm{d}\theta=\frac{1}{2\pi r^k}L(r,\mathscr{L}(z))\leqslant\frac{1}{r^{k-1}}\frac{1}{(1-r)^{2n\alpha+n\beta}}.$$

This completes the proof. □

**Theorem 3.3** *If $\gamma$ is a complex number with $\Re(\gamma)>0$ and*

$$\sum_{i=1}^{n}(\alpha_i+\beta_i)\leqslant\begin{cases}\dfrac{1+B}{2+A+B}\Re(\gamma), & \text{if } 0<\Re(\gamma)<1;\\[2mm] \dfrac{1+B}{2+A+B}, & \text{if } \Re(\gamma)\geqslant 1,\end{cases} \tag{3.5}$$

*then the integral operator $\mathscr{L}(z)=\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)$ satisfies*

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big|\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}\Big|\leqslant 1,$$

*where $B\neq-1,\alpha_i\geqslant 0,\beta_i\geqslant 0, f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$ for all $i=1,2,\ldots,n$.*

**Proof** We begin by setting

$$\mathscr{L}(z)=\int_0^z\Big[\prod_{i=1}^{n}(f_i'(t))^{\alpha_i}\Big(\frac{g_i(t)}{t}\Big)^{\beta_i}\Big]\mathrm{d}t, \tag{3.6}$$

where $f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$ for all $i\in\mathbb{Z}^+$. From (3.6), we know that

$$\mathscr{L}'(z)=\prod_{i=1}^{n}(f_i'(z))^{\alpha_i}\Big(\frac{g_i(z)}{z}\Big)^{\beta_i} \tag{3.7}$$

and $\mathscr{Z}(0)=\mathscr{Z}'(0)-1=0$. It is not difficult to see that (3.7) provides

$$\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}=\sum_{i=1}^{n}\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\sum_{i=1}^{n}\beta_i\big(\frac{zg_i'(z)}{g_i(z)}-1\big). \tag{3.8}$$

Next, setting $z=re^{i\theta}$ and using (3.8) with Lemma 2.1, we obtain

$$\begin{aligned}\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big|\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}\Big| &= \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big|\sum_{i=1}^{n}\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\sum_{i=1}^{n}\beta_i(\frac{zg_i'(z)}{g_i(z)}-1)\Big|\\
&\leqslant \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big\{\sum_{i=1}^{n}\alpha_i\Big|\frac{zf_i''(z)}{f_i'(z)}+1-1\Big|+\sum_{i=1}^{n}\beta_i\Big|\frac{zg_i'(z)}{g_i(z)}-1\Big|\Big\}\\
&\leqslant \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big\{\sum_{i=1}^{n}\alpha_i\Big[\Big|\frac{zf_i''(z)}{f_i'(z)}+1\Big|+1\Big]+\sum_{i=1}^{n}\beta_i\Big[\Big|\frac{zg_i'(z)}{g_i(z)}\Big|+1\Big]\Big\}\\
&\leqslant \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big\{\sum_{i=1}^{n}\alpha_i\big(\frac{1+Ar}{1+Br}+1\big)+\sum_{i=1}^{n}\beta_i\big(\frac{1+Ar}{1+Br}+1\big)\Big\}\\
&\leqslant \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big\{\sum_{i=1}^{n}\alpha_i\big(\frac{1+A}{1+B}+1\big)+\sum_{i=1}^{n}\beta_i\big(\frac{1+A}{1+B}+1\big)\Big\}\\
&= \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\frac{2+A+B}{1+B}\sum_{i=1}^{n}(\alpha_i+\beta_i).\end{aligned} \tag{3.9}$$

In fact, we need to discuss with $\Re(\gamma)$ for different cases:

**Case 1** If $0<\Re(\gamma)<1$. Then we easily observe that the function

$$1-|z|^{2\Re(\gamma)}\leqslant 1-|z|^2\leqslant 1 \tag{3.10}$$

for $|z|<1$.

**Case 2** If $\Re(\gamma)\geqslant 1$, then we have

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\leqslant 1-|z|^2\leqslant 1 \tag{3.11}$$

for $|z|<1$. Thus, following the (3.9), (3.10) and (3.11) and using the hypothesis (3.5), we get

$$\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\Big|\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}\Big|\leqslant\begin{cases}\frac{1}{\Re(\gamma)}\frac{2+A+B}{1+B}\sum_{i=1}^{n}(\alpha_i+\beta_i), & \text{if } 0<\Re(\gamma)<1,\\ \frac{2+A+B}{1+B}\sum_{i=1}^{n}(\alpha_i+\beta_i), & \text{if } \Re(\gamma)\geqslant 1,\end{cases}\leqslant 1.$$

This completes the proof of Theorem 3.3. $\Box$

**Remark 3.4** We define the another more general nonlinear integral operator $\mathscr{H}_{\gamma,\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z):\mathcal{A}^n\to\mathcal{A}$ as

$$\mathscr{H}_{\gamma,\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=\Big\{\gamma\int_0^z t^{\gamma-1}\Big[\prod_{i=1}^{n}(f_i'(t))^{\alpha_i}\big(\frac{g_i(t)}{t}\big)^{\beta_i}\Big]\mathrm{d}t\Big\}^{\frac{1}{\gamma}}.$$

By applying Lemma 2.2 and the above Theorem 3.3 for the function $\mathscr{Z}(z)$, then it is easy to prove that the nonlinear integral operator $\mathscr{H}_{\gamma,\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)\in S$.

**Theorem 3.5** *Let $f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$ for all $i=1,2,\ldots,n$. If $B-(B-A)\sum_{i=1}^n(\alpha_i+\beta_i)>0$, then the integral operator $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)\in K$ for $|z|<r_0$, where $r_0$ is given by $r_0=\min\{\frac{1}{B-(B-A)\sum_{i=1}^n(\alpha_i+\beta_i)},1\}$.*

**Proof** We can give that

$$\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=\mathscr{L}(z)=\int_0^z\Big[\prod_{i=1}^n(f_i'(t))^{\alpha_i}\big(\frac{g_i(t)}{t}\big)^{\beta_i}\Big]\mathrm{d}t,$$

Furthermore, (3.8) shows

$$\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}=\sum_{i=1}^n\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\sum_{i=1}^n\beta_i\big(\frac{zg_i'(z)}{g_i(z)}-1\big).\tag{3.12}$$

Let $z=re^{i\theta}$ $(0\leqslant\theta\leqslant 2\pi)$. Since $f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$ for all $i=1,2,\ldots,n$, then (3.12) and Lemma 2.1 give

$$\begin{aligned}\Re\big\{\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}+1\big\}=&\sum_{i=1}^n\alpha_i\Re\big(\frac{zf_i''(z)}{f_i'(z)}+1\big)+\sum_{i=1}^n\beta_i\Re\big(\frac{zg_i'(z)}{g_i(z)}\big)+1-\sum_{i=1}^n(\alpha_i+\beta_i)\\ \geqslant&\sum_{i=1}^n\alpha_i\frac{1-Ar}{1-Br}+\sum_{i=1}^n\beta_i\frac{1-Ar}{1-Br}+1-\sum_{i=1}^n(\alpha_i+\beta_i)\\ =&\big(\frac{1-Ar}{1-Br}-1\big)\sum_{i=1}^n(\alpha_i+\beta_i)+1\\ =&\frac{[(B-A)\sum_{i=1}^n(\alpha_i+\beta_i)-B]r+1}{1-Br}.\end{aligned}\tag{3.13}$$

Clearly the right hand side of (3.13) is positive for $|z|<r_0$. Hence $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)\in K$ for $|z|<r_0$, where $r_0$ is given as the condition with Theorem 3.5. □

**Theorem 3.6** *If $f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$ for all $i=1,2,\ldots,n$, then $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)\in V_k(0,b)$, where $b>0$ and $k=2+\frac{4+2A+2B}{b(1+B)}\sum_{i=1}^n(\alpha_i+\beta_i)$ $(B\neq-1)$.*

**Proof** It is clear from (1.4) that

$$\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)=\mathscr{L}(z)=\int_0^z\Big[\prod_{i=1}^n(f_i'(t))^{\alpha_i}\big(\frac{g_i(t)}{t}\big)^{\beta_i}\Big]\mathrm{d}t.\tag{3.14}$$

Furthermore, using (3.14) gives that

$$1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}=1+\frac{1}{b}\sum_{i=1}^n\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\frac{1}{b}\sum_{i=1}^n\beta_i\big(\frac{zg_i'(z)}{g_i(z)}-1\big)\tag{3.15}$$

and

$$\Re\big\{1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}\big\}=1+\frac{1}{b}\sum_{i=1}^n\alpha_i\Re\big\{\frac{zf_i''(z)}{f_i'(z)}\big\}+\frac{1}{b}\sum_{i=1}^n\beta_i\Re\big(\frac{zg_i'(z)}{g_i(z)}-1\big).\tag{3.16}$$

In view of $f_i(z)\in K[A,B]$ and $g_i(z)\in S^*[A,B]$, then from (3.16) and Lemma 2.1 we obtain

$$\int_0^{2\pi}\big|\Re\big\{1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}\big\}\big|\mathrm{d}\theta$$

$$\begin{aligned}
&\leqslant 2\pi+\frac{1}{b}\sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\big|\Re\{\frac{zf_i''(z)}{f_i'(z)}\}\big|\mathrm{d}\theta+\frac{1}{b}\sum_{i=1}^{n}\beta_i\int_0^{2\pi}\big|\Re(\frac{zg_i'(z)}{g_i(z)}-1)\big|\mathrm{d}\theta\\
&\leqslant 2\pi+\frac{1}{b}\sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\big|\Re\{\frac{zf_i''(z)}{f_i'(z)}+1-1\}\big|\mathrm{d}\theta+\frac{1}{b}\sum_{i=1}^{n}\beta_i\int_0^{2\pi}\big(\big|\Re(\frac{zg_i'(z)}{g_i(z)}\big|+1\big)\mathrm{d}\theta\\
&\leqslant 2\pi+\frac{1}{b}\sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\big(\big|\Re\{\frac{zf_i''(z)}{f_i'(z)}+1\}\big|+1)\mathrm{d}\theta+\frac{1}{b}\sum_{i=1}^{n}\beta_i\int_0^{2\pi}\big(\big|\Re(\frac{zg_i'(z)}{g_i(z)}\big|+1\big)\mathrm{d}\theta\\
&\leqslant 2\pi+\frac{1}{b}\sum_{i=1}^{n}\alpha_i\big(\frac{1+Ar}{1+Br}+1\big)\int_0^{2\pi}\mathrm{d}\theta+\frac{1}{b}\sum_{i=1}^{n}\beta_i\big(\frac{1+Ar}{1+Br}+1\big)\int_0^{2\pi}\mathrm{d}\theta\\
&\leqslant 2\pi+\frac{1}{b}\sum_{i=1}^{n}\alpha_i\big(\frac{1+A}{1+B}+1\big)\int_0^{2\pi}\mathrm{d}\theta+\frac{1}{b}\sum_{i=1}^{n}\beta_i\big(\frac{1+A}{1+B}+1\big)\int_0^{2\pi}\mathrm{d}\theta\\
&=2\pi+\frac{2\pi}{b}\frac{2+A+B}{1+B}\sum_{i=1}^{n}(\alpha_i+\beta_i), \qquad (3.17)
\end{aligned}$$

where $z=re^{i\theta}$ $(0\leqslant\theta\leqslant 2\pi)$. Hence, if $k=2+\frac{4+2A+2B}{b(1+B)}\sum_{i=1}^{n}(\alpha_i+\beta_i)$, then from (3.17), we have

$$\int_0^{2\pi}\big|\Re\{1+\frac{1}{b}\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}\}\big|\mathrm{d}\theta\leqslant k\pi,$$

which proves that $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)\in V_k(0,b)$. □

**Theorem 3.7** *Suppose $f_i(z)\in V_k(\rho_1,b)$ and $g_i(z)\in R_k(\rho_2,b)$ for all $i=1,2,\dots,n$, where $0\leqslant\rho_1<1$, $0\leqslant\rho_2<1$ and $b\in\mathbb{C}-\{0\}$. If $\alpha_i\geqslant 0$, $\beta_i\geqslant 0$ for all $i=1,2,\dots,n$ and*

$$0\leqslant(\rho_1-1)\sum_{i=1}^{n}\alpha_i+(\rho_2-1)\sum_{i=1}^{n}\beta_i+1<1, \qquad (3.18)$$

*then $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)\in V_k(\lambda,b)$ with*

$$\lambda=(\rho_1-1)\sum_{i=1}^{n}\alpha_i+(\rho_2-1)\sum_{i=1}^{n}\beta_i+1. \qquad (3.19)$$

**Proof** Using the definition of $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)$, we have

$$\mathscr{Z}(z)=\mathscr{H}_{\alpha_i,\beta_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)=\int_0^z\Big[\prod_{i=1}^{n}(f_i'(t))^{\alpha_i}\big(\frac{g_i(t)}{t}\big)^{\beta_i}\Big]\mathrm{d}t. \qquad (3.20)$$

Differentiating both sides of (3.20) logarithmically, we obtain

$$\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}=\sum_{i=1}^{n}\alpha_i\frac{zf_i''(z)}{f_i'(z)}+\sum_{i=1}^{n}\beta_i\big(\frac{zg_i'(z)}{g_i(z)}-1\big). \qquad (3.21)$$

By multiplying (3.21) with $\frac{1}{b}$, we easily find that

$$\begin{aligned}
\frac{1}{b}\frac{z\mathscr{Z}''(z)}{\mathscr{Z}'(z)}&=\sum_{i=1}^{n}\alpha_i\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}+\sum_{i=1}^{n}\frac{1}{b}\beta_i\big(\frac{zg_i'(z)}{g_i(z)}-1\big)\\
&=\sum_{i=1}^{n}\alpha_i\big[1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\big]+\sum_{i=1}^{n}\beta_i\big[1+\frac{1}{b}\Big(\frac{zg_i'(z)}{g_i(z)}-1\big)\Big]-\sum_{i=1}^{n}(\alpha_i+\beta_i), \qquad (3.22)
\end{aligned}$$

then (3.22) is equivalent to

$$1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}=\sum_{i=1}^{n}\alpha_i\big[1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}\big]+\sum_{i=1}^{n}\beta_i\big[1+\frac{1}{b}\big(\frac{zg_i'(z)}{g_i(z)}-1\big)\big]-\sum_{i=1}^{n}(\alpha_i+\beta_i)+1. \quad (3.23)$$

Subtracting and adding $\rho_1$ and $\rho_2$ on the right hand side of (3.23), we have

$$1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}-\lambda=\sum_{i=1}^{n}\alpha_i\big[1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}-\rho_1\big]+\sum_{i=1}^{n}\beta_i\big[(1+\frac{1}{b}\big(\frac{zg_i'(z)}{g_i(z)}-1\big)-\rho_2\big], \quad (3.24)$$

where $\lambda=(\rho_1-1)\sum_{i=1}^{n}\alpha_i+(\rho_2-1)\sum_{i=1}^{n}\beta_i+1$. Taking real part of (3.24) and then integrating from 0 to $2\pi$, we obtain

$$\begin{aligned}\int_0^{2\pi}\big|\Re\big[1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}\big]-\lambda\big|\mathrm{d}\theta\leqslant&\sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\big|\Re\big[1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}-\rho_1\big]\big|\mathrm{d}\theta+\\&\sum_{i=1}^{n}\beta_i\int_0^{2\pi}\big|\Re\big[(1+\frac{1}{b}\big(\frac{zg_i'(z)}{g_i(z)}-1\big)-\rho_2\big]\big|\mathrm{d}\theta. \quad (3.25)\end{aligned}$$

Since $f_i(z)\in V_k(\rho_1,b)$, $g_i(z)\in R_k(\rho_2,b)$ for all $i=1,2,\ldots,n$, we have

$$\int_0^{2\pi}\big|\Re\big[1+\frac{1}{b}\frac{zf_i''(z)}{f_i'(z)}-\rho_1\big]\big|\mathrm{d}\theta\leqslant(1-\rho_1)k\pi \quad (3.26)$$

and

$$\int_0^{2\pi}\big|\Re\big[(1+\frac{1}{b}\big(\frac{zg_i'(z)}{g_i(z)}-1\big)-\rho_2\big]\big|\mathrm{d}\theta\leqslant(1-\rho_2)k\pi. \quad (3.27)$$

Furthermore, applying (3.26) and (3.27) in (3.25), we obtain

$$\int_0^{2\pi}\big|\Re\big[1+\frac{1}{b}\frac{z\mathscr{L}''(z)}{\mathscr{L}'(z)}\big]-\lambda\big|\mathrm{d}\theta\leqslant[(1-\rho_1)\sum_{i=1}^{n}\alpha_i+(1-\rho_2)\sum_{i=1}^{n}\beta_i]k\pi=(1-\lambda)k\pi.$$

Hence $\mathscr{H}_{\alpha_i,\beta_i}(f_1,\ldots,f_n;g_1,\ldots,g_n)(z)\in V_k(\lambda,b)$ with $\lambda$ being given by (3.19). The proof of Theorem 3.7 is completed. □

**Remark 3.8** In fact, we can see that all the above theorems imply the corresponding results for kinds of special operators defined as Remark 1.1.

**Remark 3.9** By giving specific values to the parameters $A$ and $B$ ($-1\leqslant B<A\leqslant 1$) in Theorem 3.3 to Theorem 3.6, we can consider several interesting results with different subclasses of functions.

**Remark 3.10** Taking $\alpha_i=0$ ($i=1,\ldots,n$), $\rho_1=\rho$ and $\beta_i=0$ ($i=1,\ldots,n$), $\rho_2=\rho$ in Theorem 3.7, we obtain the results [10, Theorem 2.1] and [10, Theorem 2.5] proved by Noor et al., respectively.

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