Journal of Mathematical Research with Applications
Jul., 2016, Vol. 36, No. 4, pp. 441–449
DOI:10.3770/j.issn:2095-2651.2016.04.006
Http://jmre.dlut.edu.cn

Gröbner-Shirshov Basis for Degenerate Ringel-Hall Algebra of Type C_3

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Abstract The Gröbner-Shirshov basis of the degenerate Ringel-Hall Algebras of type C_3 is obtained by studying the generic extension monoid algebra.

Keywords Gröbner-Shirshov basis; Frobenius map; degenerate Ringel-Hall algebras; monoid algebra

MR(2010) Subject Classification 16S15; 17B37; 16G20

1. Introduction

The Gröbner-Shirshov basis theory was suggested by Shirshov [1] in 1962 for Lie algebras, by Buchberger [2] for commutative algebras and by Bergman [3] for associative algebras. It was proved to be a powerful tool for the solution of reduction problem in many algebraic structures. Ringel [4] established the concept of the generic Ringel-Hall algebra by the existence of Hall polynomials of a Dynkin quiver Q with automorphism σ. The case when the indeterminate specializes to zero is called the degenerate Ringel-Hall algebra. It was studied by Reineke in [5,6].

Fan and Zhao have given a presentation of the degenerate Ringel-Hall algebra of type B_n in [7]. It is easy to give a presentation of the degenerate Ringel-Hall algebra of type C_3. By studying the Frobenius morphism [8] and the generic extension monoid algebra [5], we obtain the Gröbner-Shirshov basis of the degenerate Ringel-Hall algebra of type C_3.

2. Some preliminaries

Let (Q,σ) be a quiver Q with an automorphism σ. $\Gamma=(\Gamma_0,\Gamma_1):=\Gamma(Q,\sigma)$ denotes the associated valued quiver, where Γ_0 and Γ_1 are the sets of σ-orbits in Q_0 and Q_1, respectively. For any $\rho: i\to j\in\Gamma_1 (i,j\in\Gamma_0)$, its tail and head are the σ-orbit of tails and heads of arrows in ρ, respectively. Denote

$$m_\rho=|\{\text{arrows in } \sigma\text{-orbit } \rho\}|,\ m_{ji}=m_\rho/\varepsilon_j \text{ and } m_{ij}=m_\rho/\varepsilon_i,$$

Received July 25, 2015; Accepted March 8, 2016
Supported by the National Natural Science Foundation of China (Grant Nos. 11471186; 11361056) and the Natural Science Foundation of Beijing (Grant No. 1162002).
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where $\varepsilon_t = |\{\text{vertices in } \sigma\text{-orbit } t\}|$ for $t \in \Gamma_0$. The valuation of Γ is given by $\{\varepsilon_t\}_{t\in\Gamma_0}, \{(m_{ji}, m_{ij})\}_{\rho\in\Gamma_1}$.

Example 2.1 Let $Q = D_4$ be the following quiver:

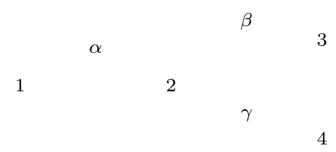

Figure 1 Quiver of type D_4

where σ is the automorphism of D_4 such that $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 4$, $\sigma(4) = 3$, $\sigma(\alpha) = \alpha$, $\sigma(\beta) = \gamma$, $\sigma(\gamma) = \beta$. Then the associated valued quiver $\Gamma = C_3$ with $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = 2$, $m_{21} = m_{12} = 1$, $m_{32} = 1$, $m_{23} = 2$ is as follows:

Figure 2 Quiver of type C_3

We now recall some concepts about Frobenius morphism, degenerate Ringel-Hall algebra, monoid algebra, and Gröbner-Shirshov basis theory.

Let $\mathbb{F}_q$ be the finite field of q elements and $\mathcal{K} = \overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$.

Definition 2.2 ([8,9]) *Let V be a vector-space over the field $\mathcal{K}$. An $\mathbb{F}_q$-linear isomorphism $F : V \longrightarrow V$ is called a Frobenius map if it satisfies:*

(i) $F(av) = a^q F(v)$ for all $v \in V$ and $a \in \mathcal{K}$;

(ii) For any $v \in V$, $F^n(v) = v$ for some $n > 0$.

Let $\mathcal{A}$ be a $\mathcal{K}$-algebra with the identity 1. A map $F_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is called a Frobenius morphism if $F_{\mathcal{A}}$ is a Frobenius map on the $\mathcal{K}$-space $\mathcal{A}$ and

$$F_{\mathcal{A}}(ab) = F_{\mathcal{A}}(a)F_{\mathcal{A}}(b), \quad \text{for all } \ a, b \in \mathcal{A}.$$

Let $A := \mathcal{K}Q$ be the path algebra of Q over $\mathcal{K}$. Then σ induces a Frobenius morphism $F = F_{Q,\sigma;q} : A \longrightarrow A$ given by $\sum_s x_s p_s \longmapsto \sum_s x_s^q \sigma(p_s)$, where $\sum_s x_s p_s$ is a $\mathcal{K}$-linear combination of paths p_s and $\sigma(p_s) = \sigma(\rho_t)\cdots\sigma(\rho_1)$ if $p_s = \rho_t \cdots \rho_1$ for arrows $\rho_1, \ldots, \rho_t \in Q_1$. Then the fixed-point algebra

$$A(q) := A^F = \{a \in A | F(a) = a\}.$$

Note that it is an algebra over the field $\mathbb{F}_q$.

Definition 2.3 ([8]) *A representation $M = (M_i, \phi_\rho)$ of Q is called an F-stable A-module if there is a Frobenius map $F_M : \bigoplus_{i\in Q_0} M_i \longrightarrow \bigoplus_{i\in Q_0} M_i$ satisfying $F_M(M_i) = M_{\sigma(i)}$ for all $i \in Q_0$ such that $F_M \circ \phi_\rho = \phi_{\sigma(\rho)} \circ F_M$ for each arrow $\rho \in Q_1$.*

An F-stable A-module is called *indecomposable* if it is nonzero and cannot be written as a direct sum of two nonzero F-stable A-modules.

Lemma 2.4 *There is a one-to-one correspondence between isomorphic classes (or isoclasses, for short) of indecomposable A^F-modules and isoclasses of indecomposable F-stable A-modules.*

We always assume that (Q,σ) is a Dynkin quiver Q with automorphism σ. It is well-known that there exists Hall polynomials of Q and $\Gamma = \Gamma(Q,\sigma)$ (see [10]). Let $\Phi^+ = \Phi^+(Q,\sigma)$ be the set of positive roots for the valued quiver $\Gamma = \Gamma(Q,\sigma)$. By [11,12], there is a bijection between the isoclasses of indecomposable $A(q)$-modules and Φ^+. Let $M_q(\alpha)$ be an indecomposable $A(q)$-module corresponding to $\alpha \in \Phi^+$. Any $A(q)$-module M can be decomposed as a direct sum of indecomposable $A(q)$-modules. That is

$$M_q(\lambda) := \bigoplus_{\alpha\in\Phi^+} \lambda(\alpha) M_q(\alpha)$$

for some function $\lambda : \Phi^+ \longrightarrow \mathbb{N}$. Thus, the isoclasses of $A(q)$-modules are indexed by the set

$$\mathcal{B} = \mathcal{B}(Q,\sigma) =: \{\lambda | \lambda : \Phi^+ \longrightarrow \mathbb{N}\} = \mathbb{N}^{\Phi^+},$$

which is independent of q. By Lemma 2.4, the set of the isoclasses of F-stable A-modules can also be identified with the set $\mathcal{B}$. The simple $A(q)$-module S_i corresponding to vertices $i \in \Gamma_0$ forms a complete set.

The generic Ringel-Hall algebra $\mathcal{H} = \mathcal{H}_{\mathfrak{q}}(\Gamma)$ (see [13]) is defined as follows. It is the free module over the polynomial ring $\mathbb{Z}[\mathfrak{q}]$ ($\mathfrak{q}$ is an indeterminate) with basis $\{u_\lambda | \lambda \in \mathcal{B}\}$ and its multiplication is

$$u_\mu u_\nu = \sum_{\lambda\in\mathcal{B}} \varphi^\lambda_{\mu,\nu}(\mathfrak{q}) u_\lambda,$$

where $\varphi^\lambda_{\mu,\nu}(\mathfrak{q}) \in \mathbb{Z}[\mathfrak{q}]$ is a Hall polynomial of Γ. It is noted that $\varphi^\lambda_{\mu,\nu}(q)$ is equal to the number of $A(q)$-submodules X of $A(q)$-module $M_q(\lambda)$ satisfying $X \cong M_q(\nu)$ and $M_q(\lambda)/X \cong M_q(\mu)$.

By specializing $\mathfrak{q}$ to 0, we obtain the degenerate Ringel-Hall $\mathbb{Z}$-algebra $\mathcal{H}_0(\Gamma)$ of $\Gamma = \Gamma(Q,\sigma)$. By [7], the set $\{u_\lambda | \lambda \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis of $\mathcal{H}_0(\Gamma)$. As a $\mathbb{Z}$-algebra, $\mathcal{H}_0(\Gamma)$ is generated by $u_i = u_{[S_i]}$, $i \in \Gamma_0$.

Let M and N be A-modules, and let $M * N$ denote the generic extension of M by N, which is unique, up to isomorphism, and whose endomorphism algebra has minimal dimension [14].

Proposition 2.5 *If M and N are two F-stable A-modules, so is $M * N$.*

By this proposition, we can define a monoid $\mathcal{M}_{Q,\sigma}$ by $[M] * [N] = [M * N]$ with the unit element [0], where $[M]$ is isoclass of F-stable A-module M. By [6,8], the monoid $\mathcal{M}_{Q,\sigma}$ of F-stable A-modules can be generated by $[S_i], i \in \Gamma_0$. For each $\lambda \in \mathcal{B}$, let $M_q(\lambda)_\mathcal{K} := M_q(\lambda) \otimes_{\mathbb{F}_q} \mathcal{K}$ be the F-stable A-module corresponding to λ, $\{[M_q(\lambda)_\mathcal{K}] | \lambda \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\mathcal{M}_{Q,\sigma}$.

Let Y be a well ordered set, Y^* the free monoid on Y, and $\mathcal{K}\langle Y\rangle$ the free associative algebra generated by Y over $\mathcal{K}$. Giving an ordering "$\prec$" on Y^* by the length-lexicographic order. For any nonzero polynomial $f \in \mathcal{K}\langle Y\rangle$ with the leading term $\overline{f}$, we denote the length of f by $l(f)$, f is called monic if the coefficient of $\overline{f}$ equals to 1.

In [15], let $f,g\in\mathcal{K}\langle Y\rangle$ be two monic polynomials and $\omega\in Y^*$. If $\omega=\overline{f}y_1=y_2\overline{g}$ for some $y_1,y_2\in Y^*$ such that $l(\overline{f})>l(y_2)$, then $(f,g)_\omega=fy_1-y_2g$ is called the intersection composition of f,g. If $\omega=\overline{f}=y_1\overline{g}y_2$ for some $y_1,y_2\in Y^*$, then $(f,g)_\omega=f-y_1gy_2$ is called the inclusion composition of f,g.

Let $G\subset\mathcal{K}\langle Y\rangle$ be the set of monic polynomials. A composition $(f,g)_\omega$ is said to be trivial with respect to G if

$$(f,g)_\omega=\sum k_iy_ig_iy_i',$$

where $k_i\in\mathcal{K},g_i\in G,y_i,y_i'\in Y^*$ and $\overline{y_ig_iy_i'}<\omega$.

G is called a Gröbner-Shirshov basis if any composition of polynomials from G is trivial with respect to G.

3. Presentation of degenerate Ringel-Hall algebra $\mathcal{H}_0(C_3)$

We fix $Q=(Q,\sigma)$ and $\Gamma=\Gamma(Q,\sigma)$ as in Example 2.1. Set

$$\begin{aligned}
X_1&=u_1u_3-u_3u_1,\\
X_2&=u_1^2u_2-(\mathfrak{q}+1)u_1u_2u_1+\mathfrak{q}u_2u_1^2,\\
X_3&=u_1u_2^2-(\mathfrak{q}+1)u_2u_1u_2+\mathfrak{q}u_2^2u_1,\\
X_4&=u_2^3u_3-(1+\mathfrak{q}+\mathfrak{q}^2)u_2^2u_3u_2+\mathfrak{q}(1+\mathfrak{q}+\mathfrak{q}^2)u_2u_3u_2^2-\mathfrak{q}^3u_3u_2^3,\\
X_5&=u_2u_3^2-(1+\mathfrak{q}^2)u_3u_2u_3+\mathfrak{q}^2u_3^2u_2,\\
X_6&=u_2^2u_3u_2u_3-(1+\mathfrak{q}+\mathfrak{q}^2)u_2u_3u_2^2u_3+\mathfrak{q}^2u_3u_2^3u_3+\mathfrak{q}u_2^2u_3^2u_2.
\end{aligned}$$

By [4], Ringel-Hall algebra $\mathcal{H}_\mathfrak{q}(C_3)$ is generated by u_1,u_1,u_3 satisfying the relations $X_i=0$ $(i=1,2,\ldots,6)$.

Set $\mathfrak{q}=0$, we get the degenerate Ringel-Hall algebra $\mathcal{H}_0(C_3)$ generated by u_1,u_1,u_3 with the following defining relations:

$$\begin{aligned}
&\text{(F1)}\quad u_1u_3=u_3u_1, &&\text{(F2)}\quad u_1^2u_2=u_1u_2u_1,\\
&\text{(F3)}\quad u_1u_2^2=u_2u_1u_2, &&\text{(F4)}\quad u_3u_2u_3=u_2u_3^2,\\
&\text{(F5)}\quad u_2^2u_3u_2=u_2^3u_3, &&\text{(F6)}\quad u_2u_3u_2^2u_3=u_2^3u_3^2.
\end{aligned}$$

Remark 3.1 The relations $X_i=0$ $(i=1,2,\ldots,5)$ are the basic relations in $\mathcal{H}_\mathfrak{q}(C_3)$ and $-\mathfrak{q}X_6=X_4u_3-u_2^2X_5$. Therefore, $X_6=0$ is automatically true. Moreover, (F4) and (F6) are equivlent to (F4) and $u_2u_3u_2^2u_3=u_2^2u_3u_2u_3$.

Consider the corresponding monoid algebra $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$. By [6], the following relations hold in $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$:

$$\begin{aligned}
&(\mathcal{F}1)\ [S_1]*[S_3]=[S_3]*[S_1]\\
&(\mathcal{F}2)\ [S_1]^{*2}*[S_2]=[S_1]*[S_2]*[S_1],\\
&(\mathcal{F}3)\ [S_1]*[S_2]^{*2}=[S_2]*[S_1]*[S_2],
\end{aligned}$$

($\mathcal{F}4$) $[S_3] * [S_2] * [S_3] = [S_2] * [S_3]^{*2}$,

($\mathcal{F}5$) $[S_2]^{*2} * [S_3] * [S_2] = [S_2]^{*3} * [S_3]$,

($\mathcal{F}6$) $[S_2] * [S_3] * [S_2]^{*2} * [S_3] = [S_2]^{*3} * [S_3]^{*2}$.

Proposition 3.2 *The monoid algebra $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$ has a presentation with generators $[S_i]$ $(1 \le i \le 3)$ and relations $(\mathcal{F}1) - (\mathcal{F}6)$.*

Proof For convenience, set $\mathbb{Z}\mathcal{M} = \mathbb{Z}\mathcal{M}_{D_4,\sigma}$. Let $\mathcal{S}$ be the free $\mathbb{Z}$-algebra with generators s_i $(1 \le i \le 3)$. Consider the ideal $\mathfrak{J}$ generated by the following elements,

$$(F'1) \quad s_1s_3 - s_3s_1, \qquad (F'2) \quad s_1^2s_2 - s_1s_2s_1,$$

$$(F'3) \quad s_1s_2^2 - s_2s_1s_2, \quad (F'4) \quad s_3s_2s_3 - s_2s_3^2,$$

$$(F'5) \quad s_2^2s_3s_2 - s_2^3s_3, \quad (F'6) \quad s_2s_3s_2^2s_3 - s_2^3s_3^2.$$

Then, a surjective monoid algebra homomorphisms $\eta : \mathcal{S} \longrightarrow \mathbb{Z}\mathcal{M}$ given by $s_i \longmapsto [S_i]$ with $1 \le i \le 3$ induces a surjective algebra homomorphism $\bar{\eta} : \mathcal{S}/\mathfrak{J} \longrightarrow \mathbb{Z}\mathcal{M}$ given by $s_i + \mathfrak{J} \longmapsto [S_i]$ $(1 \le i \le 3)$. To complete the proof, it suffices to show that $\bar{\eta}$ is injective.

Set $f_i = s_i + \mathfrak{J}$ $(1 \le i \le 3)$. Given a $\mathcal{K}C_3$-module M with dimension vector $\dim M := (a, b, c)$, we define a monomial in $\mathcal{S}/\mathfrak{J}$ as $\mathfrak{n}(M) = f_1^a f_2^b f_3^c$.

The Auslander-Reiten quiver for $\mathcal{K}D_4$ is as follows:

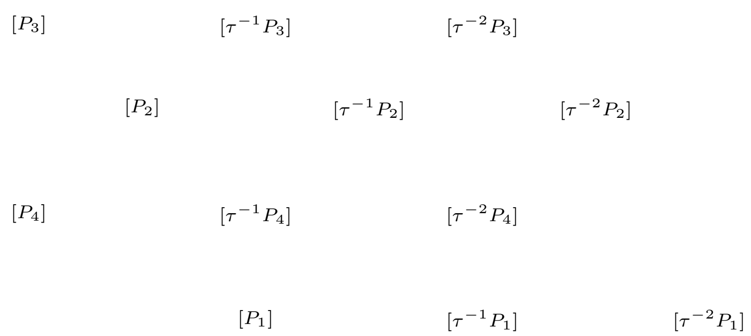

Figure 3 The AR-quiver of the path algebra $\mathcal{K}D_4$

where each P_i $(1 \le i \le 4)$ is the indecomposable projective $\mathcal{K}D_4$-module corresponding to vertex i and τ is the Auslander-Reiten translation.

Using the Frobenius morphism $F = F_{D_4,\sigma}$ introduced in Section 2, it is easy to see that P_1, P_2 are F-stable and all other P_i have F-period 2 with $P_3^{[1]} = P_4$. By folding the Auslander-Reiten quiver of $\mathcal{K}D_4$, we obtain the Auslander-Reiten quiver of $A^F = (\mathcal{K}D_4)^F \cong \mathcal{K}C_3$:

$[X_2]$ $[X_5]$ $[X_8]$

$[X_1]$ $[X_4]$ $[X_7]$

$[X_3]$ $[X_6]$ $[X_9]$

Figure 4 The AR-quiver of the path algebra $\mathcal{K}C_3$

where X_i $(1 \leq i \leq 9)$ denote all the indecomposable $\mathcal{K}C_3$-modules. Here $X_3 = P_1^F, X_2 = P_2^F, X_1 = (P_3 \oplus P_4)^F$, and $\tau = \tau_{A^F}$ is the Auslander-Reiten translation of A^F (see [9] for details). Moreover, the dimension vectors of X_i $(1 \leq i \leq 9)$ and associated monomials in $\mathcal{S}/\mathfrak{J}$ are given by

$$\begin{aligned}
&\dim X_1 = (0,0,1) \text{ and } \mathfrak{n}(X_1) = f_3,\\
&\dim X_2 = (0,1,1) \text{ and } \mathfrak{n}(X_2) = f_2 f_3,\\
&\dim X_3 = (1,1,1) \text{ and } \mathfrak{n}(X_3) = f_1 f_2 f_3,\\
&\dim X_4 = (0,2,1) \text{ and } \mathfrak{n}(X_4) = f_2^2 f_3,\\
&\dim X_5 = (1,2,1) \text{ and } \mathfrak{n}(X_5) = f_1 f_2^2 f_3,\\
&\dim X_6 = (0,1,0) \text{ and } \mathfrak{n}(X_6) = f_2,\\
&\dim X_7 = (2,2,1) \text{ and } \mathfrak{n}(X_7) = f_1^2 f_2^2 f_3,\\
&\dim X_8 = (1,1,0) \text{ and } \mathfrak{n}(X_8) = f_1 f_2,\\
&\dim X_9 = (1,0,0) \text{ and } \mathfrak{n}(X_9) = f_1.
\end{aligned}$$

Now we give an enumeration of indecomposable A^F-modules in Figure 4:

$$X_1 \prec X_2 \prec X_3 \prec X_4 \prec X_5 \prec X_6 \prec X_7 \prec X_8 \prec X_9. \qquad (*)$$

Then, by using the relations $(F'1) - (F'6)$, we compute the relations between $\mathfrak{n}(X_i)$ $(1 \leq i \leq 9)$ in $\mathcal{S}/\mathfrak{J}$:

$$\begin{aligned}
\mathfrak{n}(X_3)\mathfrak{n}(X_1) &= f_1 f_2 f_3 \cdot f_3 = f_1 f_3 f_2 f_3 \text{ (by } (F'4))\\
&= f_3 f_1 f_2 f_3 \text{ (by } (F'1))\\
&= \mathfrak{n}(X_1)\mathfrak{n}(X_3),\\
\mathfrak{n}(X_7)\mathfrak{n}(X_2) &= f_1^2 f_2^2 f_3 \cdot f_2 f_3 = f_1^2 f_2^3 f_3^2 \text{ (by } (F'5))\\
&= f_1^2 f_2 f_3 f_2^2 f_3 \text{ (by } (F'6))\\
&= f_1 f_2 f_1 f_3 f_2^2 f_3 \text{ (by } (F'2))\\
&= f_1 f_2 f_3 f_1 f_2^2 f_3 \text{ (by } (F'1))\\
&= \mathfrak{n}(X_3)\mathfrak{n}(X_5),\\
\mathfrak{n}(X_8)\mathfrak{n}(X_4) &= f_1 f_2 \cdot f_2^2 f_3 = f_1 f_2^2 f_3 f_2 \text{ (by } (F'5))\\
&= \mathfrak{n}(X_5)\mathfrak{n}(X_6).
\end{aligned}$$

By this way, we get the set R of following equalities in $\mathcal{S}/\mathfrak{J}$:

$$
\begin{aligned}
&\mathfrak{n}(X_2)\mathfrak{n}(X_1)=\mathfrak{n}(X_1)\mathfrak{n}(X_2), && \mathfrak{n}(X_3)\mathfrak{n}(X_1)=\mathfrak{n}(X_1)\mathfrak{n}(X_3),\\
&\mathfrak{n}(X_4)\mathfrak{n}(X_1)=\mathfrak{n}(X_2)\mathfrak{n}(X_2), && \mathfrak{n}(X_5)\mathfrak{n}(X_1)=\mathfrak{n}(X_2)\mathfrak{n}(X_3),\\
&\mathfrak{n}(X_6)\mathfrak{n}(X_1)=\mathfrak{n}(X_2), && \mathfrak{n}(X_7)\mathfrak{n}(X_1)=\mathfrak{n}(X_3)\mathfrak{n}(X_3),\\
&\mathfrak{n}(X_8)\mathfrak{n}(X_1)=\mathfrak{n}(X_3), && \mathfrak{n}(X_9)\mathfrak{n}(X_1)=\mathfrak{n}(X_1)\mathfrak{n}(X_9),\\
&\mathfrak{n}(X_3)\mathfrak{n}(X_2)=\mathfrak{n}(X_2)\mathfrak{n}(X_3), && \mathfrak{n}(X_4)\mathfrak{n}(X_2)=\mathfrak{n}(X_2)\mathfrak{n}(X_4),\\
&\mathfrak{n}(X_5)\mathfrak{n}(X_2)=\mathfrak{n}(X_3)\mathfrak{n}(X_4), && \mathfrak{n}(X_6)\mathfrak{n}(X_2)=\mathfrak{n}(X_4),\\
&\mathfrak{n}(X_7)\mathfrak{n}(X_2)=\mathfrak{n}(X_3)\mathfrak{n}(X_5), && \mathfrak{n}(X_8)\mathfrak{n}(X_2)=\mathfrak{n}(X_5),\\
&\mathfrak{n}(X_9)\mathfrak{n}(X_2)=\mathfrak{n}(X_3), && \mathfrak{n}(X_4)\mathfrak{n}(X_3)=\mathfrak{n}(X_3)\mathfrak{n}(X_4),\\
&\mathfrak{n}(X_5)\mathfrak{n}(X_3)=\mathfrak{n}(X_3)\mathfrak{n}(X_5), && \mathfrak{n}(X_6)\mathfrak{n}(X_3)=\mathfrak{n}(X_5),\\
&\mathfrak{n}(X_7)\mathfrak{n}(X_3)=\mathfrak{n}(X_3)\mathfrak{n}(X_7), && \mathfrak{n}(X_8)\mathfrak{n}(X_3)=\mathfrak{n}(X_7),\\
&\mathfrak{n}(X_9)\mathfrak{n}(X_3)=\mathfrak{n}(X_3)\mathfrak{n}(X_9), && \mathfrak{n}(X_5)\mathfrak{n}(X_4)=\mathfrak{n}(X_4)\mathfrak{n}(X_5),\\
&\mathfrak{n}(X_6)\mathfrak{n}(X_4)=\mathfrak{n}(X_4)\mathfrak{n}(X_6), && \mathfrak{n}(X_7)\mathfrak{n}(X_4)=\mathfrak{n}(X_5)\mathfrak{n}(X_5),\\
&\mathfrak{n}(X_8)\mathfrak{n}(X_4)=\mathfrak{n}(X_5)\mathfrak{n}(X_6), && \mathfrak{n}(X_9)\mathfrak{n}(X_4)=\mathfrak{n}(X_5),\\
&\mathfrak{n}(X_6)\mathfrak{n}(X_5)=\mathfrak{n}(X_5)\mathfrak{n}(X_6), && \mathfrak{n}(X_7)\mathfrak{n}(X_5)=\mathfrak{n}(X_5)\mathfrak{n}(X_7),\\
&\mathfrak{n}(X_8)\mathfrak{n}(X_5)=\mathfrak{n}(X_6)\mathfrak{n}(X_7), && \mathfrak{n}(X_9)\mathfrak{n}(X_5)=\mathfrak{n}(X_7),\\
&\mathfrak{n}(X_7)\mathfrak{n}(X_6)=\mathfrak{n}(X_6)\mathfrak{n}(X_7), && \mathfrak{n}(X_8)\mathfrak{n}(X_6)=\mathfrak{n}(X_6)\mathfrak{n}(X_8),\\
&\mathfrak{n}(X_9)\mathfrak{n}(X_6)=\mathfrak{n}(X_8), && \mathfrak{n}(X_8)\mathfrak{n}(X_7)=\mathfrak{n}(X_7)\mathfrak{n}(X_8),\\
&\mathfrak{n}(X_9)\mathfrak{n}(X_7)=\mathfrak{n}(X_7)\mathfrak{n}(X_9), && \mathfrak{n}(X_9)\mathfrak{n}(X_8)=\mathfrak{n}(X_8)\mathfrak{n}(X_9).
\end{aligned}
$$

Let $V_1,\ldots,V_9$ be all the non-isomorphic indecomposable A^F-modules. We assume that they are enumerated by $V_1\prec\cdots\prec V_9$ as given in $(*)$. Repeatedly applying above equalities, we get the following result:

For $1\le i<j\le 9$, there exist $1\le j_1\le j_2\le\cdots\le j_m\le 9$ such that

$$\mathfrak{n}(V_j)\mathfrak{n}(V_i)=\mathfrak{n}(V_{j_1})\mathfrak{n}(V_{j_2})\cdots\mathfrak{n}(V_{j_m}).$$

Now we are ready to prove the injectivity of

$$\bar{\eta}:\mathcal{S}/\mathfrak{J}\longrightarrow\mathbb{Z}\mathcal{M},\ s_i+\mathfrak{J}\longmapsto[S_i],\quad 1\le i\le 3.$$

Given a monomial $\omega=f_{i_1}\cdots f_{i_m}$ $(1\le i_1\le\cdots\le i_m\le 3)$, we have $\omega=f_{i_1}\cdots f_{i_m}=\mathfrak{n}(S_{i_1})\cdots\mathfrak{n}(S_{i_m})$. Applying above result repeatedly, we finally get $\omega=\mathfrak{n}(V_1)^{n_1}\cdots\mathfrak{n}(V_\mu)^{n_9}$ for some $n_1,\ldots,n_9\ge 0$. Hence, all the monomials $\mathfrak{n}(V_1)^{n_1}\cdots\mathfrak{n}(V_\mu)^{n_9}$ with $n_1,\ldots,n_9\ge 0$ span $\mathcal{S}/\mathfrak{J}$.

On the other hand, by ([6, Lemma 4.9]) $(n_1,\ldots,n_9\ge 0)$,

$$\bar{\eta}(\mathfrak{n}(V_1)^{n_1}\cdots\mathfrak{n}(V_9)^{n_9})=[V_1]^{*n_1}*\cdots*[V_9]^{*n_9}.$$

By ([5, Proposition 3.3]), the elements $[V_1]^{*n_1}*\cdots*[V_9]^{*n_9}$ with $n_1,\ldots,n_9\ge 0$ form a basis of $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$. Consequently, the morphism $\bar{\eta}$ is injective. $\square$

Proposition 3.3 *There are $\mathbb{Z}$-algebra isomorphism*

$$\Psi : \mathbb{Z}\mathcal{M}_{D_4,\sigma} \longrightarrow \mathcal{H}_0(C_3),\ [S_i] \longmapsto u_i,\ \ 1 \le i \le 3.$$

Proof By Proposition 3.2, there is a surjective $\mathbb{Z}$-algebra homomorphism $\Psi : \mathbb{Z}\mathcal{M}_{D_4,\sigma} \longrightarrow \mathcal{H}_0(C_3)$ given by $[S_i] \longmapsto u_i$ with $1 \le i \le 3$. Since $\{[M_q(\lambda)_{\mathcal{K}}] | \lambda \in \mathcal{B}\}$ and $\{u_\lambda | \lambda \in \mathcal{B}\}$ are bases for $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$ and $\mathcal{H}_0(C_3)$, respectively. Moreover, the algebra $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$ and $\mathcal{H}_0(C_3)$ have the same defining relations. So, Ψ is an isomorphism. $\Box$

4. Gröbner-Shirshov basis of $\mathcal{H}_0(C_3)$

Though the set R which is the set of equalities in $\mathcal{S}/\mathfrak{J}$ from the proof of the presentation of the monoid algebra $\mathbb{Z}\mathcal{M}_{D_4,\sigma}$ and Proposition 3.3, we give Gröbner-Shirshov basis of $\mathcal{H}_0(C_3)$.

First, we define a degree lexicographic order $\prec$ as follows:

$$u \prec v \text{ if and only if } l(u) < l(v) \text{ or } l(u) = l(v) \text{ and } u < v,$$

then it is a monomial order [16].

We have already shown that $\mathcal{H}_0(C_3)$ is an associative algebra over $\mathbb{Z}$ generated by $C = \{u_1, u_2, u_3\}$ with generating relations

$$\mathcal{F}' = \begin{cases} u_1u_3 = u_3u_1, & u_1^2u_2 = u_1u_2u_1, \\ u_1u_2^2 = u_2u_1u_2, & u_3u_2u_3 = u_2u_3^2, \\ u_2^2u_3u_2 = u_2^3u_3, & u_2u_3u_2^2u_3 = u_2^3u_3^2. \end{cases}$$

By Propositions 3.2 and 3.3, if we apply the algebra isomorphism $\Psi \circ \overline{\eta}$ to the relations in the set R, then we get a new set $\mathcal{F}''$ of relations in $\mathcal{H}_0(C_3)$ $(u_1 \succ u_2 \succ u_3)$:

$$\begin{aligned}
&u_1u_3 = u_3u_1, && u_1^2u_2 = u_1u_2u_1, \\
&u_1u_2^2 = u_2u_1u_2, && u_1u_2u_3^2 = u_3u_1u_2u_3, \\
&u_1^2u_2u_3 = u_1u_2u_3u_1, && u_1u_2^3u_3 = u_1u_2^2u_3u_2, \\
&u_1u_2^2u_3^2 = u_2u_3u_1u_2u_3, && u_1^2u_2^2u_3^2 = u_1u_2u_3u_1u_2u_3, \\
&u_1u_2^2u_3 = u_2u_1u_2u_3, && u_1^2u_2^2u_3 = u_1u_2u_1u_2u_3, \\
&u_1u_2^2u_3u_2 = u_2u_1u_2^2u_3, && u_1u_2u_1u_2^2u_3 = u_2u_1^2u_2^2u_3, \\
&u_1u_2u_3u_2u_3 = u_2u_3u_1u_2u_3, && u_1u_2u_3u_2^2u_3 = u_2^2u_3u_1u_2u_3, \\
&u_1u_2^2u_3u_2u_3 = u_1u_2u_3u_2^2u_3, && u_1u_2^2u_3u_1u_2u_3 = u_1u_2u_3u_1u_2^2u_3, \\
&u_1u_2^2u_3u_2^2u_3 = u_2^2u_3u_1u_2^2u_3, && u_1^2u_2^2u_3u_2 = u_2u_1^2u_2^2u_3, \\
&u_1^2u_2^2u_3u_1u_2 = u_1u_2u_1^2u_2^2u_3, && u_1^2u_2^2u_3u_2u_3 = u_1u_2u_3u_1u_2^2u_3, \\
&u_1^2u_2^2u_3u_1u_2u_3 = u_1u_2u_3u_1^2u_2^2u_3, && u_1^2u_2^2u_3u_2^2u_3 = u_1u_2^2u_3u_1u_2^2u_3, \\
&u_1^2u_2^2u_3u_1u_2^2u_3 = u_1u_2^2u_3u_1^2u_2^2u_3, && u_1^3u_2^2u_3 = u_1^2u_2^2u_3u_1, \\
&u_2u_3^2 = u_3u_2u_3, && u_2^3u_3 = u_2^2u_3u_2, \\
&u_2^2u_3^2 = u_2u_3u_2u_3, && u_2^2u_3u_2u_3 = u_2u_3u_2^2u_3,
\end{aligned}$$

By a routine check of compositions between the elements of $\mathcal{F}' \cup \mathcal{F}''$, we get following new set $\mathcal{F}'''$ of relations in $\mathcal{H}_0(C_3)$:

$$u_1u_2u_1u_2u_3u_2 = u_2u_1u_2u_1u_2u_3, \qquad u_1u_2u_1u_2u_3u_1u_2u_3 = u_1u_2u_3u_1u_2u_1u_2u_3,$$
$$u_1u_2u_1u_2u_1u_2u_3 = u_1u_2u_1u_2u_3u_1u_2, \quad u_2u_1u_2u_3u_2 = u_2u_1u_2^2u_3,$$
$$u_1u_2u_3u_2u_1u_2u_3 = u_2u_1u_2u_3u_1u_2u_3, \quad u_1u_2^2u_3u_2u_1u_2^2u_3 = u_2u_1u_2^2u_3u_1u_2^2u_3.$$

Let $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}'''$. Then by the construction of the set $\mathcal{F}$ of relations in $\mathcal{H}_0(C_3)$, we get our main result:

Theorem 4.1 *With notations above, $\mathcal{F}$ is a Gröbner-Shirshov basis of $\mathcal{H}_0(C_3)$.*

Acknowledgements We are very grateful to the referees for the useful comments and suggestions.

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