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Spanning Trees with Few Leaves in Almost Claw-Free Graphs

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Abstract A spanning tree with no more than 3 leaves is called a spanning 3-ended tree. In this paper, we prove that if G is a k-connected ($k \geq 2$) almost claw-free graph of order n and $\sigma_{k+3}(G) \geq n + k + 2$, then G contains a spanning 3-ended tree, where $\sigma_k(G) = \min\{\sum_{v \in S} \deg(v) : S$ is an independent set of G with $|S| = k\}$.

Keywords spanning 3-ended tree; almost claw-free graph; insertible vertex; non-insertible vertex

MR(2010) Subject Classification 68R10

1. Introduction

We consider only finite and simple graphs in this paper. For notation and terminology not defined here we refer to [1]. A subset B of $V(G)$ is a dominating set if every vertex of G is in B or adjacent to the vertices in B. The domination number of a graph G denoted by $\gamma(G)$ is the minimum cardinality of all the dominating sets of G. Let $\alpha(G)$ denote the independent number of a graph G. A graph G is claw-free if G contains no $K_{1,3}$ induced subgraph. A graph G is almost claw-free if there exists an independent set A in $V(G)$ such that $\alpha(N(v)) \leq 2$ for any vertex $v \notin A$, and $\alpha(N(v)) \leq 2 < \gamma(N(v))$ for every $v \in A$. Let $N_H(S)$ denote the set of all vertices in R adjacent to some vertex of S and $d_H(S) = |N_H(S)|$. For a subgraph R of a graph G, $G - H$ denotes the induced subgraph by $V(G) - V(H)$. For a vertex v of G, the neighborhood of v is the induced subgraph on the set of all vertices that are adjacent to v, and for convenience, we use $N(v)$ to denote both the induced subgraph and the set of vertices adjacent to v in G. Let $N[v] = N(v) \cup \{v\}$. We define $\sigma_k(G) = \min\{\sum_{v \in S} \deg(v) : S$ is an independent set of G with $|S| = k\}$. $P[a, b]$ (or aPb) denotes a path along positive orientation with end vertices a, b. For a path $P[a, b]$, $x, y \in V(P)$, let xPy denote the subpath from x to y along the positive orientation, and $yP^{-}x$ denote the subpath from y to x along the negative orientation. A graph G is hamiltonian-connected, if there exists a hamiltonian path with end vertices a, b for every pair of distinct vertices $a, b \in V(G)$.

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There are a lot of sufficient conditions on the degree sum of vertices in an independent vertex set of a graph to contain spanning k-ended trees.

Theorem 1.1 ([2]) *Let $k \geq 2$ and G be a connected graph of order $n \geq 2$. If $\sigma_2(G) \geq n-k+1$, then G contains a spanning k-ended tree.*

Kyaw [3,4] gave some degree sum conditions for $K_{1,4}$-free graphs to contain a spanning k-ended tree.

Theorem 1.2 ([3]) *Every connected $K_{1,4}$-free graph with $\sigma_4(G) \geq |G|-1$ contains a spanning 3-ended tree.*

Theorem 1.3 ([4]) *Let G be a connected $K_{1,4}$-free graph. Then*

(i) If $\sigma_3(G) \geq |G|$, then G contains a hamiltonian path.

(ii) If $\sigma_{k+1}(G) \geq |G| - \frac{k}{2}$ for an integer $k \geq 3$, then G contains a spanning k-ended tree.

On the other hand, Kano et al. [5] obtained sharp sufficient conditions for claw-free graphs to contain a spanning k-ended tree.

Theorem 1.4 ([5]) *Let $k \geq 2$ and G be a connected claw-free graph of order n. If $\sigma_{k+1}(G) \geq n-k$, then G contains a spanning k-ended tree with the maximum degree at most 3.*

Recently, Chen et al. [6] gave some degree sum conditions for k-connected $K_{1,4}$-free graphs to contain a spanning 3-ended tree.

Theorem 1.5 ([6]) *Let G be a k-connected $K_{1,4}$-free graph of order n with $k \geq 2$. If $\sigma_{k+3}(G) \geq n+2k-2$, then G contains a spanning 3-ended tree.*

Chen et al. [7] proposed if G is a k-connected almost claw-free graph of order n with $k \geq 2$, and $\sigma_{k+3}(G) \geq n+2k-2$, then G contains a spanning 3-ended tree. In this paper, we decrease the bound to improve the above result.

Theorem 1.6 *If G is a k-connected almost claw-free graph of order n with $k \geq 2$, and $\sigma_{k+3}(G) \geq n+k+2$, then G contains a spanning 3-ended tree.*

Obviously, there are a lot of almost claw-free graphs which contain $K_{1,4}$ subgraphs, so in some extent Theorem 1.6 is a generalization of Theorem 1.5.

2. Preliminaries

The properties of insertible vertices [8] and the following results are needed in the proof of Theorem 1.6.

Lemma 2.1 ([9]) *If v is a claw center of an almost claw-free graph, then $\gamma(N(v)) = 2$.*

Assume that G is a connected non-hamiltonian graph and C is a longest cycle in G with counter-clockwise direction as positive orientation. Suppose that R is a component of $G-C$ and $N_C(R) = \{u_1, u_2, \ldots, u_m\}$ such that $u_1, u_2, \ldots, u_m$ are labeled in order along the positive direction of C. Let $S_j = C(u_j, u_{j+1}]$, $1 \leq j \leq m-1$, and $S_m = C(u_m, u_1]$. A vertex u in S_j is an insertible vertex if u has two consecutive neighbors v and v^+ in $C - S_j$.

Lemma 2.2 ([8]) *For each S_j, $S_j - \{u_{j+1}\}$ contains a non-insertible vertex.*

Let v_j denote the first non-insertible vertex in $S_j - \{u_{j+1}\}$ for each $j \in [1, m]$.

Lemma 2.3 ([8]) *Let $x_i \in C[u_i, v_i], x_j \in C[u_j, v_j]$ with $1 \le i < j \le m$. Then*

(a) There is no path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap V(C) = \{x_i, x_j\}$.

(b) For any vertex u in $C[x_i^+, x_j^-]$, if $ux_i \in E(G)$, then $u^-x_j \notin E(G)$. By symmetry, for any vertex u in $C[x_j^+, x_i^-]$, if $ux_j \in E(G)$, then $u^-x_i \notin E(G)$.

(c) For any vertex u in $C[x_i, x_j]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$. By symmetry, for any vertex u in $C[x_j, x_i]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$.

Suppose for some $i \in [1, m]$, $N(v_i) \cap V(G - C - R) \neq \emptyset$ and v_i' is the second non-insertible vertex in $S_i - \{u_{i+1}\}$. Then Chen, Chen and Hu [6] gave the following result.

Lemma 2.4 ([5]) *Let $1 \le i < j \le m$, $x_i \in C[v_i^+, v_i']$ and $x_j \in C[u_j^+, v_j]$. Then*

(a) There does not exist a path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap V(C) = \{x_i, x_j\}$.

(b) For every vertex $u \in C[x_i^+, x_j^-]$, if $ux_i \in E(G)$, then $u^-x_j \notin E(G)$; Similarly, for every $u \in C[x_j^+, x_i^-]$, if $ux_j \in E(G)$, then $u^-x_i \notin E(G)$.

(c) For every vertex $u \in C[x_i, x_j]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$; By symmetry, for any vertex u in $C[x_j, x_i]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$.

3. Proof of Theorem 1.6

Suppose, to the contrary, G satisfies the conditions of Theorem 1.6 and contains no spanning 3-ended tree in G. Let $P = P[x, y]$ be a longest path in G such that P satisfies the following two conditions:

(T1) $w(G - P)$ is minimum;

(T2) $|P[x, u_1]|$ is minimum such that u_1 is the first vertex in P with $N(u_1) \cap V(G-P) \neq \emptyset$, subject to (T1).

Suppose R is a component in $G - P$, and $\{u_1, \dots, u_m\} = N_P(R)$ with $u_1, \dots, u_m$ in order along the positive direction of P. Let R_I denote an independent set in R.

Let G' denote a graph with $V(G') = V(G) \cup \{u_0\}, E(G') = E(G) \cup \{u_0u : u \in V(G)\}$. Then $C = u_0P[x, y]u_0$ is a maximum cycle in G'. We define the counter-clockwise orientation as the positive direction of C. Let S_i denote the segment $C(u_i, u_{i+1}]$ for $0 \le i \le m-1$, and $S_m = C(u_m, u_0]$. By Lemma 2.2, let v_i denote the first non-insertible vertex in S_i for $i \in [0, m]$, and $U = \{v_0, v_1, \dots, v_m\}$. By Lemma 2.3(a), U is an independent set.

C can be divided into disjoint intervals $S = C[a, b]$ with $a, b^+ \notin N(U)$ and $C[a^+, b] \subseteq N(U)$. We call the intervals U-segments. If $a = b$, then $C[a^+, b] = \emptyset$, i.e., if $|S| = 1$, then $d_U(S) = 0$. By the definition of U-segment, for any U-segment S, there exists $l \in [0, m]$ such that $S \subseteq C[v_l, v_{l+1}^-]$ (subscripts expressed modulo $m+1$).

Though Claims 1–5 in the following proof have been proved in [7], we give them a proof here for the sake of completeness.

Claim 1 $x = v_0$ and $y \notin N(v_i)$ for any $i \in [0, m-1]$.

Proof Suppose x is an insertible vertex such that $xu, xu^+ \in E(G)$, where $u, u^+ \in C - S_0$. If $u \neq y$, then we can get a path $P' = P[x^+, u]xP[u^+, y]$, which contradicts (T2). If $u = y$, then let $P' = P[x^+, y]x$, which contradicts (T2). Thus $x = v_0$. Suppose $v_i y \in E(G)$, for some $i \in [0, m-1]$. Obviously, $u_0 = y^+$. Since $yv_i, y^+v_i \in E(G)$ and $y \in C - S_i$, v_i is an insertible vertex, a contradiction. □

Claim 2 For any vertex $u \in V(P)$, if $N[u]$ is claw-free, then $d_U(u) \leq 1$.

Proof Suppose u is in some U-segment S with $S \subseteq C[v_i, v_{i+1}^-], i \in [0, m]$, and $v_{i_1}, v_{i_2} \in N_U(v)$ with $0 \leq i_1 < i_2 \leq m$. Then by Lemma 2.3(c), $u^-u^+ \notin E(G)$. Obviously, at least one vertex in $\{v_{i_1}, v_{i_2}\}$ is not in $C[v_i, v_{i+1}^-]$. Without loss of generality, suppose $v_{i_1} \notin C[v_i, v_{i+1}^-]$. If $u \notin C[u_{i+1}, v_{i+1}^-]$, then $v_{i_1} \neq v_{i+1}$, and $v_{i_1}v^-, v_{i_1}v^+ \notin E(G)$ by v_{i_1} is a non-insertible vertex. Thus $G[u, u^-, u^+, v_{i_1}] = K_{1,3}$, a contradiction. Suppose $v_{i_1} = v_{i+1}$. If $u \in C[v_i, u_{i+1})$, then by the previous proof, we can get a contradiction. If $u \in C[u_{i+1}, v_{i+1}^-]$, then we consider v_{i_2} and by the previous proof, we can get a contradiction. □

Claim 3 $d_U(u) \leq 2$ for any vertex $u \in V(P)$, and if $d_U(u) = 2$, then u is a center of a claw.

Proof Without loss of generality, suppose u is in U-segment S and $S \subseteq C[v_0, v_1^-]$. If $|S| = 1$, then $d_U(u) = 0$. Suppose $|S| \geq 2$ and $S = \{x_0, x_1, x_2, \ldots, x_h\}$, where $x_0, x_1, x_2, \ldots, x_h$ are in order along the positive direction of C. Then $x_0 \notin N(U)$, $x_i \in N(U)$ for $i \in [1, h]$. For some $i \in [1, h]$, suppose $v_{i_1}, v_{i_2}, v_{i_3} \in N_U(x_i)$ with $0 \leq i_1 < i_2 < i_3 \leq m$. Then x_i is a claw center. By Lemma 2.1, suppose y_1, y_2 are the two distinct domination vertices of $N(x_i)$. Then $N[y_1], N[y_2]$ are claw-free and at least two vertices in $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ are incident with y_1 or y_2. Without loss of generality, suppose $v_{i_1}y_1, v_{i_2}y_1 \in E(G)$. Then $y_1 \in V(P)$, and $y_1^-y_1^+ \notin E(G)$ by Lemma 2.3(c). Suppose $S_j = C(u_j, u_{j+1}]$ containing y_1, $0 \leq j \leq m$. Obviously, at least one vertex in $\{v_{i_1}, v_{i_2}\}$ is not in S_j. Without loss of generality, suppose $v_{i_1} \notin S_j$. Since v_{i_1} is a non-insertible vertex and $v_{i_1}y_1 \in E(G)$, $y_1^-v_{i_1}, y_1^+v_{i_1} \notin E(G)$. Thus $G[y_1, y_1^-, y_1^+, v_{i_1}] = K_{1,3}$, a contradiction. If $d_U(u) = 2$, then by Claim 2, u is a claw center. □

Claim 4 For any U-segment S not containing y, S contains at most one vertex u with $d_U(u) = 2$, and $d_U(S) \leq |S|$.

Proof Without loss of generality, suppose $S = \{x_0, x_1, x_2, \ldots, x_h\} \subseteq C[v_i, v_{i+1}^-], 0 \leq i \leq m$, where $x_0, x_1, x_2, \ldots, x_h$ are in order along the positive direction of C. By Claim 3, suppose that x_j is the first vertex in S with $d_U(x_j) = 2, 1 \leq j \leq h$, and $\{v_{i_1}, v_{i_2}\} = N_U(x_j)$, where $0 \leq i_1 < i_2 \leq m$. By Claim 3, x_j is a center of a claw. Then $N[x_j^+]$ is claw-free, and by Claim 2, $d_U(x_j^+) \leq 1$. Thus if $j \leq h \leq j+1$, then we are done. Suppose $h > j+1$ and $N_U(x_{j+1}) = \{v_{i_3}\}$. Since $G[x_{j+1}, x_{j+2}, x_j, v_{i_3}] \neq K_{1,3}$, $E(G)$ contains at least one edge in $\{x_jv_{i_3}, x_{j+2}v_{i_3}, x_jx_{j+2}\}$. If $x_jv_{i_3}$ or $x_{j+2}v_{i_3} \in E(G)$, then $v_{i_3} = v_i$, which contradicts Lemma 2.3(b) since $x_jv_{i_1}, x_jv_{i_2}, x_{j+1}v_{i_3} \in E(G)$. Thus $v_{i_3} \neq v_i$, and $x_jx_{j+2} \in E(G)$. Then $N[x_{j+2}]$ is claw-free and by Claim 2, $d_U(x_{j+2}) \leq$

1. Thus if $h = j + 2$, then we are done. Suppose $h > j + 2$ and $\{v_{i_4}\} = N_V(x_{j+2})$. Since $v_{i_3} \neq v_i$, by Lemma 2.3(b) $v_{i_4} \neq v_i$. Then $v_{i_4}x_{j+3} \notin E(G)$. Since $G[x_{j+2}, x_j, x_{j+3}, v_{i_4}] \neq K_{1,3}$, $v_{i_4}x_j$ or $x_jx_{j+3} \in E(G)$. If $v_{i_4}x_j \in E(G)$, then $v_{i_4} \in \{v_{i_1}, v_{i_2}\}$, and by Lemma 2.3(b), $v_{i_3} = v_{i_4}$. It follows that $v_{i_3}x_{j+1}, v_{i_3}x_{j+2} \in E(G)$, a contradiction. Thus $x_jx_{j+3} \in E(G)$, and then $N[x_{j+3}]$ is claw-free. By Claim 2, $d_U(x_{j+3}) \leq 1$. Thus if $h = j + 3$, then we are done. If $h > j + 3$, then proceeding in the above manners to the set $L = \{x_{j+4}, \dots, x_h\}$, we can get $N[u]$ is claw-free for any vertex u in L, and then by Claim 2, $d_U(u) \leq 1$. It follows that S has exactly one vertex x_j with $d_U(x_j) = 2$, and then $d_U(S) \leq |S|$. $\square$

Claim 5 Suppose the U-segment S_0 contains y. Then $d_U(u) \leq 1$ for any vertex $u \in S_0 - \{u_0\}$.

Proof If $S_0 = \{y, u_0\}$, then $d_U(y) = 0$, and we are done. Suppose $|S_0| \geq 3$. Then by Claim 1, $N_U(y) = \{v_m\}$. Thus by Lemma 2.3(b), $N_U(u) \subseteq \{v_m\}$ for any vertex $u \in S_0 - \{u_0\}$. $\square$

Claim 6 For the vertices in $U = \{v_0, \dots, v_m\}$, $\sum_{i=0}^m d_P(v_i) \leq |P| - 1$.

Proof Obviously, $V(P) = \bigcup_{i=0}^{m-1} V(P[v_i, v_{i+1}^-]) \cup V(P[v_m, y])$. Recall that $\sum_{i=0}^m d_P(v_i) = d_U(P)$. Then $\sum_{i=0}^m d_P(v_i) = \sum_{i=0}^{m-1} d_U(P[v_i, v_{i+1}^-]) + d_U(P[v_m, y])$. By Claim 5, $d_U(P[v_m, y]) \leq |P[v_m, y]| - 1$. By Claim 4, $d_U(P[v_i, v_{i+1}^-]) \leq |P[v_i, v_{i+1}^-]|$ for $1 \leq i \leq m-1$. Thus $\sum_{i=0}^m d_P(v_i) \leq \sum_{i=0}^{m-1} |P[v_i, v_{i+1}^-]| + |P[v_m, y]| - 1 = |P| - 1$. $\square$

Claim 7 Suppose $z_1, z_2 \in V(G-P)$ are two nonadjacent vertices. Then $|N_P(z_1) \cap N_P(z_2)| \leq 2$.

Proof Obviously, $N_P(z_1) \cap N_P(z_2) \subseteq \{u_1, u_2, \dots, u_m\}$. Suppose $|N_P(z_1) \cap N_P(z_2)| \geq 3$ and without loss of generality, suppose $u_1, u_2, u_3 \in N_P(z_1) \cap N_P(z_2)$. Obviously, $G[u_1, z_1, z_2, u_1^-] = K_{1,3}$. Similarly, u_2, u_3 are claw-centers. Thus $\{u_1, u_2, u_3\}$ is an independent set. Since $z_1u_1 \in E(G)$, $N[z_1]$ contains no claw. Then $G[z_1, u_1, u_2, u_3] \neq K_{1,3}$. Thus $E(G)$ contains at least one edge in $\{u_1u_2, u_1u_3, u_2u_3\}$, which contradicts the independent set $\{u_1, u_2, u_3\}$. $\square$

Claim 8 For every component R of $G - P$, $|N_P(R)| = k$, and R is hamiltonian-connected.

Proof By Lemma 2.3(a), for $0 \leq i \neq j \leq m$, $N_{G-P}(v_i) \cap N_{G-P}(v_j) = \emptyset$, and then $\sum_{i=0}^m d_{G-P}(v_i) \leq n - |P| - |R|$. Since G is k-connected, $m \geq k$. Suppose $m > k$. If $m \geq k+2$, then $\{v_0, v_1, \dots, v_m\}$ is an independent set with order at least $k + 3$. By Claim 6, we can get

$$\sum_{i=0}^m d(v_i) = \sum_{i=0}^m d_P(v_i) + \sum_{i=0}^m d_{G-P}(v_i) \leq (|P| - 1) + (n - |P| - |R|) = n - 1 - |R|,$$

which contradicts $\sigma_{k+3}(G) \geq n+k+2$. Suppose $m = k+1$ and $u \in V(R)$. Then $\{u, v_0, v_1, \dots, v_m\}$ is an independent set with order $k + 3$. Since $d(u) = d_P(u) + d_R(u) \leq m + |R| - 1 = k + |R|$,

$$\begin{aligned}\sum_{i=0}^{k+1} d(v_i) + d(u) &= \sum_{i=0}^{k+1} d_P(v_i) + \sum_{i=0}^{k+1} d_{G-P}(v_i) + d(u) \\ &\leq (|P| - 1) + (n - |P| - |R|) + k + |R| = n + k - 1,\end{aligned}$$

which contradicts $\sigma_{k+3}(G) \geq n + k + 2$. It follows that $m = k$.

Suppose that R is not hamiltonian-connected. Then by Ore's theorem in [10], there exists two nonadjacent vertices z_1 and z_2 such that $d_R(z_1)+d_R(z_2)\le |R|$. By Claims 7 and 8, $d_P(z_1)+d_P(z_2)\le k+2$. Since $\{z_1,z_2,v_0,v_1,\dots,v_k\}$ is an independent set with order $k+3$,

$$\begin{aligned}\sum_{i=0}^{k} d(v_i)+d(z_1)+d(z_2) &= \sum_{i=0}^{k} d_P(v_i)+\sum_{i=0}^{k} d_{G-P}(v_i)+(d_P(z_1)+d_P(z_2))+(d_R(z_1)+d_R(z_2))\\ &\le (|P|-1)+(n-|P|-|R|)+(k+2)+|R|=n+k+1,\end{aligned}$$

which contradicts $\sigma_{k+3}(G)\ge n+k+2$. □

Claim 9 Let $u_i,u_j\in N_P(R), 1\le i\ne j\le k$. Then $G[V(R)\cup\{u_i,u_j\}]$ contains a hamiltonian path with ended vertices u_i and u_j.

Proof By Claim 8, R is hamiltonian-connected. If $|R|=1$, then we are done. Suppose $|R|\ge 2$. If $N_R(u_i)=N_R(u_j)=\{u\}$, then $N_P(R)-\{u_i,u_j\}\cup\{u\}$ is a vertex cut of G with order $k-1$, a contradiction with the k-connectedness of G. Thus $|N_R(u_i)\cup N_R(u_j)|\ge 2$ and then the claim holds. □

If $G-P$ contains only component of R, then by Claim 8, G contains a spanning 3-ended tree. Thus we assume that $v(G-P)\ge 2$ and R' is a component in $G-P-R$.

Claim 10 $N(v_i)\cap V(R')\ne\emptyset$ for some $1\le i\le k$.

Proof By Claim 1, $N(v_0)\cap V(R')=\emptyset$. Suppose $N(v_i)\cap V(R')=\emptyset$ for any $i\in[1,k]$. Let $z_1\in V(R), z_2\in V(R')$. Then $\{z_1,z_2,v_0,v_1,\dots,v_k\}$ is an independent set of order $k+3$. By Claims 7 and 8, $d_P(z_1)+d_P(z_2)\le k+2$. Obviously, $\sum_{i=0}^{k} d_{G-P}(v_i)\le n-|P|-|R|-|R'|$. Then

$$\begin{aligned}\sum_{i=0}^{k} d(v_i)+d(z_1)+d(z_2) &= \sum_{i=0}^{k} d_P(v_i)+\sum_{i=0}^{k} d_{G-P}(v_i)+d_P(z_1)+d_P(z_2)+d_R(z_1)+d_{R'}(z_2)\\ &\le (|P|-1)+(n-|P|-|R|-|R'|)+(k+2)+|R|-1+|R'|-1\\ &= n+k-1,\end{aligned}$$

which contradicts $\sigma_{k+3}(G)\ge n+k+2$. □

By Claim 10, we assume $N(v_i)\cap V(R')\ne\emptyset$ for some $i\in[1,k]$. By Lemma 2.3(a), $N(v_j)\cap V(R')=\emptyset$ for $j\in[0,k]-\{i\}$.

By the proof in [5], we can get the following two results.

Claim 11 There exists a second non-insertible vertex v_i' in $S_i-\{u_{i+1}\}$ and $v_i'\notin N(R')$.

Proof Suppose $S_i-\{u_{i+1}\}$ contains only one non-insertible vertex v_i. Then we can get a path $P_1[u_{i+1},u_i]$ such that $V(P_1)=V(C)-\{v_i\}$ by inserting all the vertices in $S_i-\{v_i\}$ to $C[u_{i+1},u_i]$. Suppose $|V(R)|=\{u\}$. Then we get a cycle $C'=P_1[u_{i+1},u_i]uu_{i+1}$. Let $P'=V(C')-\{u_0\}$. Then $w(G-P')<w(G-P)$, which contradicts (T1). Suppose $|V(R)|\ge 2$. If $N_R(u_i)\cup N_R(u_{i+1})=\{z\}$, then $N_P(R)\cup\{z\}-\{u_i,u_{i+1}\}$ is a vertex cut of G with order $k-1$, which contradicts Claim 8. Thus $|N_R(u_i)\cup N_R(u_{i+1})|\ge 2$. By Claim 9, there is a hamiltonian path $u_iP_2u_{i+1}$ of $R\cup\{u_i,u_{i+1}\}$. Thus we can get a cycle $C_1=u_{i+1}P_1u_iP_2u_{i+1}$ longer than C, a contradiction. □

Now, we complete Theorem 1.6. Let $z_1 \in V(R), z_2 \in V(R')$. By Claim 11 and Lemma 2.4, $U' = \{v_0, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k\}$ is an indendent set. By Lemma 2.4 and the preceding proof, U' has the same properties as U. Obviously, $U' \cup \{z_1, z_2\}$ is an independent set of order $k+3$ in G. Obviously, $\sum_{u \in U'} d_{G-P}(u) \le n - |P| - |R| - |R'|$. By Claims 7 and 8, $d_P(z_1) + d_P(z_2) \le k+2$. Obviously, $d_R(z_1) \le |R| - 1, d_{R'}(z_2) \le |R'| - 1$. Then we can get

$$\begin{aligned}\sum_{u\in U'} d(u) + d(z_1) + d(z_2) &= \sum_{u\in U'} d_P(u) + \sum_{u\in U'} d_{G-P}(u) + d_P(z_1) + d_P(z_2) + d_R(z_1) + d_{R'}(z_2)\\ &\le (|P|-1) + (n - |P| - |R| - |R'|) + (k+2) + (|R|-1) + (|R'|-1)\\ &= n+k-1,\end{aligned}$$

which contradicts $\sigma_{k+3}(G) \ge n+k+2$. It follows that Theorem 1.6 holds.

References

[1] J. A. BONDY, U. S. R. MURTY. *Graph Theory With Applications*. Pure Appl. Math., 2004, **20**(1): 10–13.
[2] H. BROERSMA, H. TUINSTRA. *Independent trees and Hamilton cycles*. J. Graph Theory, 1998, **29**: 227–237.
[3] A. KYAW. *Spanning trees with at most 3 leaves in $K_{1,4}$-free graphs*. Discrete Math., 2009, **309**(20): 6146–6148.
[4] A. KYAW. *Spanning trees with at most k leaves in $K_{1,4}$-free graphs*. Discrete Math., 2011, **311**(20): 2135–2142.
[5] M. KANO, A. KYAW, H. MATSUDA, et al. *Spanning trees with a bounded number of leaves in a claw-free graph*. Ars Combin., 2012, **103**: 137–154.
[6] Yuan CHEN, Guantao CHEN, Zhiquan HU. *Spanning 3-ended trees in k-connected $K_{1,4}$-free graphs*. Sci. China Math., 2014, **57**(8): 1579–1586.
[7] Xiaodong CHEN, Meijn XU, Yanjun LIU. *Spanning 3-ended trees in almost claw-free graphs*. Discrete Dyn. Nat. Soc., 2015, Art. ID 476182, 5 pp.
[8] Guantao CHEN, R. SCHELP. *Hamiltonicity for $K_{1,4}$-free graphs*. J. Graph Theory, 1995, **20**(4): 423–439.
[9] Z. RYJACEK. *Almost claw-free graphs*. J. Graph Theory, 1994, **18**(5): 469–477.
[10] O. ORE. *Hamiltonian connected graphs*. J. Math Pure Appl., 1963, **42**: 21–27.