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The Central Extension of an Elementary Abelian p-Group by a Miniaml Non-Abelian p-Group

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Abstract Assume that N, F and G are groups. If there exsits $\tilde{N}$, a normal subgroup of G such that $\tilde{N} \cong G$ and $G/\tilde{N} \cong F$, then G is called a central extension of N by F. In this paper, the central extension of N by a minimal non-abelian p-group is determined, where N is an elementary abelian p-group of order p^3. Together with our previous work, all central extensions of N by a minimal non-abelian p-group is determined, where N is an elementary abelian p-group.

Keywords central extension; minimal non-abelian p-groups; congruent

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1. Introduction

Finite p-groups are an important class of finite groups. After the classification of finite simple groups was finally completed, the study of finite p-groups becomes more and more active. Many leading group theorists, for example, Glauberman, Janko, etc., have turned their attentions to the study of finite p-groups. As Janko mentioned in the Foreword of [1], to study p-groups with "large" abelian subgroups is another approach to finite p-groups. A well-known important result is the classification of finite p-groups with a cyclic subgroup of index p, which was obtained by Burnside [2]. Tuan [3] studied finite p-groups with an abelian subgroup of index p. Another important concept in finite p-groups is minimal non-abelian p-groups. A non-abelian group G is said to be minimal non-abelian if every proper subgroup of G is abelian. Minimal non-abelian groups were classified in [4], and in more detail for finite p-groups in [5]. Recently the author and his colleagues classified finite p-groups with a minimal non-abelian subgroup of index p.

Groups in this paper are finite p-groups. Notation and terminology are consistent with that in [6–8]. Assume that N, F and G are groups. If there exsits $\tilde{N}$, a normal subgroup of G such that $\tilde{N} \cong G$ and $G/\tilde{N} \cong F$, then G is called a central extension of N by F.

In this paper, the central extension of N by a minimal non-abelian p-group is determined, where N is an elementary abelian p-group of order p^3. Together with our previous work, all central extensions of N by a minimal non-abelian p-group is determined, where N is an elementary abelian p-group.

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2. Preliminaries

In this paper, p is always a prime. We use F_p to denote the finite field containing p elements. F_p^* is the multiplicative group of F_p. $(F_p^*)^2 = \{a^2|a \in F_p^*\}$ is a subgroup of F_p^*. $F_p^2 = (F_p^*)^2 \cup \{0\}$. For a finite non-abelian p-group G, we use $p^{I_{\min}}$ and $p^{I_{\max}}$ to denote the minimal index and the maximal index of $\mathcal{A}_1$-subgroups of G, respectively. For a square matrix A, $|A|$ denotes the determinant of A. We need the following lemmas.

Lemma 2.1 ([9, Lemma 2.2]) *Suppose that G is a finite non-abelian p-group. Then the following conditions are equivalent:*

(1) G is an $\mathcal{A}_1$-group; (2) $d(G) = 2$ and $|G'| = p$; (3) $d(G) = 2$ and $\Phi(G) = Z(G)$.

Lemma 2.2 ([8, Lemma 2.1]) *Suppose that p is odd, $\{1, \eta\}$ is a transversal for $(F_p^*)^2$ in F_p^*. Then the following matrices form a transversal for the congruence classes of invertible matrices of order 2 over F_p:*

(1) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *(2)* $\begin{pmatrix} \nu & 1 \\ -1 & 0 \end{pmatrix}$, *(3)* $\begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}$, *(4)* $\begin{pmatrix} 1 & 1 \\ -1 & r \end{pmatrix}$,

where $\nu = 1$ or η, $r = 1, 2, \ldots, p-2$.

Lemma 2.3 ([6, Lemma 4.3]) *The following matrices form a transversal for the congruence classes of invertible matrices of order 2 over F_2.*

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *(2)* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *(3)* $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Lemma 2.4 ([8, Lemma 2.3]) *Suppose that p is a prime ($p = 2$ is possible). For odd p, $\{1, \eta\}$ is a transversal for $(F_p^*)^2$ in F_p^*. Then the following matrices form a transversal for the congruence classes of non-invertible matrices of order 2 over F_p:*

(1) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *(2)* $\begin{pmatrix} 0 & 0 \\ 0 & \nu \end{pmatrix}$, *(3)* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *where $\nu = 1$ or η.*

Theorem 2.5 ([10, Theorem 2.3]) *A p-group G is metacyclic if and only if $G/\Phi(G')G_3$ is metacyclic.*

3. The central extension of C_p^3 by a miniaml non-abelian p-group

Theorem 3.1 *Suppose that G is a finite p-group. If there exists $N \cong C_p^3$ such that $N \leq Z(G) \cap G'$ and G/N is minimal abelian, then*

(1) $N = \Phi(G')G_3$; (2) G/N is not metacyclic; (3) $G_3 \cong C_p^2$ and $G' \cong C_{p^2} \times C_p \times C_p$.

Proof (1) By Theorem 2.1, $|(G/N)'| = p$. It follows that $\Phi(G')G_3 \leq N$. Since $|(G/\Phi(G')G_3)'| = p$, $\Phi(G')G_3$ is maximal in G'. Since $N < G'$, $N = \Phi(G')G_3$.

(2) If G/N is metacyclic, then, by (1), $G/\Phi(G')G_3$ is metacyclic. By Theorem 2.5, G is also metacyclic. It follows that G' is cyclic. Since $N \leq G'$, N is also cyclic, which contradicts $N \cong C_p^3$.

(3) It is obvious that $d(G) = 2$. Let $G = \langle a, b\rangle$ where $[a, b] = c$. Then $G' = \langle c, G_3\rangle$ and

$G_3 = \langle [c,a],[c,b]\rangle$. Hence $\Phi(G')G_3 = \langle c^p,[c,a],[c,b]\rangle$. Since $\Phi(G')G_3 \cong C_p^3$, we have $G_3 \cong C_p^2$ and $G' \cong C_{p^2}\times C_p\times C_p$. □

Theorem 3.2 *Suppose that G and $\bar{G}$ are finite p such that $\Phi(G')G_3 \cong C_p^3$ and $G/\Phi(G')G_3 \cong M_p(n,m,1)$, where $n \ge m \ge 2$ and $n \ge 3$ for $p=2$. Then $G \cong \bar{G}$ if and only if there exists $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21}p^{n-m} & y_{22}\end{pmatrix}$, an invertible matrix over F_p, such that $w(\bar{G}) = Y_1 w(G) Y^T$ and $v(\bar{G}) = Y_1 v(G)$, where $Y_1 = \begin{pmatrix} y_{11} & y_{12}p^{n-m} \\ y_{21} & y_{22}\end{pmatrix}$.*

Proof Suppose that $w(G)$,$v(G)$, $w(\bar{G})$ and $v(\bar{G})$ are characteristic matrices and characteristic vectors corresponding to generators a,b and $\bar{a},\bar{b}$, respectively. Let θ be an isomorphism from $\bar{G}$ to G. We may let

$$\bar{a}^\theta \equiv a^{x_{11}}b^{x_{12}}c^{x_{13}} \bmod \Phi(G')G_3,\quad \bar{b}^\theta \equiv a^{x_{21}p^{n-m}}b^{x_{22}}c^{x_{23}} \bmod \Phi(G')G_3,$$

where $X := \begin{pmatrix} x_{11} & x_{12} \\ x_{21}p^{n-m} & x_{22}\end{pmatrix}$ is an invertible matrix over F_p. By calculation, we have

$$\bar{c}^\theta = [\bar{a},\bar{b}]^\theta = [\bar{a}^\theta,\bar{b}^\theta] \equiv [a^{x_{11}}b^{x_{12}}, a^{x_{21}p^{n-m}}b^{x_{22}}] \equiv c^{|X|} \mod \Phi(G')G_3$$

and

$$\bar{x}^\theta = [\bar{b},\bar{c}]^\theta = [\bar{b}^\theta,\bar{c}^\theta] = [a^{x_{21}p^{n-m}}b^{x_{22}}, c^{|X|}] = x^{|X|x_{22}}y^{-|X|x_{21}p^{n-m}},$$
$$\bar{y}^\theta = [\bar{c},\bar{a}]^\theta = [\bar{c}^\theta,\bar{a}^\theta] = [c^{|X|}, a^{x_{11}}b^{x_{12}}] = x^{-|X|x_{12}}y^{|X|x_{11}}.$$

By transforming $\bar{x}^{\bar{w}_{11}}\bar{y}^{\bar{w}_{12}}\bar{c}^{\bar{w}_{13}p} = \bar{a}^{p^n}$ by θ, we have

$$(\bar{w}_{11},\bar{w}_{12})\begin{pmatrix} |X|x_{22} & -|X|x_{21}p^{n-m} \\ -|X|x_{12} & |X|x_{11}\end{pmatrix} = (x_{11}, x_{12}p^{n-m})\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix} \tag{3.1}$$

and

$$|X|\bar{w}_{13} = (x_{11},x_{12}p^{n-m})\begin{pmatrix} w_{13} \\ w_{23}\end{pmatrix}. \tag{3.2}$$

By transforming $\bar{x}^{\bar{w}_{21}}\bar{y}^{\bar{w}_{22}}\bar{c}^{\bar{w}_{23}p} = \bar{b}^{p^m}$ by θ, we have

$$(\bar{w}_{21},\bar{w}_{22})\begin{pmatrix} |X|x_{22} & -|X|x_{21}p^{n-m} \\ -|X|x_{12} & |X|x_{11}\end{pmatrix} = (x_{21}, x_{22})\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix} \tag{3.3}$$

and

$$|X|\bar{w}_{23} = (x_{21},x_{22})\begin{pmatrix} w_{13} \\ w_{23}\end{pmatrix}. \tag{3.4}$$

By Eqs. (3.1) and (3.3),

$$|X|\begin{pmatrix} \bar{w}_{11} & \bar{w}_{12} \\ \bar{w}_{21} & \bar{w}_{22}\end{pmatrix}\begin{pmatrix} x_{22} & -x_{21}p^{n-m} \\ -x_{12} & x_{11}\end{pmatrix} = \begin{pmatrix} x_{11} & x_{12}p^{n-m} \\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix}. \tag{3.5}$$

By Eqs. (3.2) and (3.4),

$$|X|\begin{pmatrix} \bar{w}_{13} \\ \bar{w}_{23}\end{pmatrix} = \begin{pmatrix} x_{11} & x_{12}p^{n-m} \\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix} w_{13} \\ w_{23}\end{pmatrix}. \tag{3.6}$$

Let

$$Y=|X|^{-1}X=|X|^{-1}\begin{pmatrix} x_{11} & x_{12}\\ x_{21}p^{n-m} & x_{22}\end{pmatrix}\begin{pmatrix} y_{11} & y_{12}\\ y_{21}p^{n-m} & y_{22}\end{pmatrix}$$

and $Y_1=\begin{pmatrix} y_{11} & y_{12}p^{n-m}\\ y_{21} & y_{22}\end{pmatrix}$. Right multiplying Y^T on Eq. (3.5), we have

$$\begin{pmatrix} \bar{w}_{11} & \bar{w}_{12}\\ \bar{w}_{21} & \bar{w}_{22}\end{pmatrix}=Y_1\begin{pmatrix} w_{11} & w_{12}\\ w_{21} & w_{22}\end{pmatrix}Y^T. \tag{3.7}$$

By Eq. (3.6),

$$v(\bar{G})=Y_1v(G). \tag{3.8}$$

Conversely, if there exists an invertible matrix $Y=\begin{pmatrix} y_{11} & y_{12}\\ y_{21}p^{n-m} & y_{22}\end{pmatrix}$ such that the Eqs. (3.7) and (3.8) hold, then, let $X=|Y|^{-1}Y=\begin{pmatrix} x_{11} & x_{12}\\ x_{21}p^{n-m} & x_{22}\end{pmatrix}$. By using above argument, it is easy to check the map $\theta:\bar{a}\mapsto a^{x_{11}}b^{x_{12}}, \bar{b}\mapsto a^{x_{21}p^{n-m}}b^{x_{22}}$ is an isomorphism from $\bar{G}$ to G. □

Theorem 3.2 *Let G be a finite p-group such that $\Phi(G')G_3\cong C_p^3$, $\Phi(G')G_3\le Z(G)$ and $G/\Phi(G')G_3\cong M_p(n,m,1)$, where $n\ge m\ge 2$. Then G is one of the following non-isomorphic groups:*

(A1) $\langle a,b\mid a^4=b^4=c^4=d^2=e^2=1,[a,b]=c,[c,a]=d,[c,b]=e,[d,a]=[d,b]=[e,a]=[e,b]=1\rangle$;

(A2) $\langle a,b\mid a^8=b^4=c^4=d^2=1,[a,b]=c,[c,a]=d,[c,b]=a^4,[d,a]=[d,b]=1\rangle$;

(A3) $\langle a,b\mid a^8=b^4=c^4=d^2=1,[a,b]=c,[c,a]=a^4,[c,b]=d,[d,a]=[d,b]=1\rangle$;

(A4) $\langle a,b\mid a^8=b^4=c^4=d^2=1,[a,b]=c,[c,a]=d,[c,b]=a^4d,[d,a]=[d,b]=1\rangle$;

(A5) $\langle a,b\mid a^8=b^8=c^4=d^2=1,a^4=b^4,[a,b]=c,[c,a]=a^4,[c,b]=d,[d,a]=[d,b]=1\rangle$;

(A6) $\langle a,b\mid a^8=b^8=c^4=d^2=e^2=1,a^4=b^4=c^2,[a,b]=c,[c,a]=d,[c,b]=e\rangle$;

(A7) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=b^4,[c,b]=a^4\rangle$;

(A8) $\langle a,b\mid a^8=b^8=c^4=d^2=1,b^4=c^2,[a,b]=c,[c,a]=d,[c,b]=a^4,[d,a]=[d,b]=1\rangle$;

(A9) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=a^4,[c,b]=b^4\rangle$;

(A10) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=b^4c^2,[c,b]=a^4\rangle$;

(A11) $\langle a,b\mid a^8=b^8=c^4=d^2=1,a^4=c^2,[a,b]=c,[c,a]=b^4c^2,[c,b]=d,[d,a]=[d,b]=1\rangle$;

(A12) $\langle a,b\mid a^8=b^8=c^4=d^2=1,a^4=c^2,[a,b]=c,[c,a]=d,[c,b]=b^4c^2,[d,a]=[d,b]=1\rangle$;

(A13) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=a^4c^2,[c,b]=b^4c^2\rangle$;

(A14) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=b^4c^2,[c,b]=a^4c^2\rangle$;

(A15) $\langle a,b\mid a^8=b^8=c^4=1,[a,b]=c,[c,a]=a^4b^4,[c,b]=a^4c^2\rangle$;

(B1) $\langle a,b,c\mid a^{p^{n+1}}=b^{p^{n+1}}=c^{p^2}=1,[a,b]=c,[c,a]=a^{p^n},[c,b]=b^{p^n}\rangle$, *where $p>2$, $n\ge 2$;*

(B2) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = a^{p^n} b^{\nu p^n}, [c,b] = b^{p^n}\rangle$, *where* $p > 2$, $n \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p;

(B3) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = b^{\nu p^n}, [c,b] = a^{-p^n}\rangle$, *where* $p > 2$, $n \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p;

(B4) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a]^{1+r} = a^{p^n} b^{p^n}, [c,b]^{1+r} = a^{-rp^n} b^{p^n}\rangle$, *where* $p > 2$, $n \geq 2$, $r = 1, 2, \ldots, p-2$;

(B5) $\langle a,b,c \mid a^{2^{n+1}} = b^{2^{n+1}} = c^4 = 1, [a,b] = c, [c,a] = b^{2^n}, [c,b] = a^{2^n}\rangle$, *where* $n \geq 3$;

(B6) $\langle a,b,c \mid a^{2^{n+1}} = b^{2^{n+1}} = c^4 = 1, [a,b] = c, [c,a] = a^{2^n}, [c,b] = b^{2^n}\rangle$, *where* $n \geq 3$;

(B7) $\langle a,b,c \mid a^{2^{n+1}} = b^{2^{n+1}} = c^4 = 1, [a,b] = c, [c,a] = a^{2^n} b^{2^n}, [c,b] = a^{2^n}\rangle$, *where* $n \geq 3$;

(B8) $\langle a,b,c,d \mid a^{p^{n+1}} = b^{p^n} = c^{p^2} = d^p = 1, [a,b] = c, [c,a] = a^{p^n}, [c,b] = d, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$;

(B9) $\langle a,b,c,d \mid a^{p^n} = b^{p^{n+1}} = c^{p^2} = d^p = 1, [a,b] = c, [c,a] = b^{\nu p^n}, [c,b] = d, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p;

(B10) $\langle a,b,c,d,e \mid a^{p^n} = b^{p^n} = c^{p^2} = d^p = e^p = 1, [a,b] = c, [c,a] = d, [c,b] = e, [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$;

(C1) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = b^{sp^n} c^{-sp}, [c,b] = a^{-\nu p^n} b^{st\nu p^n} c^{-stp}\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p, $s \in F_p^*$, $t = 0, 1, \ldots, \frac{p-1}{2}$;

(C2) $\langle a,b,c,d \mid a^{p^{n+1}} = b^{p^{n+1}} = d^p = 1, c^p = b^{p^n}, [a,b] = c, [c,a] = d, [c,b] = a^{-\nu p^n} d^{t\nu}, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p, $t = 0, 1, \ldots, \frac{p-1}{2}$;

(C3) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = a^{p^n}, [c,b] = a^{sp^n} b^{p^n} c^{-p}\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $s \in F_p$;

(C4) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{n+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = a^{p^n}, [c,b] = b^{sp^n} c^{-sp}\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $s = 2, 3, \ldots, \frac{p-1}{2}$;

(C5) $\langle a,b,c,d \mid a^{p^{n+1}} = b^{p^{n+1}} = d^p = 1, c^p = b^{p^n}, [a,b] = c, [c,a] = a^{p^n}, [c,b] = d, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$;

(C6) $\langle a,b,c,d \mid a^{p^n} = b^{p^{n+1}} = c^{p^2} = d^p = 1, [a,b] = c, [c,a] = d, [c,b] = b^{p^n} c^{-p}, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$;

(C7) $\langle a,b,c,d \mid a^{p^n} = b^{p^{n+1}} = c^{p^2} = d^p = 1, [a,b] = c, [c,a] = b^{sp^n} c^{-sp}, [c,b] = d, [d,a] = [d,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$, $s \in F_p^*$;

(C8) $\langle a,b,c,d,e \mid a^{p^n} = b^{p^{n+1}} = d^p = e^p = 1, c^p = b^{p^n}, [a,b] = c, [c,a] = d, [c,b] = e, [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle$, *where* $n \geq 3$ *for* $p = 2$ *and* $n \geq 2$;

(D1) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{m+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = a^{p^n}, [c,b] = b^{sp^m}\rangle$, *where* $n > m \geq 2$, $s \in F_p^*$;

(D2) $\langle a,b,c \mid a^{p^{n+1}} = b^{p^{m+1}} = c^{p^2} = 1, [a,b] = c, [c,a] = b^{\nu_1 p^m}, [c,b] = a^{-\nu_2 p^n}\rangle$, *where* $n > m \geq 2$, $\nu_1, \nu_2 = 1$ *or a fixed quadratic non-residue modular* p;

(D3) $\langle a,b,c,d \mid a^{p^{n+1}} = b^{p^m} = c^{p^2} = d^p = 1, [a,b] = c, [c,a] = d, [c,b] = a^{-\nu p^n}, [d,a] = [d,b] = 1\rangle$, *where* $n > m \geq 2$, $\nu = 1$ *or a fixed quadratic non-residue modular* p;

(D4) $\langle a,b,c,d \mid a^{p^n}=b^{p^{m+1}}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=b^{\nu p^m},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $\nu=1$ *or a fixed quadratic non-residue modular* p;

(D5) $\langle a,b,c,d \mid a^{p^n}=b^{p^{m+1}}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=d,[c,b]=b^{p^m},[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$;

(D6) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^m}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=a^{p^n},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$;

(D7) $\langle a,b,c,d,e \mid a^{p^n}=b^{p^m}=c^{p^2}=d^p=e^p=1,[a,b]=c,[c,a]=d,[c,b]=e,[d,a]=[d,b]=[e,a]=[e,b]=1\rangle$, *where* $n>m\geq 2$;

(E1) $\langle a,b,c \mid a^{p^{n+1}}=b^{p^{m+1}}=c^{p^2}=1,[a,b]=c,[c,b]=a^{-sp^n}b^{st\nu p^m}c^{sp},[c,a]=b^{\nu p^m}\rangle$, *where* $n>m\geq 2$, $\nu=1$ *or a fixed quadratic non-residue modular* p, $s\in F_p^*$, $t=0,1,\dots,\frac{p-1}{2}$;

(E2) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^{m+1}}=d^p=1,c^p=a^{p^n}b^{-t\nu p^m},[a,b]=c,[c,a]=b^{\nu p^m},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $\nu=1$ *or a fixed quadratic non-residue modular* p, $t=0,1,\dots,\frac{p-1}{2}$;

(E3) $\langle a,b,c \mid a^{p^{n+1}}=b^{p^{m+1}}=c^{p^2}=1,[a,b]=c,[c,a]=a^{p^n}c^{-p},[c,b]=b^{sp^m}\rangle$, *where* $n>m\geq 2$, $s\in F_p^*$;

(E4) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^m}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=a^{p^n}c^{-p},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$;

(E5) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^{m+1}}=d^p=1,c^p=a^{p^n}b^{sp^m},[a,b]=c,[c,a]=d,[c,b]=b^{p^m},[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $s\in F_p$;

(E6) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^m}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=d,[c,b]=a^{-sp^n}c^{sp},[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $s\in F_p^*$;

(E7) $\langle a,b,c,d,e \mid a^{p^{n+1}}=b^{p^m}=d^p=e^p=1,c^p=a^{p^n},[a,b]=c,[c,a]=d,[c,b]=e,[d,a]=[d,b]=[e,a]=[e,b]=1\rangle$, *where* $n>m\geq 2$;

(F1) $\langle a,b,c \mid a^{p^{n+1}}=b^{p^{m+1}}=c^{p^2}=1,[a,b]=c,[c,a]=a^{p^n},[c,b]=b^{sp^m}c^{-sp}\rangle$, *where* $n>m\geq 2$, $s\in F_p^*$;

(F2) $\langle a,b,c \mid a^{p^{n+1}}=b^{p^{m+1}}=c^{p^2}=1,[a,b]=c,[c,a]=b^{sp^m}c^{-sp},[c,b]=a^{-\nu p^n}\rangle$, *where* $n>m\geq 2$, $s\in F_p^*$, $\nu=1$ *or a fixed quadratic non-residue modular* p;

(F3) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^{m+1}}=d^p=1,c^p=b^{p^m},[a,b]=c,[c,a]=d,[c,b]=a^{-\nu p^n},[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $\nu=1$ *or a fixed quadratic non-residue modular* p;

(F4) $\langle a,b,c,d \mid a^{p^n}=b^{p^{m+1}}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=b^{sp^m}c^{-sp},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$, $s\in F_p^*$;

(F5) $\langle a,b,c,d \mid a^{p^n}=b^{p^{m+1}}=c^{p^2}=d^p=1,[a,b]=c,[c,a]=d,[c,b]=b^{p^m}c^{-p},[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$;

(F6) $\langle a,b,c,d \mid a^{p^{n+1}}=b^{p^m}=d^p=1,c^p=b^{p^m},[a,b]=c,[c,a]=a^{p^n},[c,b]=d,[d,a]=[d,b]=1\rangle$, *where* $n>m\geq 2$;

(F7) $\langle a,b,c,d,e \mid a^{p^n}=b^{p^{m+1}}=d^p=e^p=1,c^p=b^{p^m},[a,b]=c,[c,a]=d,[c,b]=e,[d,a]=[d,b]=[e,a]=[e,b]=1\rangle$, *where* $n>m\geq 2$.

Proof Case 1 $n=m$.

If $p=n=m=2$, then $|G|=2^8$. By checking the list of groups of order 2^8, we get the groups of type (A1)–(A15). In the following we may assume that $n>2$ for $p=2$.

Subcase 1.1 $v(G) = (0,0)^T$.

Assume that G and $\bar{G}$ are two groups described in the theorem with $v(G) = v(\bar{G})(0,0)^T$. By Theorem 3.2, $G \cong \bar{G}$ if and only if there exists $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, an invertible matrix over F_p such that $w(\bar{G}) = Yw(G)Y^T$. That is $w(\bar{G})$ and $w(G)$ are mutually congruent. By Lemmas 2.2–2.4, we get the groups of type (B1)–(B10).

Subcase 1.2 $v(G) \neq (0,0)^T$.

If $w_{13} \neq 0$, then, let $Y_1 = \begin{pmatrix} -w_{23}w_{13}^{-1} & w_{13}^{-1} \\ w_{13}^{-1} & 0 \end{pmatrix}$, $Y_1 v(G) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $w_{13} = 0$, then $w_{23} \neq 0$. Let $Y_1 = \begin{pmatrix} w_{23}^{-1} & 0 \\ 0 & w_{23}^{-1} \end{pmatrix}$. Then $Y_1 v(G) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let G and $\bar{G}$ be two groups with $v(G) = v(\bar{G}) = (0,1)^T$. By Theorem 3.2, $G \cong \bar{G}$ if and only if there exists $Y = \begin{pmatrix} y_{11} & 0 \\ y_{21} & 1 \end{pmatrix}$, an invertible matrix over F_p such that $w(\bar{G}) = Yw(G)Y^T$.

By suitably choosing y_{21}, we can simplify $w(G)$ to be one of the following types:

(a) $\begin{pmatrix} w_{11} & w_{12} \\ 0 & w_{22} \end{pmatrix}$ where $w_{11} \neq 0$,

(b) $\begin{pmatrix} 0 & w_{12} \\ -w_{12} & w_{22} \end{pmatrix}$ where $w_{12} \neq 0$,

(c) $\begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}$ where $w_{12} \neq 0$ and $w_{21} \neq -w_{12}$,

(d) $\begin{pmatrix} 0 & 0 \\ w_{21} & 0 \end{pmatrix}$ where $w_{21} \neq 0$ and (e) $\begin{pmatrix} 0 & 0 \\ 0 & w_{22} \end{pmatrix}$.

In the following, we assume that both $w(G)$ and $w(\bar{G})$ are such matrices. It is easy to check that (i) different types give non-isomorphic groups, (ii) $G \cong \bar{G}$ if and only if there exists $y_{11} \in F_p^*$ such that $w(\bar{G}) = Yw(G)Y^T$ where $Y = \mathrm{diag}(y_{11}, 1)$. By Table 1, we get the groups of Type (C1)–(C8).

$w(G)$	y_{11}	Remark 1	$w(\bar{G})$	Group	Remark 2
(a)	z^{-1}	$w_{11} = \nu z^2$	$\begin{pmatrix} \nu & w_{12}z^{-1} \\ 0 & w_{22} \end{pmatrix}$	(C1) if $w_{22} \neq 0$ (C2) if $w_{22} = 0$	$s = (w_{22})^{-1}$ $t = w_{12}z^{-1}$
(b)	w_{12}^{-1}		$\begin{pmatrix} 0 & 1 \\ -1 & w_{22} \end{pmatrix}$	(C3)	
(c)	w_{12}^{-1}		$\begin{pmatrix} 0 & 1 \\ w_{21}w_{12}^{-1} & 0 \end{pmatrix}$	(C4) if $w_{21} \neq 0$ (C5) if $w_{21} = 0$	$s = -(w_{21})^{-1}w_{12}$
(d)	$-w_{21}^{-1}$		$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$	(C6)	
(e)			$\begin{pmatrix} 0 & 0 \\ 0 & w_{22} \end{pmatrix}$	(C7) if $w_{22} \neq 0$ (C8) if $w_{22} = 0$	$s = w_{22}^{-1}$

Table 1 Subcase 1.2 in Theorem 3.3

Case 2 $n > m$.

Subcase 2.1 $v(G) = (0,0)^T$.

Let G and $\bar{G}$ be two groups such that $v(G) = v(\bar{G}) = (0,0)^T$. By Theorem 3.2, $G \cong \bar{G}$ if and only if there exists $Y = \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{pmatrix}$, an invertible matrix over F_p such that $w(\bar{G}) = Y_1 w(G) Y^T$, where $Y_1 = \begin{pmatrix} y_{11} & 0 \\ y_{21} & y_{22} \end{pmatrix}$.

By suitably choosing y_{21} and y_{12}, that is, using an elementary row operation and an elementary column operation, we can simplify $w(G)$ to be such a matrix, in which every column and every row have at most one non-zero entry. In the following, we assume that both $w(G) = (w_{ij})$ and $w(\bar{G}) = (\bar{w}_{ij})$ are such matrices. It is easy to check that (i) for all possible subscripts i, j, $\bar{w}_{ij} \neq 0$ if and only if $w_{ij} \neq 0$; (ii) $G \cong \bar{G}$ if and only if there exists $Y = \mathrm{diag}(y_{11}, y_{22})$, an invertible matrix over F_p, such that $w(\bar{G}) = Y w(G) Y$.

If $w(G) = \begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}$ where $w_{12} w_{21} \neq 0$, then letting $Y = \mathrm{diag}(w_{12}^{-1}, 1)$, we have $w(\bar{G}) = Y w(G) Y = \begin{pmatrix} 0 & 1 \\ w_{21} w_{12}^{-1} & 0 \end{pmatrix}$. Hence we get the group of type (D1) where $s = -w_{21}^{-1} w_{12}$. It is easy to see that different s gives non-isomorphic groups.

If $w(G) = \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}$ where $w_{12} w_{21} \neq 0$, then letting $Y = \mathrm{diag}(y_{12}, y_{22})$, we have $w(\bar{G}) = Y w(G) Y = \begin{pmatrix} w_{11} y_{11}^2 & 0 \\ 0 & w_{22} y_{22}^2 \end{pmatrix}$. Hence we can simplify $w(G)$ to be $\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$ where $\nu_1, \nu_2 = 1$ or a fixed quadratic non-residue modular p. Thus we get the group of type (D2). It is easy to see that different ν_1 or ν_2 gives non-isomorphic groups.

If $w(G)$ is invertible, then $w(G)$ is one of the above types. If $w(G)$ is of rank 1, then $w(G)$ is one of the following types:

(a) $\begin{pmatrix} w_{11} & 0 \\ 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 \\ 0 & w_{22} \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 0 \\ w_{21} & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & w_{12} \\ 0 & 0 \end{pmatrix}$.

By similar arguments as above, we get the groups of type (D3)–(D6), respectively. If $w(G) = 0$, then G is the group of type (D7).

Subcase 2.2 $v(G) \neq (0,0)^T$.

If $w_{13} \neq 0$, then, letting $Y_1 = \begin{pmatrix} w_{13}^{-1} & 0 \\ w_{13}^{-1} w_{23} & -1 \end{pmatrix}$, we have $Y_1 v(G) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $w_{13} = 0$, then $w_{23} \neq 0$. Let $Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & w_{23}^{-1} \end{pmatrix}$. Then $Y_1 v(G) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By Theorem 3.2, $v(G) = (1,0)^T$ and $(0,1)^T$ respectively are mutually non-isomorphic.

Subcase 2.2.1 $v(G) = (1,0)^T$.

By calculation, $\begin{pmatrix} y_{11} & 0 \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if and only if $y_{21} = 0$ and $y_{11} = 1$. Suppose that G and $\bar{G}$ are two groups described in that theorem with $v(G) = v(\bar{G})(1,0)^T$. By Theorem 3.2, $G \cong \bar{G}$ if and only if there exist $Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & y_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & y_{12} \\ 0 & y_{22} \end{pmatrix}$ where y_{22} such that $w(\bar{G}) = Y_1 w(G) Y^T$.

By suitable choosing y_{12}, that is, using an elementary column operation, we can simplify $w(G)$ to be one of the following types:

(a) $\begin{pmatrix} w_{11} & w_{12} \\ 0 & w_{22} \end{pmatrix}$ where $w_{22} \neq 0$, (b) $\begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}$ where $w_{12} \neq 0$,
(c) $\begin{pmatrix} w_{11} & 0 \\ w_{21} & 0 \end{pmatrix}$ where $w_{21} \neq 0$, (d) $\begin{pmatrix} w_{11} & 0 \\ 0 & 0 \end{pmatrix}$.

In the following, we may assume that both $w(G)$ and $w(\bar{G})$ are such matrix. It is easy to check that (i) different types give non-isomorphic groups;(ii) $G \cong \bar{G}$ if and only if there exists $Y = \mathrm{diag}(1, y_{22})$, an invertible matrix over F_p, such that $w(\bar{G}) = Y w(G) Y$. By Table 2, we get the groups of types (E1)–(E7).

$w(G)$	y_{22}	$w(\bar{G})$	Group	Remark
(a) where $w_{22} = \nu z^2$	z^{-1}	$\begin{pmatrix} w_{11} & w_{12}z^{-1} \\ 0 & \nu \end{pmatrix}$	(E1) if $\mathrm{w}_{11} \neq 0$ (E2) if $\mathrm{w}_{11} = 0$	$s = (w_{11})^{-1}$ $t = w_{12}z^{-1}$
(b)	w_{12}^{-1}	$\begin{pmatrix} 0 & 1 \\ w_{21}w_{12}^{-1} & 0 \end{pmatrix}$	(E3) if $\mathrm{w}_{21} \neq 0$ (E4) if $\mathrm{w}_{21} = 0$	$s = -w_{21}^{-1}w_{12}$
(c)	$-w_{21}^{-1}$	$\begin{pmatrix} w_{11} & 0 \\ -1 & 0 \end{pmatrix}$	(E5)	$s = w_{11}$
(d)		$\begin{pmatrix} w_{11} & 0 \\ 0 & 0 \end{pmatrix}$	(E6) if $\mathrm{w}_{11} \neq 0$ (E7) if $\mathrm{w}_{11} = 0$	$s = w_{11}^{-1}$

Table 2 Subcase 2.2.1 in Theorem 3.3

Subcase 2.2.2 $v(G) = (0,1)^T$.

By calculation, $\begin{pmatrix} y_{11} & 0 \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if and only if $y_{22} = 1$. Let G and $\bar{G}$ be two groups described in theorem with $v(G) = v(\bar{G}) = (0,1)^T$. By Theorem 3.2, $G \cong \bar{G}$ if and only if there exist $Y = \begin{pmatrix} y_{11} & y_{12} \\ 0 & 1 \end{pmatrix}$ and $Y_1 = \begin{pmatrix} y_{11} & 0 \\ y_{21} & 1 \end{pmatrix}$, invertible matrices over F_p, such that $w(\bar{G}) = Y_1 w(G) Y^T$.

By suitably choosing y_{21} and y_{12}, that is, using an elementary row operation and an elementary column operation, we can simplify $w(G)$ to be such a matrix, in which every column and every row have at most one non-zero entry. In the following, we assume that both $w(G) = (w_{ij})$ and $w(\bar{G}) = (\bar{w}_{ij})$ are such matrices. It is easy to check that (i) for all possible subscripts i, j, $\bar{w}_{ij} \neq 0$ if and only if $w_{ij} \neq 0$; (ii) $G \cong \bar{G}$ if and only if there exists $Y = \mathrm{diag}(y_{11}, 1)$, an invertible matrix over F_p, such that $w(\bar{G}) = Y w(G) Y$.

If $w(G) = \begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}$ where $w_{12}w_{21} \neq 0$, then letting $Y = \mathrm{diag}(w_{12}^{-1}, 1)$, we have

$w(\bar{G}) = Yw(G)Y = \begin{pmatrix} 0 & 1 \\ w_{21}w_{12}^{-1} & 0 \end{pmatrix}$. Hence we get the group of type (F1) where $s = -w_{21}^{-1}w_{12}$. It is easy to see that different s gives non-isomorphic groups.

If $w(G) = \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}$ where $w_{12}w_{21} \neq 0$, then letting $Y = \mathrm{diag}(y_{12}, 1)$, we have $w(\bar{G}) = Yw(G)Y = \begin{pmatrix} w_{11}y_{11}^2 & 0 \\ 0 & w_{22} \end{pmatrix}$. Hence we can simplify $w(G)$ to be $\begin{pmatrix} \nu & 0 \\ 0 & w_{22} \end{pmatrix}$ where $\nu = 1$ or a fixed quadratic non-residue modular p. Thus we get the group of type (F2). It is easy to see that different ν gives non-isomorphic groups.

If $w(G)$ is invertible, then $w(G)$ is one of the above types. If $w(G)$ is of rank 1, then $w(G)$ is one of the following types:

(a) $\begin{pmatrix} w_{11} & 0 \\ 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 \\ 0 & w_{22} \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 0 \\ w_{21} & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & w_{12} \\ 0 & 0 \end{pmatrix}$.

By similar arguments as above, we get the groups of type (F3)–(F6), respectively. If $w(G) = 0$, then G is the group of type (F7). $\Box$

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