Journal of Mathematical Research with Applications
Jul., 2016, Vol. 36, No. 4, pp. 467–476
DOI:10.3770/j.issn:2095-2651.2016.04.009
Http://jmre.dlut.edu.cn

A Unicity Theorem Related to Multiple Values and Derivatives of Meromorphic Functions on Annuli

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Abstract In this paper, we first obtain the famous Xiong Inequality of meromorphic functions on annuli. Next we get a uniqueness theorem of meromorphic function on annuli concerning to their multiple values and derivatives by using the inequality.

Keywords uniqueness theorem; Nevanlinna value; annuli; meromorphic function

MR(2010) Subject Classification 30D35; 39B12

1. Introduction and main results

It is assumed that the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as the characteristic function $T(r,f)$, the proximate function $m(r,f)$, the counting function $N(r,f)$, and so on [1,2].

The uniqueness of meromorphic functions in the complex plane $\mathbb{C}$ is an important subject in the value distribution theory. In 1926, Nevanlinna [3] proved his famous five-value theorem: For two nonconstant meromorphic functions f and g in $\mathbb{C}$, if they have the same inverse images (ignoring multiplicities) for five distinct values, then $f(z)\equiv g(z)$. After this work, the uniqueness of meromorphic functions with shared values in $\mathbb{C}$ attracted many investigations (references, see the book [4] or some recent papers [5–7]). Here we shall mainly study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. By the Doubly Connected Mapping Theorem [8] each doubly connected domain is conformally equivalent to the annulus $\{z: r<|z|<R\}$, $0\le r<R\le+\infty$. We consider only two cases: $r=0$, $R=+\infty$ simultaneously and $0<r<R<+\infty$. In the latter case the homothety $z\mapsto\frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z:\frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R}{r}}$. In two cases every annulus is invariant with respect to the inversion $z\mapsto\frac{1}{z}$.

Recently, Khrystiyanyn and Kondratyuk [9,10] proposed the Nevanlinna theory for meromorphic functions on annulus (see also an important paper [11]). We will show the basic notions of the Nevanlinna value on annulus in the next section. Thus, it is interesting to consider the

Received March 13, 2015; Accepted June 8, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11126327), the Natural Science Foundation of Guangdong Province (Nos. 2016A030313002; 2015A030313644) and the Training Plan for the Outstanding Young Teachers in Higher Education of Guangdong Province (Grant No. Yq2013159).
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uniqueness theory of meromorphic functions on annulus. The main purpose of this paper is to deal with this subject.

2. Basic notions in the Nevanlinna theory on annuli

Let $f(z)$ be a family of meromorphic function on the annuli $A(R_0)=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. We recall the classical notations of Nevanlinna value as follows

$$N(R,f)=\int_0^R\frac{n(t,f)-n(0,f)}{t}\mathrm{d}t+n(0,f)\log R,$$
$$m(R,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|\mathrm{d}\theta,$$
$$T(R,f)=N(R,f)+m(R,f)$$

where $\log^+x=\max\{\log x,0\}$, $n(t,f)$ is the counting function of poles of f in $\{z:|z|\le t\}$.

Here we show the notations of Nevanlinna value on the annuli. Let f be a nonconstant meromorphic function on the annulus $A(R_0)=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. Denote

$$m(R,\frac{1}{f-a})=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(Re^{i\theta}-a)|}\mathrm{d}\theta,\quad m(R,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|\mathrm{d}\theta,$$

where $a\in C$ and $\frac{1}{R_0}<R<R_0$. Let

$$m_0(R,\frac{1}{f-a})=m(R,\frac{1}{f-a})+m(\frac{1}{R},\frac{1}{f-a}),\quad 1<R<R_0$$

and

$$m_0(R,f)=m(R,f)+m(\frac{1}{R},f),\quad 1<R<R_0.$$

Put

$$N_1(R,\frac{1}{f-a})=\int_{\frac{1}{R}}^1\frac{n_1(t,\frac{1}{f-a})}{t}\mathrm{d}t,\quad N_2(R,\frac{1}{f-a})=\int_1^R\frac{n_2(t,\frac{1}{f-a})}{t}\mathrm{d}t,$$

where $n_1(t,\frac{1}{f-a})$ is the counting function of zeros of the function $f-a$ in $\{z:t<|z|\le1\}$ and $n_2(t,\frac{1}{f-a})$ is the counting function of zeros of the function $f-a$ in $\{z:1<|z|\le t\}$. Denote also

$$N_1(R,f)=\int_{\frac{1}{R}}^1\frac{n_1(t,f)}{t}\mathrm{d}t,\quad N_2(R,f)=\int_1^R\frac{n_2(t,f)}{t}\mathrm{d}t,$$

where $n_1(t,f)$ is the counting function of poles of the function f in $\{z:t<|z|\le1\}$ and $n_2(t,f)$ is the counting function of poles of the function f in $\{z:1<|z|\le t\}$. Let

$$N_0(R,\frac{1}{f-a})=N_1(R,\frac{1}{f-a})+N_2(R,\frac{1}{f-a}),$$
$$N_0(R,f)=N_1(R,f)+N_2(R,f).$$

Denote

$$\overline{N}_0(R,\frac{1}{f-a})=\int_{\frac{1}{R}}^1\frac{\overline{n}_1(R,\frac{1}{f-a})}{t}\mathrm{d}t+\int_1^R\frac{\overline{n}_2(R,\frac{1}{f-a})}{t}\mathrm{d}t$$

$$= \overline{N}_1(R, \frac{1}{f-a}) + \overline{N}_2(R, \frac{1}{f-a})$$

where $\overline{n}_1(R, \frac{1}{f-a})$ is the counting function of zeros of the function of $f-a$ in $\{z : t < |z| \le 1\}$ (ignoring multiplicity) and $\overline{n}_2(R, \frac{1}{f-a})$ is the counting function of zeros of the function of $f-a$ in $\{z : 1 < |z| \le t\}$ (ignoring multiplicity).

In addition, we use $\overline{n}_1^{k)}(t, \frac{1}{f-a})$ (or $\overline{n}_1^{(k}(t, \frac{1}{f-a})$) to denote the counting function of zeros of the functions $f-a$ with multiplicities $\le k$ (or $> k$) in $\{z : t < |z| \le 1\}$, and we use $\overline{n}_2^{k)}(t, \frac{1}{f-a})$ (or $\overline{n}_2^{(k}(t, \frac{1}{f-a})$) to denote the counting function of zeros of the functions $f-a$ with multiplicities $\le k$ (or $> k$) in $\{z : 1 < |z| \le R\}$, each point counted only once.

Similarly, we can give the notations $\overline{N}_1^{k)}(t, f)$, $\overline{N}_2^{k)}(t, f)$, $\overline{N}_0^{k)}(t, f)$, $\overline{N}_1^{(k}(t, f)$, $\overline{N}_2^{(k}(t, f)$, $\overline{N}_0^{(k}(t, f)$.

We first define the Nevanlinna characteristic of f on $A(R_0)$ by

$$T_0(R, f) = m_0(R, f) - 2m(1, f) + N_0(R, f), \quad 1 < R_0 \le +\infty.$$

Then, we can define the deficiency by

$$\delta_0(a, f) = \delta_0(0, f-a) = \liminf_{r \mapsto R_0} \frac{m_0(r, \frac{1}{f-a})}{T_0(r, f)} = 1 - \limsup_{r \mapsto R_0} \frac{N_0(r, \frac{1}{f-a})}{T_0(r, f)}$$

and the reduced deficiency by

$$\Theta_0(a, f) = \Theta_0(0, f-a) = 1 - \limsup_{r \mapsto R_0} \frac{\overline{N}_0(r, \frac{1}{f-a})}{T_0(r, f)}.$$

Suppose that f, g are two meromorphic functions on $A(R_0)$, where $1 < R_0 \le +\infty$. Then

$$m_0(R, fg) \le m_0(R, f) + m_0(R, g) + O(1). \tag{2.1}$$

Lemma 2.1 (Generalization of Jensen's theorem [9, Theorem 1]) *Let f be a nonconstant meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Then*

$$N_0(R, \frac{1}{f}) - N_0(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\frac{1}{Re^{i\theta}})| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \frac{1}{\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

for every R such that $1 < R < R_0$.

Lemma 2.2 ([9]) *Let f be a nonconstant meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Then*

(i) $T_0(R, f) = T_0(R, \frac{1}{f})$,

(ii) $\max\{T_0(R, f_1 \cdot f_2), T_0(R, \frac{f_1}{f_2}), T_0(R, f_1 + f_2)\} \le T_0(R, f_1) + T_0(R, f_2) + O(1)$.

By Lemma 2.2, the first fundamental theorem on the annulus $A(R_0)$ is immediately obtained.

Lemma 2.3 (The first fundamental theorem [9, Theorem 2]) *Let f be a nonconstant meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let $T_0(R, f)$*

be its Nevanlinna characteristic functions. Then

$$T_0(R, \frac{1}{f-a}) = T_0(R, f) + O(1), \quad 1 < R < R_0,$$

for every fixed $a \in \mathbb{C}$.

Lemma 2.4 (Lemma on the logarithmic derivative [10, Theorem 1]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$ *and let* $\lambda \geq 0$. *Then*

(1) In the case $R_0 = +\infty$,

$$m_0(R, \frac{f'}{f}) = O\big(\log(RT_0(R, f))\big)$$

for $R \in (1, +\infty)$ *except for the set* $\triangle_R$ *such that* $\int_{\triangle_R} R^{\lambda-1}\mathrm{d}R < +\infty$;

(2) In the case $R_0 < +\infty$,

$$m_0(R, \frac{f'}{f}) = O\big(\log(\frac{T_0(R, f)}{R_0 - R})\big)$$

for $R \in (1, R_0)$ *except for the set* $\triangle_R{}'$ *such that* $\int_{\triangle_R{}'} \frac{1}{(R_0-R)^{\lambda-1}}\mathrm{d}R < +\infty$.

Lemma 2.5 (The second fundamental theorem [12, Theorem 2.2]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, a_2, \ldots, a_p$ *be* p *distinct finite complex numbers and* $\lambda \geq 0$. *Then*

$$m_0(R, f) + \sum_{\nu=1}^{p} m_0(R, \frac{1}{f - a_\nu}) \leq 2T_0(R, f) - N_0^{(1)}(R, f) + S(R, f),$$

where

$$N_0^{(1)}(R, f) = N_0(R, \frac{1}{f'}) + 2N_0(R, f) - N_0(R, f'),$$

and

(1) In the case $R_0 = +\infty$,

$$S(R, f) = O(\log(RT_0(R, f)))$$

for $R \in (1, +\infty)$ *except for the set* $\triangle_R$ *such that* $\int_{\triangle_R} R^{\lambda-1}\mathrm{d}R < +\infty$;

(2) In the case $R_0 < +\infty$,

$$S(R, f) = O(\log(\frac{T_0(R, f)}{R_0 - R}))$$

for $R \in (1, R_0)$ *except for the set* $\triangle_R{}'$ *such that* $\int_{\triangle_R{}'} \frac{1}{(R_0-R)^{\lambda-1}}\mathrm{d}R < +\infty$.

Khrystiyanyn and Kondratyuk also obtained the second fundamental theorem on the annulus A. We show here the reduced form due to Cao, Yi and Xu.

Lemma 2.6 (The reduced second fundamental theorem [13,14]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let*

$a_1, a_2, \ldots, a_p$ *be* p *distinct finite complex numbers and* $\lambda \geq 0$. *Then*

$$(q-2)T_0(R,f) < \sum_{j=1}^{q} \overline{N}_0(R, \frac{1}{f-a_j}) + S(R,f).$$

From the lemma on the logarithmic derivative and the second fundamental theorem, it is easy to get the following theorem. Then there are

Lemma 2.7 ([15]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Then*

$$m_0(R, \frac{f^{(k)}}{f}) = S(R, f^{(k)}) = S(R,f).$$

Lemma 2.8 ([12]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* a *be an arbitrary complex number and* k *be a positive integer. Then*

(i) $\overline{N}_0(R, \frac{1}{f-a}) \leq \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f-a}) + \frac{1}{k+1}N_0(R, \frac{1}{f-a})$,

(ii) $\overline{N}_0(R, \frac{1}{f-a}) \leq \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f-a}) + \frac{1}{k+1}T_0(R,f) + O(1)$.

Then, we can introduce other interesting forms of the second fundamental theorem on annulus about the these notations as follows, which are similar to those on the complex plane $\mathbb{C}$.

Lemma 2.9 ([12, Theorem 2.3]) *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, a_2, \ldots, a_q$ *be* q *distinct complex numbers in the extended complex plane* $\overline{C} = C \cup \{\infty\}$, *let* $k_1, k_2, \ldots, k_q$ *be* q *positive integers and let* $\lambda \geq 0$. *Then*

(i) $(q-2)T_0(R,f) < \sum_{j=1}^{q} \frac{k_j}{k_j+1}\overline{N}_0^{k_j)}(R, \frac{1}{f-a_j}) + \sum_{j=1}^{q} \frac{1}{k_j+1}N_0(R, \frac{1}{f-a_j}) + S(R,f)$,

(ii) $(q-2-\sum_{j=1}^{q} \frac{1}{k_j+1})T_0(R,f) < \sum_{j=1}^{q} \frac{k_j}{k_j+1}\overline{N}_0^{k_j)}(R, \frac{1}{f-a_j}) + S(R,f)$

where

$$N_0^{(1)}(R,f) = N_0(R, \frac{1}{f'}) + 2N_0(R,f) - N_0(R,f').$$

and $S(R,f)$ *satisfies the properties (i) and (ii) mentioned in Lemma 2.5.*

3. Multiple values and uniqueness of meromorphic functions on annuli

Let f be a nonconstant meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a be a complex number in the extended complex plane $\overline{C} = \mathbb{C} \cup \{\infty\}$. Write $E(a,f) = \{z \in A(R_0) : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,f)$. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f-a$ with multiplicity no greater than k, in which each zero is counted only once.

Definition 3.1 *Let* f *be a nonconstant meromorphic function on the annulus* $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *We call* f *admissible or transcendental if*

$$\limsup_{R\to\infty} \frac{T_0(R,f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty$$

or

$$\limsup_{R\to R_0} \frac{T_0(R,f)}{-\log(R_0-R)} = \infty, \quad 1 < R < R_0 < +\infty.$$

Thus for a transcendental or admissible meromorphic function on the annulus $A(R_0)$, $S(R,f) = o(T_0(R,f))$ holds for all $1 < R_0 \le \infty$ except for the set $\triangle_R$ or the set $\triangle_R{}'$ mentioned in Lemma 2.4, respectively.

To prove a unicity theorem related to multiple values and derivatives of meromorphic functions on annuli, we need to get the following Xiong inequality of meromorphic functions on annuli.

Lemma 3.2 *Let $f(z)$ be an admissible or transcendental meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let a be a finite complex number and $b_1, b_2, \ldots, b_q$ be q distinct finite non-zero complex numbers and k be a natural number. Then we have*

$$\begin{aligned} qT_0(R,f) \le &\overline{N}_0(R,f) + qN_0(R,\frac{1}{f-a}) + \sum_{j=1}^{q} N_0(R,\frac{1}{f^{(k)}-b_j}) - (q-1)N_0(R,\frac{1}{f^{(k)}}) - \\ &N_0(R,\frac{1}{f^{(k+1)}}) + S(R,f). \end{aligned} \tag{3.1}$$

Proof From [11], we have

$$\begin{aligned} T_0(R,f') =&T_0(R,f\frac{f'}{f}) \le T_0(R,f) + T_0(R,\frac{f'}{f}) + O(1) \\ =&T_0(R,f) + m_0(R,\frac{f'}{f}) + N_0(R,\frac{f'}{f}) - 2m(1,\frac{f'}{f}) + O(1) \\ =&T_0(R,f) + \overline{N}_0(R,f) + S(R,f) \\ =&2T_0(R,f) + S(R,f). \end{aligned} \tag{3.2}$$

Hence, by Lemma 2.4 and (3.2), we have

$$S(R,f^{(k)}) = O(\log RT_0(R,f^{(k)})) = O(\log RT_0(R,f)) = S(R,f). \tag{3.3}$$

$$m_0(R,\frac{f^{(k)}}{f-a_i}) = S(R,f). \tag{3.4}$$

From Lemma 2.4, (3.3) and (3.4), we have

$$m_0(R,\frac{f^{(k)}}{\prod_{i=1}^{p}(f-a_i)}) = S(R,f^{(k)}), \quad m_0(R,\frac{f^{(k+1)}}{f^{(k)}\prod_{j=1}^{q}(f^{(k)}-b_j)}) = S(R,f^{(k)})$$

and

$$\frac{1}{\prod_{i=1}^{p}(f-a_i)^n} = \{\frac{f^{(k)}}{\prod_{i=1}^{p}(f-a_i)}\}^n \cdot \frac{f^{(k+1)}}{f^{(k)}\prod_{j=1}^{q}(f^{(k)}-b_j)} \cdot \frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}.$$

Then

$$nm_0(R,\frac{1}{\prod_{i=1}^{p}(f-a_i)}) \le m_0(R,\frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}) + S(R,f^{(k)}). \tag{3.5}$$

From Lemmas 2.1 and 2.4, (2.1) and (3.3), then

$$\begin{aligned} m_0(R, \frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}) =& N_0(R, \frac{(f^{(k)})^{n-1}f^{(k+1)}}{\prod_{j=1}^{q}(f^{(k)}-b_j)}) - N_0(R, \frac{\prod_{j=1}^{q}(f^{(k)}-b_j)}{(f^{(k)})^{n-1}f^{(k+1)}}) + S(R, f^{(k)}) \\ =& \overline{N}_0(R, f) - (q-n)N_0(R, f^{(k)}) + \sum_{j=1}^{q} N_0(R, \frac{1}{f^{(k)}-b_j}) - \\ & (n-1)N_0(R, \frac{1}{f^{(k)}}) - N_0(R, \frac{1}{f^{(k+1)}}) + S(R, f^{(k)}). \end{aligned} \tag{3.6}$$

From Lemma 2.1 and (3.2), (3.3), the left of (3.6) can be replaced by

$$\begin{aligned} nm_0(R, \frac{1}{\prod_{i=1}^{p}(f-a_i)}) =& nT_0(R, \prod_{i=1}^{p}(f-a_i)) - nN_0(R, \frac{1}{\prod_{i=1}^{p}(f-a_i)}) + O(1) \\ =& npT_0(R, f) - n\sum_{i=1}^{p} N_0(R, \frac{1}{f-a_i}) + S(R, f^{(k)}). \end{aligned} \tag{3.7}$$

Put (3.6) and (3.7) into (3.5), then we have

$$\begin{aligned} npT_0(R, f) \le & \overline{N}_0(R, f) + n\sum_{i=1}^{p} N_0(R, \frac{1}{f(z)-a_i}) + \sum_{j=1}^{q} N_0(R, \frac{1}{f^{(k)}-b_j}) - \\ & (q-n)N_0(R, f^{(k)}) - (n-1)N_0(R, \frac{1}{f^{(k)}}) - N(R, \frac{1}{f^{(k+1)}}) + S(R, f). \end{aligned}$$

Let $n = q$, $p = 1$, we get the inequality (3.1). The proof of Lemma 3.2 is completed. □

By Lemma 3.2, we can get the following lemma.

Lemma 3.3 *Let $f(z)$ be an admissible or transcendental meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$ where $1 < R_0 \le +\infty$ and b_1, b_2, b_3 are three distinct finite non-zero complex numbers. Then, we have*

$$3T_0(R, f) < \overline{N}_0(R, f) + 3N_0(R, \frac{1}{f}) + \sum_{j=1}^{3} \overline{N}_0(R, \frac{1}{f^{(k)}-b_j}) + S(R, f).$$

We now show our main result below which is an analog of a result on the plane $\mathbb{C}$ obtained by Yi [16] (see [4, Theorem 3.36]).

Theorem 3.4 *Let f_1, f_2 be two admissible or transcendental meromorphic functions on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let b_1, b_2, b_3 be three distinct complex numbers in the extended complex plane $\overline{C} = \mathbb{C} \cup \{\infty\}$, let k be a positive integer or ∞ and let n be a positive integer satisfying*

$$\overline{E}_{k)}(b_j, f_1^{(n)}) = \overline{E}_{k)}(b_j, f_2^{(n)}), \quad j = 1, 2, 3. \tag{3.8}$$

Furthermore, let

$$C_i = 3(k+1)\delta_0(0, f_i) + (2nk + 3n + k + 1)\Theta_0(\infty, f_i) - (2nk + 3n + 3k + 4), \quad i = 1, 2.$$

If

$$\min\{C_1, C_2\} \ge 0, \tag{3.9}$$

$$\max\{C_1, C_2\} > 0, \tag{3.10}$$

Then, $f_1(z) \equiv f_2(z)$.

Proof By Lemma 3.8, we have

$$3T_0(R, f_1) < \overline{N}_0(R, f_1) + 3N_0(R, \frac{1}{f_1}) + \sum_{j=1}^{3} \overline{N}_0(R, \frac{1}{f_1^{(n)} - b_j}) + S(R, f). \tag{3.11}$$

Note that

$$T_0(R, f_1^{(n)}) \le T_0(R, f_1) + n\overline{N}_0(R, f_1) + S(R, f_1).$$

We deduce that

$$\begin{aligned}\overline{N}_0(R, \frac{1}{f_1^{(n)} - b_j}) <& \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + \frac{1}{k+1}T_0(R, f_1^{(n)}) + O(1)\\ <& \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + \frac{1}{k+1}T_0(R, f_1) +\\ & \frac{n}{k+1}\overline{N}_0(R, f_1) + S(R, f_1).\end{aligned} \tag{3.12}$$

From (3,11) and (3.12), we can get

$$\begin{aligned}3T_0(R, f_1) <& \frac{3n+k+1}{k+1}\overline{N}_0(R, f_1) + 3N_0(R, \frac{1}{f_1}) + \frac{3}{k+1}T_0(R, f_1) +\\ & \frac{k}{k+1}\sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + S(R, f_1)\\ <& \frac{3n+k+1}{k+1}(1 - \Theta_0(\infty, f_1))T_0(R, f_1) + 3(1 - \delta_0(0, f_1))T_0(R, f_1) +\\ & \frac{3}{k+1}T_0(R, f_1) + \frac{k}{k+1}\sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + S(R, f_1).\end{aligned}$$

Hence

$$\begin{aligned}&(3\delta_0(0, f_1) + \frac{3n+k+1}{k+1}\Theta_0(\infty, f_1) - \frac{3n+k+4}{k+1})T_0(R, f_1)\\ &< \frac{k}{k+1}\sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + S(R, f_1).\end{aligned}$$

Then

$$\{2k + 2nk(1 - \Theta_0(\infty, f_1)) + C_1\}T_0(R, f_1) < k\sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) + S(R, f_1). \tag{3.13}$$

By (3.8), we have

$$\begin{aligned}\sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) =& \sum_{j=1}^{3}\overline{N}_0^{k)}(R, \frac{1}{f_2^{(n)} - b_j})\\ \le& 3T_0(R, f_2^{(n)}) + O(1) \le 3(n+1)T_0(R, f_2) + S(R, f_2).\end{aligned} \tag{3.14}$$

Now that $C_1 \geq 0$. The inequalities (3.13) and (3.14) give

$$T_0(R, f_1) = O(T_0(R, f_2)), \quad R \to \infty, \ R \notin E. \tag{3.15}$$

Similarly, we have

$$T_0(R, f_2) = O(T_0(R, f_1)), \quad R \to \infty, \ R \notin E. \tag{3.16}$$

If $f_1^{(n)} \not\equiv f_2^{(n)}$, then from (3.8) and Lemma 2.2(ii) and Lemma 2.3, we have

$$\begin{aligned}
\sum_{j=1}^{3} \overline{N}_0^{k)}(R, \frac{1}{f_1^{(n)} - b_j}) = \sum_{j=1}^{3} \overline{N}_0^{k)}(R, \frac{1}{f_2^{(n)} - b_j}) &\leq N_0(R, \frac{1}{f_1^{(n)} - f_2^{(n)}}) \\
&\leq T_0(R, f_1^{(n)}) + T_0(R, f_2^{(n)}) + O(1) \\
&\leq T_0(R, f_1) + n\overline{N}_0(R, f_1) + S(R, f_1) + T_0(R, f_2) + n\overline{N}_0(R, f_2) + S(R, f_2) \\
&\leq T_0(R, f_1) + n(1 - \Theta_0(\infty, f_1))T_0(R, f_1) + S(R, f_1) + T_0(R, f_2) + \\
&\quad n(1 - \Theta_0(\infty, f_2))T_0(R, f_2) + S(R, f_2).
\end{aligned}$$

Substituting the above inequality into (3.13) gives

$$\begin{aligned}
&[k + nk(1 - \Theta_0(\infty, f_1)) + C_1]T_0(R, f_1) \\
&\quad < [k + nk(1 - \Theta_0(\infty, f_2))]T_0(R, f_2) + S(R, f_1) + S(R, f_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&[k + nk(1 - \Theta_0(\infty, f_2)) + C_2]T_0(R, f_2) \\
&\quad < [k + nk(1 - \Theta_0(\infty, f_1))]T_0(R, f_1) + S(R, f_1) + S(R, f_2).
\end{aligned}$$

From the above two inequalities, we have

$$C_1T_0(R, f_1) + C_2T_0(R, f_2) < S(R, f_1) + S(R, f_2). \tag{3.17}$$

By (3.9), (3.10), (3.15) and (3.16), the above inequality cannot hold, then $f_1^{(n)} \equiv f_2^{(n)}$, thus $f_1(z) \equiv f_2(z) + p(z)$, where $p(z)$ is a polynomial of at most degree $n - 1$.

From (3.9), we can see that $\delta_0(0, f_i) > 0, \Theta_0(\infty, f_i) > 0$ $(i = 1, 2)$. Therefore $f_i(z)$ $(i = 1, 2)$ must be transcendental meromorphic functions.

Hence $T_0(R, p) = o(T_0(R, f_i))$ $(i = 1, 2)$. If $p(z) \not\equiv 0$, then

$$\begin{aligned}
&\Theta_0(0, f_1) + \Theta_0(p, f_1) + \Theta_0(\infty, f_1) \geq \delta_0(0, f_1) + \delta_0(p, f_1) + \Theta_0(\infty, f_1) \\
&\quad = \delta_0(0, f_1) + \delta_0(0, f_2) + \Theta_0(\infty, f_1) \\
&\quad \geq \frac{2nk + 3n + 3k + 4}{3(k+1)} - (\frac{2nk + 3n + k + 1}{3(k+1)} - 1)\Theta_0(\infty, f_1) + \\
&\quad\quad \frac{2nk + 3n + 3k + 4}{3(k+1)} - \frac{2nk + 3n + k + 1}{3(k+1)}\Theta_0(\infty, f_2) \\
&\quad \geq \frac{1}{3(k+1)}[(2nk + 3n + 3k + 4) - (2nk + 3n - 2k - 2) + \\
&\quad\quad (2nk + 3n + 3k + 4) - (2nk + 3n + k + 1)] \\
&\quad = \frac{7k + 9}{3(k+1)} > 2.
\end{aligned}$$

This is impossible. Hence $p(z) \equiv 0$, thus $f_1(z) \equiv f_2(z)$.

From [12, Theorem 3.1] and Theorem 3.4, we can get the following corollary.

Corollary 3.5 *Let $f(z)$ be an admissible or transcendental meromorphic function on the annulus $A(R_0) = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$ and satisfying $6\delta_0(0, f) + (5n+2)\Theta_0(\infty, f) > 5n + 7$ for a positive integer n. Then $f(z)$ can be uniquely determined by $\overline{E}_{1)}(a_j, f)$ $(j = 1, 2, 3)$ or $\overline{E}_{1)}(b_j, f)$ $(j = 1, 2, 3)$, where a_j $(j = 1, 2, 3)$ and b_j $(j = 1, 2, 3)$ are two groups of finite non-zero complex numbers, and $a_i \ne a_j, b_i \ne b_j$ $(i \ne j)$.*

Acknowledgements We thank the referees for their time and comments.

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