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A Note on the Partial Stability of Retarded System Using Liapunov Functions

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Abstract Sufficient conditions for the stability with respect to part of the functional differential equation variables are given. These conditions utilize Lyapunov functions to determine the uniform stability and uniform asymptotic stability of functional differential equations. These conditions for the partial stability develop the Razumikhin theorems on uniform stability and uniform asymptotic stability of functional differential equations. An example is presented which demonstrates these results and gives insight into the new stability conditions.

Keywords retarded functional differential equation; solution; stability

MR(2010) Subject Classification 34K20

1. Introduction

The study of the stability of the retarded functional differential equation (RFDE) with respect to part of the variables, or the partial stability problem, naturally arises. The partial stability problem seeks to determine if an equilibrium is stable with respect to a specific subset of the RFDE variables. RFDEs are a general type of equations and they include ordinary differential equations and differential difference equations [1]. There are many areas where these types of the partial stability problems arise, and the partial stability problems were studied for ordinary differential equations (ODEs), and difference equations. For some examples and results in the area see, for example, [1–15] and the references cited therein. Vorotnikov and Martyshenko [2] considered the stability problem with respect to a part of variables of the zero equilibrium position for ODEs. As compared to known assumptions, more general assumptions are made on the initial values of variables non-controlled in the course of studying stability. In addition, a stability problem is considered with respect to a part of variables of the "partial" equilibrium position, with similar assumptions made for initial values of variables that do not define the given equilibrium position. On the results of ODEs, Fisher and Bhattacharya [3] proposed a methodology for algorithmic construction of Lyapunov functions for problems concerning the stability of an equilibrium with respect to part of the system variables. Conditions for stability with respect to part of the variables are developed that allow for Lyapunov functions to be determined in terms of a sum of squares. Asymptotic stability conditions in terms of sum of squares polynomials are developed for autonomous and non-autonomous systems. An example

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is presented which demonstrates the methodology. Hancock and Hill [5] used set invariance methods to ensure that the 'auxiliary' variables remain on a restricted domain, and then use this framework to develop new results for both local and global partial stability theory. On the results of the partial stability for difference equations, Xiao and Liao [14] obtained two results (sufficient conditions) for the partial stability (see Theorems 3.1 and 3.3). Boundedness problems of partial solutions of the RFDEs were also studied. For example, Zhao [16] studied uniform boundedness and uniform ultimate boundedness of solutions of RFDEs. Nonetheless, there are still very few results on the partial stability of RFDEs [1].

Liapunov functions are simpler than Liapunov functionals. In this paper, two results (sufficient conditions) for the partial stability (uniform stability and uniform asymptotic stability) of the RFDEs are given. These conditions utilize Liapunov functions to determine the partial stability. These conditions for the partial stability develop well-known Razumikhin theorems on uniform stability and uniform asymptotic stability of the RFDEs. An example is presented which demonstrates these results and gives insight into the new stability conditions.

Of course, the following definitions of uniform stability and uniform asymptotic stability, and these conditions for the Theorems 3.1 and 3.3 can still be improved. We will publish the further research results in another article.

2. Preliminaries

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $C = C([-r, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We designate the norm of an element ϕ in C by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. If

$$\sigma \in \mathbb{R}, A \geq 0 \quad \text{and} \quad x \in C([\sigma - r, \sigma + A], \mathbb{R}^n),$$

then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. $|x|$ is the norm of x. Let

$$\begin{aligned}
&x_{i\sim j} = (x_i, x_{i+1}, \ldots, x_j)^T \in \mathbb{R}^{j+1-i} \ (1 \leq i \leq j \leq n), \ x = x_{1\sim n};\\
&\mathbb{C}^i = C([-r, 0], \mathbb{R}^i), \ \phi_{i\sim j} = (\phi_i, \phi_{i+1}, \ldots, \phi_j)^T \in \mathbb{C}^{j+1-i}, \ \phi = \phi_{1\sim n};\\
&y_t = x_{1\sim m}(t + \theta), \ z_t = x_{m+1\sim n}(t + \theta) \ (1 \leq m \leq n).
\end{aligned}$$

Suppose $F : \mathbb{R} \times \mathbb{C}^m \times \mathbb{C}^{n-m} \to \mathbb{R}^n$ is continuous and consider retarded functional differential equation [1]

$$\dot{x}(t) = F(t, y_t, z_t) \quad (x_t = x(t + \theta) = (y_t^T, z_t^T)^T \in \mathbb{C}^n = C). \tag{1}$$

We will assume that there is a unique solution $x(t, t_0, \phi)$ of Eq. (1) through $(t_0, \phi) \in \mathbb{R} \times C$. Let $x(t) = x(t, t_0, \phi)$ $(x(t, t_0, \phi) = x_t(t_0, \phi) = x(t_0, \phi)(t))$.

Definition 2.1 *Suppose $F(t, 0, 0) = 0$ for all $t \in \mathbb{R}$ and $1 \leq m \leq n$. The zero solution of Eq. (1) is said to be uniformly stable with respect to $x_{1\sim m}$ if for any $t_0 \geq 0$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that $|\phi| < \delta$ $(\phi \in C)$ implies $|x_{1\sim m}(t, t_0, \phi)| < \varepsilon$ for $t \geq t_0$. The zero solution of Eq. (1)*

is said to be uniformly asymptotically stable with respect to $x_{1\sim m}$ *if it is uniformly stable with respect to* $x_{1\sim m}$ *and there is a* $b_0 > 0$ *such that, for every* $\eta > 0$*, there is a* $T(\eta) > 0$ *such that* $|\phi| < b_0$ *implies* $|x_{1\sim m}(t, t_0, \phi)| < \eta$ *for* $t \geq t_0 + T(\eta)$ *for every* $t_0 \geq 0$.

If $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ *is a continuous function, then* $\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0))$*, the derivative of* V *along the solutions of Eq. (1) is defined to be*

$$\begin{aligned}\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0)) = &\limsup_{\alpha\to 0^+}\frac{1}{\alpha}[V(t+\alpha, x_{1\sim m}(t,\phi)(t+\alpha), x_{m+1\sim n}(t,\phi)(t+\alpha)) - \\ &V(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0))].\end{aligned}$$

3. Main results

Suppose $F : \mathbb{R} \times \mathbb{C}^m \times \mathbb{C}^{n-m} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{C}^m$) $\times\, \mathbb{C}^{n-m}$ into bounded sets of $\mathbb{R}^n$ and $F(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. Suppose there are positive integers m and k with $1 \leq m \leq k \leq n$. Suppose $h, w_1, w_2, w_3 : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing functions, $h(0) = w_1(0) = w_2(0) = 0$, $h'(s) \geq 1$ for $s \geq 0$, $w_1(s), w_2(s), w_3(s) > 0$ for $s > 0$, w_2 strictly increasing. Suppose $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $g(s) \geq h(s)$ for $s \geq 0$.

Theorem 3.1 *If there is a continuous function* $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ *such that*

$$w_1(|x_{1\sim m}|) \leq V(t, x_{1\sim m}, x_{m+1\sim n}) \leq w_2(|x_{1\sim k}|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{2}$$

and

$$\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0)) \leq 0 \tag{3}$$

if there is a θ_0 *in* $[-r, 0]$ *such that*

$$h(V(t+\theta, \phi_{1\sim m}(\theta), x_{m+1\sim n})) \leq g(V(t+\theta_0, \phi_{1\sim m}(\theta_0), x_{m+1\sim n}))$$

for $\theta \in [-r,\ 0]$*, then the zero solution of Eq. (1) is uniformly stable with respect to* $x_{1\sim m}$.

Remark 3.2 The well-known Razumikhin theorems on uniform stability [1] become the consequence of Theorem 3.1 of this paper ($m = k = n$, $\theta_0 = 0$, $h(s) = g(s) = s$).

Theorem 3.3 *If there is a continuous function* $V : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ *such that*

$$w_1(|x_{1\sim m}|) \leq V(t, x_{1\sim m}, x_{m+1\sim n}) \leq w_2(|x_{1\sim k}|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{4}$$

and

$$\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0)) \leq -w_3(|\phi_{1\sim m}(0)|) \tag{5}$$

if there is a θ_0 *in* $[-r, 0]$ *such that*

$$h(V(t+\theta, \phi_{1\sim m}(\theta), x_{m+1\sim n})) \leq g(V(t+\theta_0, \phi_{1\sim m}(\theta_0), x_{m+1\sim n}))$$

for $\theta \in [-r,\ 0]$*, then the zero solution of Eq. (1) is uniformly asymptotically stable with respect to* $x_{1\sim m}$.

Remark 3.4 The well-known Razumikhin theorems on uniform asymptotic stability [1] become the consequence of Theorem 3.3 of this paper ($m = k = n$, $\theta_0 = 0$, $h(s) = s$, $g(s)$ is a continuous function, $g(s) \geq s$ for $s \geq 0$ ($p(s)$ is a continuous, nondecreasing function, $p(s) > s$ for $s > 0$, and $p(V(t, \phi(0))) > V(t + \theta, \phi(\theta))$)).

4. The proof of the theorems

Now we are in the position to prove our Theorem 3.1.

Proof of Theorem 3.1 For any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$, such that $w_2(\delta) < w_1(\varepsilon)$. If $|\phi| < \delta$ ($\phi \in C$), then using $|x_{1\sim k}| \leq |x|$, the inequality (2), and the fact that w_2 is nondecreasing, we have

$$V(t, x_{1\sim m}, x_{m+1\sim n}) \leq w_2(|x_{1\sim k}|) \leq w_2(|x|), \quad t \geq t_0, \tag{6}$$

and using $x(t_0, t_0, \phi) = \phi$ and the inequality (6), we get

$$V(t_0, x_{1\sim m}(t_0, t_0, \phi), x_{m+1\sim n}(t_0, t_0, \phi)) \leq w_2(|\phi|) \leq w_2(\delta). \tag{7}$$

If

$$\mathbb{V}(t, \phi_{1\sim m}, \phi_{m+1\sim n}) = \sup_{\theta \in [-r,\, 0]} V(t + \theta, \phi_{1\sim m}(\theta), \phi_{m+1\sim n}(0)) \tag{8}$$

for $t \in \mathbb{R}$, $\phi_{1\sim m} \in \mathbb{C}^m$, $\phi_{m+1\sim n} \in \mathbb{C}^{n-m}$, then there is a θ_0 in $[-r, 0]$ such that

$$\mathbb{V}(t, \phi_{1\sim m}, \phi_{m+1\sim n}) = V(t + \theta_0, \phi_{1\sim m}(\theta_0), \phi_{m+1\sim n}(0)) \tag{9}$$

and

$$\begin{aligned} V(t + \theta, \phi_{1\sim m}(\theta), \phi_{m+1\sim n}(0)) &\leq \sup_{\theta \in [-r,\, 0]} V(t + \theta, \phi_{1\sim m}(\theta), \phi_{m+1\sim n}(0)) \\ &= \mathbb{V}(t, \phi_{1\sim m}, \phi_{m+1\sim n}) = V(t + \theta_0, \phi_{1\sim m}(\theta_0), \phi_{m+1\sim n}(0)). \end{aligned} \tag{10}$$

Using the inequality (10), $g(s) \geq h(s)$ for $s \geq 0$, and the fact that h is nondecreasing, we have

$$\begin{aligned} &h(V(t + \theta, \phi_{1\sim m}(\theta), \phi_{m+1\sim n}(0))) \leq h(V(t + \theta_0, \phi_{1\sim m}(\theta_0), \phi_{m+1\sim n}(0))) \\ &\quad \leq g(V(t + \theta_0, \phi_{1\sim m}(\theta_0), \phi_{m+1\sim n}(0))), \quad \theta \in [-r, 0]. \end{aligned} \tag{11}$$

From (3) and (11), we obtain

$$\dot{V}(t, \phi_{1\sim m}(0), \phi_{m+1\sim n}(0)) \leq 0 \tag{12}$$

for all $t \geq t_0$. Using (2), (7), (12) and our choice of $w_2(\delta)$, we have [1]

$$\begin{aligned} &w_1(|x_{1\sim m}(t, t_0, \phi)|) \leq V(t, x_{1\sim m}(t, t_0, \phi), x_{m+1\sim n}(t, t_0, \phi)) \\ &\quad \leq V(t_0, x_{1\sim m}(t_0, t_0, \phi), x_{m+1\sim n}(t_0, t_0, \phi)) \leq w_2(\delta) < w_1(\varepsilon), \quad t \geq t_0. \end{aligned} \tag{13}$$

Using (13) and the fact that w_1 is nondecreasing, we have

$$|x_{1\sim m}(t, t_0, \phi)| < \varepsilon, \quad t \geq t_0. \tag{14}$$

The proof of the Theorem 3.1 is therefore completed. □

Proof of Theorem 3.3 From (4) and (5), we obtain (2) and (3); that is, Theorem 3.3 implies uniform stability. By Definition 2.1, for $\varepsilon = 1$, there is a $\delta_0 = \delta(\varepsilon) = \delta(1)$ such that $|\phi| < \delta_0$ ($\phi \in C$) implies

$$|x_{1\sim m}(t, t_0, \phi)| < 1 \tag{15}$$

for $t \geq t_0$ for any $t_0 \geq 0$. To complete the proof of the theorem, choose $b_0 = \delta_0 = \delta(1)$. For every $\eta > 0$, we claim that there exists a $T(\eta) > 0$ such that $|\phi| < b_0$ implies $|x_{1\sim m}(t, t_0, \phi)| < \eta$ for $t \geq t_0 + T(\eta)$ for every $t_0 \geq 0$. If this were no so, then there would exist an $\eta_0 > 0$, a $\phi_0 \in C$, $|\phi_0| < b_0$, a constant $\alpha_0 > 0$, and a sequence $\{t_j\}$, $t_j \to +\infty$ as $j \to +\infty$, $t_j - t_{j-1} \geq \alpha_0$ ($j = 1, 2, \ldots$) such that

$$|x_{1\sim m}(t_j, t_0, \phi_0)| \geq \eta_0. \tag{16}$$

Since (15) and the fact that F takes $\mathbb{R}\times$ (bounded sets of $\mathbb{C}^m$) $\times\, \mathbb{C}^{n-m}$ into bounded sets of $\mathbb{R}^n$, there exists a constant $M > 0$ such that

$$|\dot{x}(t, t_0, \phi_0)| \leq M, \quad t \geq t_0, \ |\phi_0| < b_0. \tag{17}$$

Let

$$\beta_0 = \min\{\frac{\eta_0}{4M}, \frac{\alpha_0}{4}\}. \tag{18}$$

If $t \in [t_j - \beta_0, t_j + \beta_0]$, then using (16)–(18), and the mean value theorem, we have

$$\begin{aligned}
&|x_{1\sim m}(t, t_0, \phi_0)| = |x_{1\sim m}(t_j, t_0, \phi_0) + (\dot{x}_1(\xi_1, t_0, \phi_0), \dot{x}_2(\xi_2, t_0, \phi_0), \ldots, \dot{x}_m(\xi_m, t_0, \phi_0))^T (t - t_j)| \\
&\quad \geq |x_{1\sim m}(t_j, t_0, \phi_0)| - |(\dot{x}_1(\xi_1, t_0, \phi_0), \dot{x}_2(\xi_2, t_0, \phi_0), \ldots, \dot{x}_m(\xi_m, t_0, \phi_0))^T| \times |t - t_j| \\
&\quad \geq \eta_0 - M \times 2\beta_0 \geq \eta_0 - (\eta_0/2) = \eta_0/2
\end{aligned} \tag{19}$$

for $t \in [t_j - \beta_0, t_j + \beta_0]$. From (5) and (11) (see (8)–(10)), we obtain

$$\dot{V}(t, x_{1\sim m}(t, t_0, \phi_0), x_{m+1\sim n}(t, t_0, \phi_0)) \leq -w_3(|x_{1\sim m}(t, t_0, \phi_0)|) \tag{20}$$

for all $t \geq t_0$. Using (19) and the fact that w_3 is nondecreasing, we have

$$-w_3(|x_{1\sim m}(t, t_0, \phi_0)|) \leq -w_3(\eta_0/2), \quad t \in [t_j - \beta_0, t_j + \beta_0]. \tag{21}$$

From $|\phi_0| < b_0$ ($\phi_0 \in C$), we obtain

$$V(t_0, x_{1\sim m}(t_0, t_0, \phi_0), x_{m+1\sim n}(t_0, t_0, \phi_0)) \leq w_2(|\phi_0|) \leq w_2(b_0) \quad \text{(see (6) and (7))}. \tag{22}$$

Using (4), (20)–(22), and the properties of the function w_1, we have

$$\begin{aligned}
0 \leq\, & w_1(|x_{1\sim m}(t_j + \beta_0, t_0, \phi_0)|) \\
\leq\, & V(t_j + \beta_0, x_{1\sim m}(t_j + \beta_0, t_0, \phi_0), x_{m+1\sim n}(t_j + \beta_0, t_0, \phi_0)) \\
=\, & V(t_0, x_{1\sim m}(t_0, t_0, \phi_0), x_{m+1\sim n}(t_0, t_0, \phi_0)) + \\
& \int_{t_0}^{t_j+\beta_0} \dot{V}(\tau, x_{1\sim m}(\tau, t_0, \phi_0), x_{m+1\sim n}(\tau, t_0, \phi_0)) \mathrm{d}\tau \leq w_2(b_0) + \\
& \int_{t_0}^{t_j+\beta_0} \dot{V}(\tau, x_{1\sim m}(\tau, t_0, \phi_0), x_{m+1\sim n}(\tau, t_0, \phi_0)) \mathrm{d}\tau \leq w_2(b_0) + \\
& \int_{t_0}^{t_j+\beta_0} -w_3(|x_{1\sim m}(\tau, t_0, \phi_0)|) \mathrm{d}\tau \leq w_2(b_0) +
\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{j}\int_{t_i-\beta_0}^{t_i+\beta_0} -w_3(|x_{1\sim m}(\tau,t_0,\phi_0)|)\mathrm{d}\tau \le w_2(b_0)+\\&\sum_{i=1}^{j}\int_{t_i-\beta_0}^{t_i+\beta_0} -w_3(\eta_0/2)\mathrm{d}\tau = w_2(b_0)-2j\beta_0 w_3(\eta_0/2).\end{aligned}\tag{23}$$

If J is the smallest integer $\ge w_2(b_0)/[2\beta_0 w_3(\eta_0/2)]$, then

$$w_2(b_0)-2J\beta_0 w_3(\eta_0/2)\le 0.\tag{24}$$

If $j\ge J+1$, then using (23) and (24), we have

$$\begin{aligned}0&\le V(t_j+\beta_0,x_{1\sim m}(t_j+\beta_0,t_0,\phi_0),x_{m+1\sim n}(t_j+\beta_0,t_0,\phi_0))\\&\le w_2(b_0)-2j\beta_0 w_3(\eta_0/2)\le w_2(b_0)-2\beta_0(J+1)w_3(\eta_0/2)\\&=w_2(b_0)-2\beta_0 Jw_3(\eta_0/2)-2\beta_0 w_3(\eta_0/2)\le 0-2\beta_0 w_3(\eta_0/2)<0,\end{aligned}$$

which is a contradiction. Therefore, there is a $b_0>0$ ($b_0=\delta_0=\delta(1)$) such that, for every $\eta>0$, there is a $T(\eta)>0$ such that $|\phi|<b_0$ implies $|x_{1\sim m}(t,t_0,\phi)|<\eta$ for $t\ge t_0+T(\eta)$ for every $t_0\ge 0$. This proves the uniform asymptotic stability. We complete the proof of the theorem. $\Box$

5. Example

We give the following example in order to demonstrate the Theorems 3.1 and 3.3 in this paper and give insight into the new stability conditions. It is easy to see that the well-known Razumikhin theorems on uniform stability and uniform asymptotic stability of RFDEs cannot apply to the following example.

Example 5.1 Consider the second-order equation

$$\begin{aligned}&\dot{x}(t)=y^{2m_1-1}(t),\\&\dot{y}(t)=-f(x(t))-\Phi(t,x(t),y(t))+\int_{-r}^{0}G(t+\theta,x(t+\theta),y(t+\theta))\mathrm{d}\theta,\end{aligned}\tag{25}$$

where $m_1=1,2,\ldots,M_1$. If $m_1=1$, $G(t+\theta,x(t+\theta),y(t+\theta))=y(t+\theta)\cdot \mathrm{d}f(x(t+\theta))/\mathrm{d}x(t+\theta)$, then Eq. (25) includes the second-order scalar equation

$$\ddot{x}(t)+\Phi(t,x(t),\dot{x}(t))+f(x(t-r))=0.\tag{26}$$

Eq. (26) includes the equation of controlling a ship [20, p.149], the sunflower equation [20, p.151] and the Zhao example [19, Eq. (2)]. Eq. (25) also is generalization of the Burton example [20, p.278]

$$\begin{aligned}&\dot{x}(t)=y(t),\\&\dot{y}(t)=-g(x(t))-\psi(x(t),y(t))y(t)+\int_{-r}^{0}g^*(x(t+s))y(t+s)\mathrm{d}s.\end{aligned}\tag{27}$$

We make the following assumptions on Eq. (25):

(a) $\Phi:\mathbb{R}^3\to\mathbb{R}$ is continuous, Φ takes $\mathbb{R}\times$ (bounded sets of $\mathbb{R}^2$) into bounded sets, $\Phi(t,0,0)=0$ for all $t\in\mathbb{R}$, and there is a constant $H>0$, such that $(\Phi(t,x,y)/y)\ge H$ for all

$t, x, y \in \mathbb{R}$ $(y \neq 0)$.

(b) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$ and $xf(x) > 0$ $(x \neq 0)$.

(c) $G : \mathbb{R}^3 \to \mathbb{R}$ is continuous, $G(t, 0, 0) = 0$ for all $t \in \mathbb{R}$, and there is a constant $L > 0$, such that $|G(t, x, y)| \leq L|y|$ for all $t, x, y \in \mathbb{R}$.

(d) $Lr = H$ $(Lr < H)$.

It is always assumed that a uniqueness result holds for the solutions of Eq. (25). Under the above hypotheses, we will show that the zero solution of Eq. (25) is uniformly stable (uniformly asymptotically stable) with respect to y.

If $V(x, y) = \int_0^x f(s)\mathrm{d}s+(y^{2m_1}/(2m_1))$, $h(s) = g(s) = s$, $h(V(x, y(t+\theta))) \leq g(V(x, y(t)))$ $(\theta_0 = 0)$, $\theta \in [-r, 0]$, then $|y(t + \theta)| \leq |y(t)|$ and

$$\begin{aligned}
&\dot{V}(x(t), y(t)) = f(x(t))\dot{x}(t) + y^{2m_1-1}(t)\dot{y}(t)\\
&= f(x(t))y^{2m_1-1}(t) + y^{2m_1-1}(t)\Big[- f(x(t)) - \Phi(t, x(t), y(t))+\\
&\quad \int_{-r}^{0} G(t + \theta, x(t + \theta), y(t + \theta))\mathrm{d}\theta\Big]\\
&= -y^{2m_1-1}(t)\Phi(t, x(t), y(t)) + y^{2m_1-1}(t)\int_{-r}^{0} G(t + \theta, x(t + \theta), y(t + \theta))\mathrm{d}\theta\\
&\leq -Hy^{2m_1}(t) + |y(t)|^{2m_1-1}\int_{-r}^{0} |G(t + \theta, x(t + \theta), y(t + \theta))|\mathrm{d}\theta\\
&\leq -Hy^{2m_1}(t) + |y(t)|^{2m_1-1}\int_{-r}^{0} L|y(t + \theta)|\mathrm{d}\theta\\
&\leq -Hy^{2m_1}(t) + |y(t)|^{2m_1-1}\int_{-r}^{0} L|y(t)|\mathrm{d}\theta\\
&= -(H - Lr)y^{2m_1}(t).
\end{aligned}$$

Therefore, the Theorem 3.1 (Theorem 3.3) implies the zero solution of Eq. (25) is uniformly stable (uniformly asymptotically stable) with respect to y.

6. Conclusion

This paper mainly focuses on the partial stability analysis of the RFDEs. Two new sufficient criteria are given to guarantee the partial stability (uniform stability and uniform asymptotic stability) of the RFDEs. These results can be applied widely in more areas.

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