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A Note on Homogenization of the Hyperbolic-Parabolic Equations in Domains with Holes

Zhanying YANG[1,*], Xianhe ZHAO[2]
1. *School of Mathematics and Statistics, South-Central University for Nationalities, Hubei* 430074*, P. R. China;*
2. *College of Mathematics and Information Science, Henan Normal University, Henan* 453007*, P. R. China*

Abstract In this paper, we study a class of hyperbolic-parabolic problems in periodically perforated domains with a homogeneous Neumann condition on the boundary of holes. We focus on the homogenization of these equations, which generalizes those achieved by Bensoussan-Lions-Papanicolau and Migorski. The proof is based on the periodic unfolding method in perforated domains.

Keywords hyperbolic-parabolic equations; perforated domains; homogenization; periodic unfolding method

MR(2010) Subject Classification 35B27; 35M10

1. Introduction

In this paper, we consider the hyperbolic-parabolic equation with homogenous Dirichlet-Neumann boundary in the perforated domain, namely:

$$\begin{cases} \alpha_\varepsilon u_\varepsilon'' + \beta_\varepsilon u_\varepsilon' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f_\varepsilon, & \text{in } \Omega_\varepsilon^* \times (0,T), \\ u_\varepsilon = 0, & \text{on } \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0, & \text{on } \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u_\varepsilon^0, & \text{in } \Omega_\varepsilon^*, \\ \alpha_\varepsilon u_\varepsilon'(x,0) = \sqrt{\alpha_\varepsilon} u_\varepsilon^1, & \text{in } \Omega_\varepsilon^*, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz continuous boundary, S_ε is a set of ε-periodic holes of size ε and $\Omega_\varepsilon^* = \Omega \backslash S_\varepsilon$, n_ε is the outward unit normal vector field defined on ∂S_ε. Let α_ε and β_ε be two coefficients such that:

$$\begin{cases} \alpha_\varepsilon, \beta_\varepsilon \in L^\infty(\Omega), \\ \alpha_\varepsilon \geq 0 \text{ a.e. in } \Omega, \\ \beta_\varepsilon \geq c > 0 \text{ a.e. in } \Omega. \end{cases}$$

For the initial data, we always assume $u_\varepsilon^0 \in H_0^1(\Omega)$, $u_\varepsilon^1 \in L^2(\Omega_\varepsilon^*)$ and $f_\varepsilon \in L^2(0,T; L^2(\Omega_\varepsilon^*))$.

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* Corresponding author
E-mail address: yangzhanying8011@163.com (Zhanying YANG); zhaoxianhe989@163.com (Xianhe ZHAO)

The coefficient matrix A^ε satisfies the following assumptions:

$$\begin{cases} A^\varepsilon \in M(\alpha,\beta,\Omega), \\ A^\varepsilon \text{ symmetric}, \end{cases} \tag{1.2}$$

where $M(\alpha,\beta,\Omega)$ is the classical set of the $n\times n$ matrix-valued functions defined in Section 2 (see [1]). This problem models many kinds of phenomena arising in electricity and magnetism, in the theory of elasticity, in vibrations theory and in hydrodynamics [2,3].

To the best knowledge of the author, the study of problem (1.1) was initiated by Bensoussan, Lions and Papanicolau [4], where the homogenization was given for the fixed domains. In the special case $\alpha_\varepsilon = 1$ and $\beta_\varepsilon = 0$, Cioranescu and Donato [5] investigated the classical homogenization result in perforated domains. The corresponding corrector result was studied in Nabil [6]. Recently, using the unfolding method, Donato and the first author [7] studied the homogenization and corrector results under some weaker assumptions.

In [8], Migorski carried out a study of the problem (1.1), and derived the homogenization in the perforated domains. Subsequently, Timofte [9] further extended the homogenization to the nonlinear case. In the case that $\alpha_\varepsilon = 0$ and $\beta_\varepsilon = 1$, Donato and Nabil [10] gave the homogenization and corrector results. Also, they presented these results for the corresponding semilinear problems in [11]. Observe that the above results were achieved for the classical case $A^\varepsilon(x) = A(x/\varepsilon)$ with A being symmetric, periodic, bounded and uniformly elliptic.

This paper is devoted to the homogenization of problem (1.1) under some conditions weaker than usual. For this purpose, we need to introduce some necessary assumptions. In what follows, we suppose that the coefficient matrix A^ε satisfies

$$\mathcal{T}_\varepsilon(A^\varepsilon) \to A \;\text{ strongly in } (L^1(\Omega\times Y))^{n\times n}, \tag{1.3}$$

where $\mathcal{T}_\varepsilon$ is the unfolding operator in fixed domains [12]. These assumptions recover the classical periodic coefficient case mentioned above.

For the coefficients α_ε and β_ε, we assume that $\alpha_\varepsilon > 0$ and

$$\begin{cases} \text{(i)} \;\; \|\alpha_\varepsilon\|_{L^\infty(\Omega)} \le C \text{ and } \|\beta_\varepsilon\|_{L^\infty(\Omega)} \le C, \\ \text{(ii)} \;\; \mathcal{T}_\varepsilon(\alpha_\varepsilon) \to \alpha(x,y) \quad \text{strongly in } L^2(\Omega\times Y) \text{ with } \alpha^* = \mathcal{M}_{Y^*}(\alpha) \ge c > 0, \\ \text{(iii)} \;\; \mathcal{T}_\varepsilon(\beta_\varepsilon) \to \beta(x,y) \quad \text{strongly in } L^2(\Omega\times Y), \\ \text{(iv)} \;\; \sqrt{\alpha_\varepsilon} \rightharpoonup \gamma \;\text{ weakly in } L^2(\Omega), \end{cases} \tag{1.4}$$

where C is a constant independent of ε. Note that this assumption is slightly weaker than that in [8].

For the initial data, we make the following assumptions:

$$\begin{cases} \text{(v)} \;\; u_\varepsilon^0 \rightharpoonup u^0 \text{ weakly in } \; H_0^1(\Omega), \\ \text{(vi)} \;\; \widetilde{u_\varepsilon^1} \to u^1 \text{ strongly in } L^2(\Omega), \\ \text{(vii)} \;\; \widetilde{f_\varepsilon} \rightharpoonup f \text{ weakly in } \; L^2(0,T;L^2(\Omega)). \end{cases} \tag{1.5}$$

The main purpose of this paper is to derive the homogenization result under these assumptions, which are weaker than those imposed in Migorski [8]. Our method is also quite different. The work of Migorski [8] was done by the Tartar's oscillating test function method, while our study mainly relies on the time-dependent periodic unfolding method in perforated domains [7]. This method was originally introduced in Cioranescu, Damlamian and Griso [12] (see [13] for more details) and extended to perforated domains in Cioranescu, Donato and Zaki [14,15] (see Cioranescu, Damlamian, Donato, et al. [16] for more general situations and see [7] for the time-dependent case).

Now, we state our homogenization result, where we use some notations to be defined in the next section.

Theorem 1.1 *Let A^ε be a matrix satisfying (1.2) and (1.3). Suppose that u_ε is the solution of problem (1.1) with (1.4) and (1.5). Then there exist $u \in L^2(0,T;H_0^1(\Omega))$ with $u' \in L^2(0,T;L^2(\Omega))$ and $\widehat{u} \in L^2(0,T;L^2(\Omega, H^1_{\mathrm{per}}(Y^*)))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, such that*

$$\begin{cases} \mathcal{T}_\varepsilon^*(u_\varepsilon) \to u \ \text{ strongly in } L^2(0,T;L^2(\Omega,H^1(Y^*))), \\ \mathcal{T}_\varepsilon^*(u_\varepsilon') \rightharpoonup u' \ \text{ weakly in } L^2(0,T;L^2(\Omega\times Y^*)), \\ \mathcal{T}_\varepsilon^*(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u} \ \text{ weakly in } L^2(0,T;L^2(\Omega\times Y^*)), \\ \|u_\varepsilon - u\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))} \to 0. \end{cases} \tag{1.6}$$

The pair $(u,\widehat{u})$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$ is the unique solution of the following problem:

$$\begin{cases} \theta \displaystyle\int_0^T \int_\Omega \alpha^* u \Psi \varphi'' \mathrm{d}x\mathrm{d}t + \theta \int_0^T \int_\Omega \beta^* u \Psi \varphi' \mathrm{d}x\mathrm{d}t + \\ \dfrac{1}{|Y|} \displaystyle\int_0^T \int_{\Omega\times Y^*} A(\nabla u + \nabla_y \widehat{u})(\nabla \Psi + \nabla_y \Phi)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ = \displaystyle\int_0^T \int_\Omega f \Psi \varphi \mathrm{d}x\mathrm{d}t \\ \text{for any } \Psi \in H_0^1(\Omega),\ \Phi \in L^2(\Omega; H^1_{\mathrm{per}}(Y^*)) \text{ and } \varphi \in \mathcal{D}(0,T), \\ u = 0 \ \text{ on } \ \Omega \times (0,T), \\ u(x,0) = u^0, \ \ u'(x,0) = \dfrac{\gamma}{\theta\alpha^*} u^1 \ \text{ in } \ \Omega. \end{cases} \tag{1.7}$$

We also have

$$\widehat{u} = \sum_{j=1}^n \frac{\partial u}{\partial x_j} \chi_j \tag{1.8}$$

with $\chi_j \in L^\infty(\Omega; H^1_{\mathrm{per}}(Y^*))$ $(j = 1, \ldots, n)$ being the solution of the cell problem

$$\begin{cases} -\mathrm{div}_y\big(A\nabla_y(\chi_j + y_j)\big) = 0, & \text{in } Y^*, \\ A\nabla_y(\chi_j + y_j)\cdot n_1 = 0, & \text{on } \partial S, \\ \mathcal{M}_{Y^*}(\chi_j)(x,\cdot) = 0, & \chi_j(x,\cdot) \ \ Y\text{-periodic}. \end{cases}$$

Moreover, u is the unique solution of the following homogenized wave equation

$$\begin{cases} \alpha^* u'' + \beta^* u' - \mathrm{div}(A^0 \nabla u) = \theta^{-1} f, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u^0, \ u'(x,0) = \frac{\gamma}{\theta\alpha^*} u^1, & \text{in } \Omega, \end{cases} \tag{1.9}$$

where the homogenized matrix $A^0 = (a^0_{ij})_{1\le i,j\le n}$ *is defined by*

$$a^0_{ij}(x) = \mathcal{M}_{Y^*}\Big(a_{ij} + \sum_{k=1}^{n} a_{ik}\frac{\partial \chi_j}{\partial y_k}\Big). \tag{1.10}$$

In addition, we have the following convergences:

$$\begin{cases} \textit{(i) } \widetilde{u}_\varepsilon \rightharpoonup \theta u \ \ \textit{weakly}^* \textit{ in } L^\infty(0,T;L^2(\Omega)),\\ \textit{(ii) } A^\varepsilon\widetilde{\nabla u_\varepsilon} \rightharpoonup \theta A^0\nabla u \ \ \textit{weakly}^* \textit{ in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{1.11}$$

We would like to mention that the effective matrix field A^0 depends on x, while the classical matrix is constant [17].

2. Preliminaries

In this section, we briefly recall some results related to the unfolding method in perforated domains. For some details associated to the unfolding theory, we refer the reader to [13] for the case of fixed domains and to [16] for the case of perforated domains.

2.1. Some notations

Let $b = (b_1,\dots,b_n)$ be a basis in $\mathbb{R}^n$. Set

$$\mathcal{G} = \Big\{\xi\in\mathbb{R}^n:\ \xi = \sum_{i=1}^{n} k_i b_i,\ (k_1,\dots,k_n)\in\mathbb{Z}^n\Big\},$$
$$Y = \Big\{\xi\in\mathbb{R}^n:\ \xi = \sum_{i=1}^{n} y_i b_i,\ (y_1,\dots,y_n)\in(0,1)^n\Big\}.$$

Suppose that $\Omega\subset\mathbb{R}^n$ is an open and bounded set with Lipschitz continuous boundary $\partial\Omega$, and S is a closed proper subset of $\overline{Y}$ with Lipschitz continuous boundary. Denote ε by the general term of a sequence of positive real numbers which converge to zero. Define the perforated domain $\Omega^*_\varepsilon = \Omega\setminus\tau_\varepsilon(\varepsilon S)$, where $\tau_\varepsilon(\varepsilon S) = \bigcup_{\xi\in\mathcal{G}}\varepsilon(\xi+S)$.

We make the following assumption:

$$\tau_\varepsilon(\varepsilon S)\cap\partial\Omega = \emptyset.$$

This implies $\partial\Omega^*_\varepsilon = \partial\Omega\cup\partial S_\varepsilon$, where S_ε is the subset of $\tau_\varepsilon(\varepsilon S)$ contained in Ω.

Now we recall some notations related to the unfolding method introduced in [13] and [16]. Let

$$\widehat{\Omega}_\varepsilon = \text{interior}\Big\{\bigcup_{\xi\in\Xi_\varepsilon}\varepsilon(\xi+\overline{Y})\Big\},\quad \Lambda_\varepsilon = \Omega\setminus\widehat{\Omega}_\varepsilon,$$

where $\Xi_\varepsilon = \{\xi\in\mathcal{G}|\varepsilon(\xi+Y)\subset\Omega\}$. Set

$$\widehat{\Omega}^*_\varepsilon = \widehat{\Omega}_\varepsilon\setminus S_\varepsilon \ \text{ and } \ \Lambda^*_\varepsilon = \Omega^*_\varepsilon\setminus\widehat{\Omega}^*_\varepsilon.$$

In what follows, we will use the following notations:

- $|D|$ denotes the Lebesgue measure of a measurable set D in $\mathbb{R}^n$;
- $Y^* = Y\setminus\bar{S}$;

• $\theta = \frac{|Y^*|}{|Y|}$;

• $\mathcal{M}_{\mathcal{O}}(v) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} v\mathrm{d}x$. For convenience, let $v^* := \mathcal{M}_{Y^*}(v)$;

• $\widetilde{g}$ is the zero extension to Ω (resp., $\Omega \times (0,T)$) of any function g defined on Ω_ε^* (resp., $\Omega_\varepsilon^* \times (0,T)$);

• V^ε is defined by

$$V^\varepsilon = \{v \in H^1(\Omega_\varepsilon^*) \mid v = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_{1\varepsilon})}$;

• $M(\alpha, \beta, \mathcal{O})$ is the set of the $n \times n$ matrix-valued functions in $L^\infty(\mathcal{O})$ such that

$$(B(x)\lambda, \lambda) \geq \alpha|\lambda|^2, \quad |B(x)\lambda| \leq \beta|\lambda|$$

for any $\lambda \in \mathbb{R}^n$ and a.e., on $\mathcal{O}$, where $\alpha, \beta \in \mathbb{R}$ and $0 < \alpha < \beta$;

• The notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set $\mathcal{O}$, since no ambiguity will arise;

• c and C denote generic constants which do not depend on ε.

2.2. The time-dependent unfolding operator in perforated domains

In this subsection, we list some results associated to the unfolding operator which will be used in this paper. For other properties and related comments, we refer the reader to [13] and [16].

For $z \in \mathbb{R}^n$, we use $[z]_Y$ to denote the unique integer combination $\sum_{j=1}^n k_j b_j$ of the period such that $z - [z]_Y \in Y$. Let $\{z\}_Y = z - [z]_Y \in Y$ a.e., for $z \in \mathbb{R}^n$. Then we have

$$x = \varepsilon\big([\frac{x}{\varepsilon}]_Y + \{\frac{x}{\varepsilon}\}_Y\big) \quad \text{for } x \in \mathbb{R}^n.$$

Definition 2.1 *For $p \in [1, +\infty)$ and $q \in [1, \infty]$, let ϕ be in $L^q(0, T; L^p(\Omega_\varepsilon^*))$. The unfolding operator $\mathcal{T}_\varepsilon^* : L^q(0, T; L^p(\Omega_\varepsilon^*)) \mapsto L^q(0, T; L^p(\Omega \times Y^*))$ is defined as follows:*

$$\mathcal{T}_\varepsilon^*(\phi)(x, y, t) = \begin{cases} \phi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y, t\big), & \text{a.e., for } (x, y, t) \in \widehat{\Omega}_\varepsilon \times Y^* \times (0, T), \\ 0, & \text{a.e., for } (x, y, t) \in \Lambda_\varepsilon \times Y^* \times (0, T). \end{cases}$$

Remark 2.2 Let $\mathcal{T}_\varepsilon$ be the unfolding operator for the fixed domain $\Omega \times (0, T)$ (see [18]). Then we have

$$\mathcal{T}_\varepsilon^*(\omega|_{\Omega_\varepsilon^* \times (0,T)}) = \mathcal{T}_\varepsilon(\omega)|_{\Omega \times Y^* \times (0,T)},$$

where ω is defined on $\Omega \times (0, T)$. For simplicity, we always write $\mathcal{T}_\varepsilon^*(\omega)$ instead of $\mathcal{T}_\varepsilon^*(\omega|_{\Omega_\varepsilon^* \times (0,T)})$.

The following propositions contain some basic properties associated to the unfolding and the averaging operators.

Proposition 2.3 *Let $p \in [1, +\infty)$ and $q \in [1, \infty]$.*

(i) $\mathcal{T}_\varepsilon^*$ is linear and continuous from $L^q(0, T; L^p(\Omega_\varepsilon^*))$ to $L^q(0, T; L^p(\Omega \times Y^*))$.

(ii) Let $w \in L^q(0, T; L^p(\Omega_\varepsilon^*))$. For a.e. $t \in (0, T)$, we have

$$\|\mathcal{T}_\varepsilon^*(w)\|_{L^p(\Omega \times Y^*)} = |Y|^{1/p}\|w\|_{L^p(\widehat{\Omega}_\varepsilon^*)} \leq |Y|^{1/p}\|w\|_{L^p(\Omega_\varepsilon^*)}.$$

(iii) For $w, v \in L^q(0,T;L^p(\Omega_\varepsilon^*))$, $\mathcal{T}_\varepsilon^*(vw) = \mathcal{T}_\varepsilon^*(v)\mathcal{T}_\varepsilon^*(w)$.

(iv) For $\psi \in L^p(\Omega_\varepsilon^*)$ and $\varphi \in L^q(0,T)$, $\mathcal{T}_\varepsilon^*(\psi\varphi) = \varphi\mathcal{T}_\varepsilon^*(\psi)$.

(v) For $p, q \in [1,\infty)$, let $\{\omega_\varepsilon\}$ be a sequence in $L^q(0,T;L^p(\Omega))$ such that

$$\omega_\varepsilon \to \omega \text{ strongly in } L^q(0,T;L^p(\Omega)).$$

Then $\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \to \omega$ strongly in $L^q(0,T;L^p(\Omega\times Y^*))$.

(vi) For $p \in (1,\infty)$ and $q \in (1,\infty]$, let $\{\omega_\varepsilon\}$ be a sequence in $L^q(0,T;L^p(\Omega_\varepsilon^*))$ such that

$$\|\omega_\varepsilon\|_{L^q(0,T;L^p(\Omega_\varepsilon^*))} \le C.$$

If $\mathcal{T}_\varepsilon^*(\omega_\varepsilon) \rightharpoonup \widehat{\omega}$ weakly in $L^q(0,T;L^p(\Omega\times Y^*))$, then we have

$$\widetilde{\omega}_\varepsilon \rightharpoonup \theta\mathcal{M}_{Y^*}(\widehat{\omega}) \text{ weakly in } L^q(0,T;L^p(\Omega)).$$

For $q=\infty$, the weak convergences above are replaced by the weak* convergences, respectively.

Proposition 2.4 *For $q \in [1,+\infty]$, let ϕ_ε be in $L^q(0,T;L^1(\Omega_\varepsilon^*))$ and satisfy*

$$\int_0^T\int_{\Lambda_\varepsilon^*}|\phi_\varepsilon|\mathrm{d}x\mathrm{d}t \to 0.$$

Then

$$\int_0^T\int_{\Omega_\varepsilon^*}\phi_\varepsilon\mathrm{d}x\mathrm{d}t - \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\mathcal{T}_\varepsilon^*(\phi_\varepsilon)\mathrm{d}x\mathrm{d}y\mathrm{d}t \to 0.$$

In particular, we have the following result:

For $p, q \in (1,+\infty)$, let $\{\varphi_\varepsilon\}$ and $\{\psi_\varepsilon\}$ be two sequences in $L^q(0,T;L^p(\Omega_\varepsilon^*))$ and $L^{q'}(0,T;L^{p'}(\Omega_\varepsilon^*))$ $(1/p+1/p'=1,\ 1/q+1/q'=1)$, respectively. Suppose that

$$\mathcal{T}_\varepsilon^*(\varphi_\varepsilon) \to \varphi \text{ strongly in } L^q(0,T;L^p(\Omega\times Y^*)),$$
$$\mathcal{T}_\varepsilon^*(\psi_\varepsilon) \rightharpoonup \psi \text{ weakly in } L^{q'}(0,T;L^{p'}(\Omega\times Y^*)).$$

Then for any $\phi \in \mathcal{D}(\Omega)$, we have

$$\int_0^T\int_{\Omega_\varepsilon^*}\varphi_\varepsilon\psi_\varepsilon\phi\mathrm{d}x\mathrm{d}t \to \frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*}\varphi\psi\phi\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Finally, we complete this section with the following convergence theorem which plays a crucial role in proving our homogenization result.

Theorem 2.5 *Let $\{w_\varepsilon\}$ be a sequence in $L^2(0,T;V^\varepsilon)$ such that*

$$\|\nabla w_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))} \le C \quad\text{and}\quad \Big\|\frac{\partial w_\varepsilon}{\partial t}\Big\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))} \le C.$$

Then there exist $w \in L^2(0,T;H_0^1(\Omega))$ with $\frac{\partial w}{\partial t} \in L^2(0,T;L^2(\Omega))$ and $\widehat{w} \in L^2(0,T;L^2(\Omega;H^1_{\rm per}(Y^*)))$ with $M_{Y^*}(\widehat{w}) \equiv 0$, such that, up to a subsequence,

(i) $\mathcal{T}_\varepsilon^*(w_\varepsilon) \to w$ strongly in $L^2(0,T;L^2(\Omega;H^1(Y^*)))$,

(ii) $\mathcal{T}_\varepsilon^*(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y\widehat{w}$ weakly in $L^2(0,T;L^2(\Omega\times Y^*))$,

(iii) $\mathcal{T}_\varepsilon^*(\frac{\partial w_\varepsilon}{\partial t}) \rightharpoonup \frac{\partial w}{\partial t}$ weakly in $L^2(0,T;L^2(\Omega\times Y^*))$,

(iv) $\|w_\varepsilon - w\|_{L^2(0,T;L^2(\Omega_\varepsilon^*))} \to 0$.

In fact, the proof can be directly obtained by the same arguments as those in the proof of Theorem 2.19 in [7] (see also the proof of Theorem 2.12 in [16]).

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Our proof is fundamentally based on the periodic unfolding method. Our starting point is the following variational formulation of problem (1.1):

Find $u_\varepsilon \in L^2(0,T;V^\varepsilon)$ with $u'_\varepsilon \in L^2(0,T;L^2(\Omega^*_\varepsilon))$ and $\alpha_\varepsilon u''_\varepsilon \in L^2(0,T;(V^\varepsilon)')$ such that

$$\begin{cases} \langle \alpha_\varepsilon u''_\varepsilon, v\rangle_{(V^\varepsilon)',V^\varepsilon} + \int_{\Omega^*_\varepsilon} \beta_\varepsilon u'_\varepsilon v \mathrm{d}x + \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u_\varepsilon \cdot \nabla v \mathrm{d}x = \int_{\Omega^*_\varepsilon} f_\varepsilon v \mathrm{d}x \\ \text{ in } \mathcal{D}'(0,T) \text{ for all } v \in V^\varepsilon, \\ u_\varepsilon(x,0) = u^0_\varepsilon, \quad \alpha_\varepsilon u'_\varepsilon(x,0) = \sqrt{\alpha_\varepsilon} u^1_\varepsilon \quad \text{ in } \Omega^*_\varepsilon. \end{cases} \tag{3.1}$$

For every fixed ε, following the classical arguments [8], we know that the problem (1.1) has a unique solution u_ε such that

$$u_\varepsilon \in L^\infty(0,T;V^\varepsilon), \ u'_\varepsilon \in L^2(0,T;L^2(\Omega^*_\varepsilon)) \text{ and } \alpha_\varepsilon u''_\varepsilon \in L^2(0,T;(V^\varepsilon)').$$

Furthermore, we have the following uniform estimates. For the proof, we refer the interested readers to [8, Theorem 4.1] and [4, Theorem 1.1]:

Lemma 3.1 *Suppose that the assumptions (1.2), (1.4) and (1.5) are satisfied. For every ε, we have the following uniform estimates:*

$$\|u_\varepsilon\|_{L^\infty(0,T;V^\varepsilon))} + \|u'_\varepsilon\|_{L^2(0,T;L^2(\Omega^*_\varepsilon))} + \|\sqrt{\alpha_\varepsilon} u'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega^*_\varepsilon))} \le C, \tag{3.2}$$

where the constant C does not depend on ε.

With Theorem 2.5 and Lemma 3.1 at our disposal, we proceed to prove Theorem 1.1.

Proof of Theorem 1.1 In view of (3.2), we use Theorem 2.5 to get that there exist $u \in L^2(0,T;H^1_0(\Omega))$ with $u' \in L^2(0,T;L^2(\Omega))$ and $\widehat{u} \in L^2(0,T;L^2(\Omega,H^1_{\text{per}}(Y^*)))$ with $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, such that, up to a subsequence (still denoted by ε), the convergences in (1.6) hold.

Let $\Psi, \phi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\text{per}}(Y^*)$. Set

$$v_\varepsilon(x) = \Psi(x) + \varepsilon\phi(x)\psi^\varepsilon(x) \quad \text{with} \quad \psi^\varepsilon(x) = \psi(\frac{x}{\varepsilon}),$$

then

$$\nabla v_\varepsilon = \nabla\Psi + \varepsilon\psi^\varepsilon\nabla\phi + \phi(\nabla_y\psi)(\frac{\cdot}{\varepsilon}).$$

From Proposition 2.3, we have

$$\begin{cases} \mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi \ \text{ strongly in } L^2(\Omega \times Y^*), \\ \mathcal{T}^*_\varepsilon(\phi\psi^\varepsilon) \to \Phi \ \text{ strongly in } L^2(\Omega \times Y^*) \text{ with } \Phi = \phi(x)\psi(y), \\ \mathcal{T}^*_\varepsilon(\nabla v_\varepsilon) \to \nabla\Psi + \nabla_y\Phi \ \text{ strongly in } L^2(\Omega \times Y^*). \end{cases} \tag{3.3}$$

Let $\varphi \in \mathcal{D}(0,T)$. By (1.3), (1.6) and (3.3), we use Proposition 2.4 to get

$$\int_0^T \int_{\Omega^*_\varepsilon} \alpha_\varepsilon u_\varepsilon v_\varepsilon \varphi'' \mathrm{d}x\mathrm{d}t \to \theta \int_0^T \int_\Omega \alpha^* u \Psi \varphi'' \mathrm{d}x\mathrm{d}t. \tag{3.4}$$

$$\int_0^T \int_{\Omega_\varepsilon^*} \beta_\varepsilon u_\varepsilon v_\varepsilon \varphi' \mathrm{d}x\mathrm{d}t \to \theta \int_0^T \int_\Omega \beta^* u \Psi \varphi' \mathrm{d}x\mathrm{d}t. \tag{3.5}$$

$$\int_0^T \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon \varphi \mathrm{d}x\mathrm{d}t \to \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \widehat{u})(\nabla \Psi + \nabla_y \Phi)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t. \tag{3.6}$$

The fact that v_ε strongly converges to Ψ in $L^2(\Omega)$, together with (1.5), leads to

$$\int_0^T \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon \varphi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega f \Psi \varphi \mathrm{d}x\mathrm{d}t. \tag{3.7}$$

Choosing $v_\varepsilon \varphi$ as test function in the variational formulation (3.1), passing to the limit and making use of (3.4)–(3.7), we obtain

$$\begin{aligned} &\theta \int_0^T \int_\Omega \alpha^* u \Psi \varphi'' \mathrm{d}x\mathrm{d}t - \theta \int_0^T \int_\Omega \beta^* u \Psi \varphi' \mathrm{d}x\mathrm{d}t + \\ &\quad \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \widehat{u})(\nabla \Psi + \nabla_y \Phi)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \theta \int_0^T \int_\Omega f \Psi \varphi \mathrm{d}x\mathrm{d}t. \end{aligned}$$

This gives the equation in (1.7), due to the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ and the density of $\mathcal{D}(\Omega) \otimes H^1_{\mathrm{per}}(Y^*)$ in $L^2(\Omega, H^1_{\mathrm{per}}(Y^*))$.

Setting $\Psi = 0$ in (1.7), we get

$$\frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} A(\nabla u + \nabla_y \widehat{u})(\nabla_y \Phi)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t = 0$$

which implies that $\mathrm{div}_y A(\nabla u + \nabla_y \widehat{u}) = 0$. Since u is independent of y and $\mathcal{M}_{Y^*}(\widehat{u}) = 0$, we obtain (1.8). Then by a standard computation [7], we have the following identity:

$$\int_{Y^*} A(\nabla u + \nabla_y \widehat{u}) \nabla \Psi \mathrm{d}y = |Y^*| A^0 \nabla u \nabla \Psi,$$

where A^0 is defined by (1.10). Substituting this and (1.8) into (1.7), we get

$$\alpha^* u'' + \beta^* u' - \mathrm{div}(A^0 \nabla u) = \theta^{-1} f \quad \text{in } \Omega \times (0, T) \tag{3.8}$$

which is exactly the equation in (1.9).

In what follows, we will check the initial conditions. Firstly, we prove the following convergence:

$$\beta_\varepsilon \widetilde{u}_\varepsilon' \rightharpoonup \theta \beta^* u' \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \tag{3.9}$$

In fact, the second condition of (1.4) implies that $\{\beta_\varepsilon \widetilde{u}_\varepsilon'\}$ is bounded in $L^2(0, T; L^2(\Omega))$. On the other hand, by (1.4) and (1.6), we use Proposition 2.4 to get that

$$\int_0^T \int_\Omega \beta_\varepsilon \widetilde{u}_\varepsilon' \phi \varphi \mathrm{d}x\mathrm{d}t \to \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y^*} \beta u' \phi \varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t$$

holds for $\phi \in \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(0, T)$. This gives (3.9). In the same way, we have the convergence:

$$\alpha_\varepsilon \widetilde{u}_\varepsilon' \rightharpoonup \theta \alpha^* u' \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \tag{3.10}$$

From (3.8) and (3.9), we know $\theta\alpha^* u'' \in L^2(0,T;H^{-1}(\Omega))$. Moreover, by the classical results, we have

$$u \in \mathcal{C}^0([0,T];L^2(\Omega)) \ \text{ and } \ \theta\alpha^* u' \in \mathcal{C}^0([0,T];H^{-1}(\Omega)).$$

Let $\Psi \in \mathcal{D}(\Omega)$ and $\varphi \in C^\infty([0,T])$ with $\varphi(0)=1$ and $\varphi(T)=0$. Choosing $\Psi\varphi$ as test function in the variational formulation (3.1) and integrating by parts, we have

$$\begin{aligned}
&-\int_0^T\int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u_\varepsilon \nabla\Psi\varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon^*} f_\varepsilon \Psi\varphi \mathrm{d}x\mathrm{d}t\\
&= \int_0^T \langle \alpha_\varepsilon u_\varepsilon'', \Psi\rangle_{(V^\varepsilon)',V^\varepsilon}\varphi \mathrm{d}t + \int_0^T \langle \beta_\varepsilon u_\varepsilon', \Psi\rangle_{(V^\varepsilon)',V^\varepsilon}\varphi \mathrm{d}t\\
&= \int_{\Omega_\varepsilon^*} \alpha_\varepsilon (u_\varepsilon'\varphi)|_0^T \Psi \mathrm{d}x - \int_0^T\int_{\Omega_\varepsilon^*} \alpha_\varepsilon u_\varepsilon' \Psi\varphi' \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon^*} \beta_\varepsilon u_\varepsilon' \Psi\varphi \mathrm{d}x\mathrm{d}t\\
&= -\int_\Omega \sqrt{\alpha_\varepsilon}\Psi \widetilde{u_\varepsilon^1}\mathrm{d}x - \int_0^T\int_{\Omega_\varepsilon^*} \alpha_\varepsilon u_\varepsilon' \Psi\varphi' \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega_\varepsilon^*} \beta_\varepsilon u_\varepsilon' \Psi\varphi \mathrm{d}x\mathrm{d}t
\end{aligned}$$

Passing to the limit, we use (3.4)–(3.7) and (3.10) to derive

$$\begin{aligned}
&-\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y^*} A(\nabla u + \nabla_y \widehat{u})(\nabla\Psi)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}t + \theta\int_0^T\int_\Omega f\Psi\varphi \mathrm{d}x\mathrm{d}t\\
&= -\int_\Omega \gamma u^1\Psi \mathrm{d}x - \theta\int_0^T\int_\Omega \alpha^* u'\Psi\varphi' \mathrm{d}x\mathrm{d}t + \theta\int_0^T\int_\Omega \beta^* u'\Psi\varphi \mathrm{d}x\mathrm{d}t\\
&= -\int_\Omega \gamma u^1\Psi \mathrm{d}x + \theta\int_\Omega \alpha^* u'(x,0)\Psi \mathrm{d}x +\\
&\quad \theta\int_0^T \langle \alpha^* u'', \Psi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi \mathrm{d}t + \theta\int_0^T\int_\Omega \beta^* u'\Psi\varphi \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Combining this with (1.7), we have $u'(x,0) = \frac{\gamma}{\theta\alpha^*}u^1$.

Choosing $\varphi \in C^\infty([0,T])$ with $\varphi(0)=\varphi(T)=\varphi'(T)=0$, $\varphi'(0)=1$ and taking $\Psi\varphi$ as test function in the variational formulation (3.1), by a similar argument, we obtain $u(x,0)=u^0$. Consequently, u solves problem (1.9).

By the standard arguments [7], we obtain the uniform ellipticity of A^0 and the uniqueness of the solution of problem (1.9). Together with (1.8), we get that the pair $(u,\widehat{u})$ with $\mathcal{M}_{Y^*}(\widehat{u})=0$ is a unique solution of problem (1.7). This implies that each convergence in Theorem 1.1 holds for the whole sequence.

Finally, we turn to the proof of (1.11). From Proposition 2.3 (iv), we get

$$\widetilde{u}_\varepsilon \rightharpoonup \theta u \ \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Due to (3.2), convergence (1.11) (i) holds for the above subsequence. Then arguing as we have done for getting (3.9), we obtain the following convergence

$$A^\varepsilon\widetilde{\nabla u_\varepsilon} \rightharpoonup \theta\mathcal{M}_{Y^*}[A(\nabla u + \nabla_y\widehat{u})] \ \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

By the standard computation (see the proof of Theorem 3.1 in [7]), we have

$$A^\varepsilon\widetilde{\nabla u_\varepsilon} \rightharpoonup \theta A^0\nabla u \ \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

Together with (3.2), we get (1.11) (ii). □

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