

Algebraic Properties of Toeplitz Operators on the Harmonic Bergman Space

Jingyu YANG^{1,2,*}, Yufeng LU², Xiaoying WANG¹

1. School of Mathematics and Statistics, Chifeng University, Inner Mongolia 024000, P. R. China;
2. Department of Mathematics, Dalian University of Technology, Liaoning 116024, P. R. China

Abstract In this paper, we first investigate the finite-rank product problem of several Toeplitz operators with quasihomogeneous symbols on the harmonic Bergman space. Next, we characterize finite rank commutators and semi-commutators of two Toeplitz operators with quasihomogeneous symbols.

Keywords Toeplitz operator; Harmonic Bergman space; quasihomogeneous; commutator; finite rank

MR(2010) Subject Classification 47B35; 47B20

1. Introduction

Let D be the open unit disk in the complex plane C and dA be the normalized area measure on D . $L^2(D, dA)$ is the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} dA(z).$$

The Bergman space $L_a^2(D)$ is the closed subspace of all analytic functions in $L^2(D, dA)$ and harmonic Bergman space $L_h^2(D)$ is the closed subspace of $L^2(D, dA)$ consisting of the harmonic functions on D . There is the relation that

$$L_h^2(D) = L_a^2(D) + \overline{L_a^2(D)},$$

where $\overline{L_a^2(D)} = \{\bar{f} | f \in L_a^2(D), f(0) = 0\}$. It is well known that each point evaluation in $L_h^2(D)$ is a bounded linear functional on D , there exists a unique function $R_z \in L_h^2(D)$ which has the reproducing property $f(z) = \langle f, R_z \rangle$ for every $f \in L_h^2(D)$.

Since $L_h^2(D) = L_a^2(D) + \overline{L_a^2(D)}$, there is a relation

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1, \quad z, w \in D. \quad (1)$$

where K_z is the reproducing kernel for $L_a^2(D)$ and given by

$$K_z(w) = \frac{1}{(1 - \bar{z}w)^2}, \quad z, w \in D.$$

Received October 24, 2015; Accepted March 9, 2016

Supported by the National Natural Science Foundation of China (Grant Nos. 11271059; 11301047) and the Scientific Research Found of Higher School of Inner Mongolia (Grant No. NJZY 13298).

* Corresponding author

E-mail address: yjy923@163.com (Jingyu YANG); 527514533@qq.com (Xiaoying WANG)

Let P be the orthogonal projection from $L^2(D, dA)$ onto $L_a^2(D)$ and Q denote the orthogonal projection from $L^2(D, dA)$ onto $L_h^2(D)$. Since $P\varphi(z) = \langle g, K_z \rangle$ for $\varphi \in L^2(D, dA)$ and $z \in D$, then by (1), we have

$$Q\varphi(z) = P\varphi(z) + \overline{P(\overline{\varphi})(z)} - P\varphi(0). \quad (2)$$

For a function $\varphi \in L^\infty(D)$, the Toeplitz operator $T_\varphi : L_h^2(D) \rightarrow L_h^2(D)$ with symbol φ is defined by

$$T_\varphi(f) = Q(\varphi f)(z) = \int_D f(w)\varphi(w)\overline{R_z(w)}dA(w).$$

In 1964, Brown and Halmos [1] proved that if $T_f T_g = 0$ on the Hardy space $H^2(T)$, then either f or g must be identically zero. In [2], Ahern and Čučković showed the result analogous to that in [1] for two Toeplitz operators with harmonic symbols on the Bergman space of unit disk. Moreover in [3] they proved that if $T_f T_g = 0$, where f is arbitrary bounded and g is radial, then either $f \equiv 0$ or $g \equiv 0$. Recently, those zero product results have been generalized to finite rank product result in [4]. Čučković and Louhichi [5] studied finite rank product of several quasihomogeneous Toeplitz operators on the Bergman space of the unit disk.

For two Toeplitz operators T_φ and T_ψ the commutator and semi-commutator are defined by $[T_\varphi, T_\psi] = T_\varphi T_\psi - T_\psi T_\varphi$, $(T_\varphi, T_\psi) = T_{\varphi\psi} - T_\varphi T_\psi$.

On the Hardy space the problem of finite rank commutator or semi-commutator has been completely solved [6,7]. On the Bergman space the problem seems to be far from solution. Guo, Sun and Zheng [8] completely characterized the finite rank commutator and semi-commutator of two Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk. Čučković and Louhichi [5] investigated the finite rank semi-commutators and commutators of Toeplitz operators with quasihomogeneous symbols on the Bergman space of the unit disk and obtained different results from the case of harmonic Toeplitz operators. Lu and Zhang [9] studied finite rank commutators and semi-commutators of two quasihomogeneous Toeplitz operators on the Bergman space of the polydisk.

The theory of Toeplitz operators on the harmonic Bergman space is quite different from that on L_a^2 . For example, Choe and Lee [10] showed that two analytic Toeplitz operators on L_h^2 commute only when their symbols and the constant function 1 are linearly dependent, but analytic Toeplitz operators always commute on L_a^2 . Motivated by Čučković and Louhichi [5] and Choe, Lee [10], we will discuss the finite rank (semi-)commutators of quasihomogeneous Toeplitz operators on the harmonic Bergman space of unit disk in this paper.

2. The Mellin transform and Mellin convolution

Mellin transform is a useful tool in the following calculations.

Definition 2.1 Let $f \in L^1([0, 1], r dr)$. The Mellin transform \hat{f} of a function f is defined by

$$\hat{f}(z) = \int_0^1 f(r)r^{z-1}dr.$$

It is clear that \hat{f} is well defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$. It is important and helpful to know that the Mellin transform \hat{f} is uniquely determined by its value on an arithmetic sequence of integers. In fact, we have the following classical theorem [11, p. 102].

Theorem 2.2 Suppose that f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes at the pairwise distinct points z_1, z_2, \dots , where

- (i) $\inf\{|z_n|\} > 0$;
- (ii) $\sum_{n \geq 1} \operatorname{Re}(\frac{1}{z_n}) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 2.3 We shall often use this theorem to show that if $f \in L^1([0, 1], r dr)$ and if there exists a sequence $(n_k)_{k \geq 0} \subset \mathbb{N}$ such that

$$\hat{f}(n_k) = 0, \quad \sum_{k \geq 0} \frac{1}{n_k} = \infty,$$

then $\hat{f}(z) = 0$ for all $z \in \{z : \operatorname{Re}(z) > 2\}$ and so $f = 0$.

If f and g are defined on $[0, 1]$, then their Mellin convolution is defined by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{r}{t}\right)g(t)\frac{dt}{t}, \quad 0 \leq r < 1.$$

The Mellin convolution theorem states that

$$\widehat{f *_M g}(s) = \hat{f}(s)\hat{g}(s),$$

and that, if f and g are in $L^1([0, 1], r dr)$, then so is $f *_M g$.

3. Finite rank product of n Toeplitz operators

We will discuss the finite rank product of Toeplitz operators with quasihomogeneous symbols in this section.

Definition 3.1 Let $p \in \mathbb{Z}$. A function $\varphi \in L^1(D, dA)$ is called a quasihomogeneous function of degree p if φ is of the form $e^{ip\theta} f$, where f is a radial function, i.e., $\varphi(re^{i\theta}) = e^{ip\theta} f(r)$.

The main reason for many researchers to study Toeplitz operators with quasihomogeneous symbols is that any function f in $L^2(D, dA)$ has the polar decomposition

$$f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0, 1], r dr)$.

Lemma 3.2 ([12, Lemma 2.1]) Let $p \in \mathbb{Z}$ and let φ be a bounded radial function. Then for each $k \in \mathbb{N}$,

$$T_{e^{ip\theta}\varphi}(z^k) = \begin{cases} 2(k+p+1)\hat{\varphi}(2k+p+2)z^{k+p}, & k \geq -p; \\ 2(-k-p+1)\hat{\varphi}(-p+2)\bar{z}^{-k-p}, & k < -p. \end{cases}$$

$$T_{e^{ip\theta}\varphi}(\bar{z}^k) = \begin{cases} 2(k-p+1)\widehat{\varphi}(2k-p+2)\bar{z}^{k-p}, & k \geq p; \\ 2(p-k+1)\widehat{\varphi}(p+2)z^{p-k}, & k < p. \end{cases}$$

Theorem 3.3 Let $p_1, \dots, p_m \in \mathbb{Z}$ and let f_1, \dots, f_m be bounded radial functions. If $T_{e^{ip_m\theta}f_m} \cdots T_{e^{ip_1\theta}f_1}$ is of finite rank M , then $f_i = 0$ for some $i \in \{1, 2, \dots, m\}$.

Proof We denote by S the product of Toeplitz operators $T_{e^{ip_m\theta}f_m} \cdots T_{e^{ip_1\theta}f_1}$. By Lemma 3.2, for $k \geq \sum_{j=i}^m |p_j|$, we have

$$\begin{aligned} S(z^k) &= 2(k+p_1+1)\widehat{f}_1(2k+p_1+2)2(k+p_1+p_2+1)\widehat{f}_2(2k+2p_1+p_2+2) \cdots \\ &\quad 2(k+p_1+\cdots+p_m+1)\widehat{f}_m(2k+2p_1+\cdots+2p_{m-1}+p_m+2)z^{k+p_1+\cdots+p_m}, \\ S(\bar{z}^k) &= 2(k-p_1+1)\widehat{f}_1(2k-p_1+2)2(k-p_1-p_2+1)\widehat{f}_2(2k-2p_1-p_2+2) \cdots \\ &\quad 2(k-p_1-\cdots-p_m+1)\widehat{f}_m(2k-2p_1-\cdots-2p_{m-1}-p_m+2)z^{k-p_1-\cdots-p_m}. \end{aligned}$$

Thus the sets $\{S(z^k) : k \geq \sum_{j=i}^m |p_j|\}$ and $\{S(\bar{z}^k) : k \geq \sum_{j=i}^m |p_j|\}$ are linearly independent sets which are included in the rang of S . Hence $\{S(z^k) : k \geq \sum_{j=i}^m |p_j|\}$ and $\{S(\bar{z}^k) : k \geq \sum_{j=i}^m |p_j|\}$ contain at most M elements.

Since $\{S(\bar{z}^k) : k \geq \sum_{j=i}^m |p_j|\}$ contain at most M elements, there exists some positive integer $n_0 > \sum_{j=i}^m |p_j|$ such that $S(\bar{z}^k) = 0, \forall k \geq n_0$, which is equivalent to

$$\widehat{f}_1(2k+p_1+2) \cdots \widehat{f}_m(2k+2p_1+2p_2+\cdots+2p_{m-1}+p_m+2) = 0, \quad \forall k \geq n_0. \tag{3}$$

Let $l = \min\{p_1, 2p_1+p_2, \dots, 2p_1+2p_2+\cdots+2p_{m-1}+p_m\}$. Then (3) implies that

$$\widehat{r^{p_1-l}f_1}(2k+l+2) \cdots \widehat{r^{2p_1+2p_2+\cdots+2p_{m-1}+p_m-l}f_m}(2k+l+2) = 0, \quad \forall k \geq n_0.$$

Since $\{2k+l+2\}_{k \geq n_0}$ is an arithmetic sequence, by Theorem 2.2 we know that there exists at least one of $f_i, i \in \{1, 2, \dots, m\}$ such that $f_i = 0$.

Since $\{S(\bar{z}^k) : k \geq \sum_{j=i}^m |p_j|\}$ contains at most M elements, using the same method derives that there exists some $i \in \{1, 2, \dots, m\}$ such that $f_i = 0$. The proof of this Theorem is completed. \square

4. Finite rank commutator

In this section, we investigate the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ and $[T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}], p, s \geq 0$. We show that these commutators are nonzero finite rank operators.

Theorem 4.1 Let p, s be non-negative integers and at least one of them is nonzero. Let f and g be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{is\theta}\psi}$ are bounded operators. If the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ has finite rank M , then M is at most equal to $s+p-1$.

Proof Let S denote the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$. Since S has finite rank on $L_h^2(D)$, we know that $\{S(z^k)\}_{k \geq 0}$ and $\{S(\bar{z}^k)\}_{k > 0}$ must have finite rank. If S has rank M , the rank of $\{S(z^k)\}_{k \geq 0}$ is equal to N_1 and the rank of $\{S(\bar{z}^k)\}_{k > 0}$ is equal to N_2 , then $M = N_1 + N_2$. Using Lemma 3.2, we obtain

$$S(z^k) = 2(k+p+s+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2)-$$

$$\begin{aligned}
 & 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p+s+2)]z^{k+p+s}, \\
 S(\bar{z}^k) = & 2(k-p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)- \\
 & 2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p-s+2)]\bar{z}^{k-p-s}, \quad k \geq p+s.
 \end{aligned}$$

Since $\{S(z^k)\}_{k \geq 0}$ has finite rank N_1 , there exists $n_0 \geq N_1$ such that

$$S(z^k) = 0, \quad \forall k \geq n_0,$$

which is equivalent to

$$\begin{aligned}
 & 2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2) = 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p+s+2), \\
 \text{for all } k \geq n_0. \text{ Using } \widehat{1}(2k+2s+2) = \frac{1}{2(k+s+1)} \text{ and } \widehat{1}(2k+2p+2) = \frac{1}{2(k+p+1)}, \text{ we have} \\
 & \widehat{r^{2p}1}(2k+2)\widehat{r^s\psi}(2k+2)\widehat{r^{2s+p}\varphi}(2k+2) = \widehat{r^{2s}1}(2k+2)\widehat{r^{2p+s}\psi}(2k+2)\widehat{r^p\varphi}(2k+2), \quad (4)
 \end{aligned}$$

for all $k \geq n_0$. Since $\{2k+2\}_{k \geq n_0}$ is arithmetic, Remark 2.3 and (4) imply that

$$\widehat{r^{2p}1}(z)\widehat{r^s\psi}(z)\widehat{r^{2s+p}\varphi}(z) = \widehat{r^{2s}1}(z)\widehat{r^{2p+s}\psi}(z)\widehat{r^p\varphi}(z),$$

for all $\text{Re } z > 2$. In particular, if $z = 2k+2$ with $k \geq 0$, we have

$$\widehat{r^{2p}1}(2k+2)\widehat{r^s\psi}(2k+2)\widehat{r^{2s+p}\varphi}(2k+2) = \widehat{r^{2s}1}(2k+2)\widehat{r^{2p+s}\psi}(2k+2)\widehat{r^p\varphi}(2k+2).$$

Hence, we have $S(z^k) = 0, \forall k \geq 0$. Therefore, the rank of $\{S(z^k)\}_{k \geq 0}$ is equal to zero.

Since $\{S(\bar{z}^k)\}_{k > 0}$ has finite rank, there exists $n_1 \geq s+p$ such that

$$S(\bar{z}^k) = 0, \quad \forall k \geq n_1,$$

which is equivalent to

$$2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2) = 2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p-s+2),$$

for all $k \geq n_1$.

Similarly to the discussion for $\{S(z^k)\}_{k \geq 0}$, we get

$$S(\bar{z}^k) = 0, \quad \forall k \geq p+s,$$

which means that the rank of $\{S(\bar{z}^k)\}_{k > 0}$ is at most equal to $s+p-1$. So the rank of S is at most equal to $s+p-1$. This completes the proof of Theorem. \square

Corollary 4.2 *Let $p > 0$ and let φ and ψ be two integrable radial functions on D such that T_φ and $T_{e^{ip\theta}\psi}$ are bounded operators. If the commutator $[T_\varphi, T_{e^{ip\theta}\psi}]$ has finite rank, its rank is at most equal to $p-1$.*

Theorem 4.3 *Let $p, s \geq 0$ and at least one of them is nonzero. Let φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$ has finite rank, then the rank is equal to $p+s-1$.*

Proof Let S denote the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$.

If $s \geq p$, using Lemma 3.2, we have

$$S(z^k) = \begin{cases} 2(k-s+p+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)- \\ 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & k \geq s, \\ 2(k-s+p+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)- \\ 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & s-p \leq k \leq s-1, \\ 2(s-p-k+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)- \\ 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)]\bar{z}^{s-p-k}, & 0 \leq k \leq s-p-1, \end{cases}$$

$$S(\bar{z}^k) = \begin{cases} 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)- \\ 2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)]\bar{z}^{k+s-p}, & k \geq p, \\ 2(k+s-p+1)[2(s+k+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)- \\ 2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)]\bar{z}^{k+s-p}, & 0 < k \leq p-1. \end{cases}$$

Since S has finite rank, similar to the discussion of Theorem 4.1, we deduce that

$$S(z^k) = 0, \forall k \geq s, \quad S(\bar{z}^k) = 0, \forall k \geq p.$$

These indicate that the rank of the commutator S is at most equal to $s+p-1$.

If $s < p$, we have

$$S(z^k) = \begin{cases} 2(k-s+p+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)- \\ 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & k \geq s, \\ 2(k-s+p+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)- \\ 2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & 0 \leq k \leq s-1, \end{cases}$$

$$S(\bar{z}^k) = \begin{cases} 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)- \\ 2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)]\bar{z}^{k+s-p}, & k \geq p, \\ 2(k+s-p+1)[2(s+k+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s-p+2)- \\ 2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)]\bar{z}^{k+s-p}, & p-s \leq k \leq p-1, \\ 2(p-k-s+1)[2(s+k+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)- \\ 2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)]z^{p-k-s}, & 0 < k \leq p-s-1. \end{cases}$$

In this case, if S has finite rank, we obtain

$$S(z^k) = 0, \forall k \geq s, \quad S(\bar{z}^k) = 0, \forall k \geq p.$$

These also imply that the rank of S is at most equal to $p+s-1$. The proof is completed. \square

Example 4.4 We give an example about Theorem 4.1.

From the proof of Theorem 4.1 we know that

$$T_{e^{ip\theta}r^m}T_{e^{is\theta}f}(z^k) = T_{e^{is\theta}f}T_{e^{ip\theta}r^m}(z^k), \quad \forall k \geq 0.$$

Next we will construct a radial function f , such that

$$T_{e^{ip\theta}r^m}T_{e^{is\theta}f}(\bar{z}^k) = T_{e^{is\theta}f}T_{e^{ip\theta}r^m}(\bar{z}^k), \quad \forall k \geq p+s, \tag{5}$$

where $p > 0$, $s > 0$ and $m \geq 0$.

Eq. (5) implies that for $k \geq p + s$,

$$\frac{k - p + 1}{2k - p + m + 2} \widehat{f}(2k - 2p - s + 2) = \frac{k - s + 1}{2k - 2s - p + m + 2} \widehat{f}(2k - s + 2).$$

Thus for $k \succeq p + s$,

$$\frac{r^{-s-2p} \widehat{f}(2k + 2 + 2p)}{r^{-s-2p} \widehat{f}(2k + 2)} = \frac{(2k + 2 - 2p)(2k + 2 - p + m - 2s)}{(2k + 2 - 2s)(2k + 2 - p + m)}.$$

Now, using Remark 2.3, we obtain that

$$\frac{r^{-s-2p} \widehat{f}(z + 2p)}{r^{-s-2p} \widehat{f}(z)} = \frac{(z - 2p)(z - p + m - 2s)}{(z - 2s)(z - p + m)}, \tag{6}$$

for all $\text{Re}z \geq 2p + 2s + 2$.

Let F be the analytic function defined for $\text{Re}z \geq 2p + 2s$ by

$$F(z) = \frac{\Gamma(\frac{z-2p}{2p})\Gamma(\frac{z-p-2s+m}{2p})}{\Gamma(\frac{z-2s}{2p})\Gamma(\frac{z-p+m}{2p})},$$

where Γ denotes the gamma function. Then by using the well-known identity $\Gamma(z + 1) = z\Gamma(z)$, (6) implies that

$$\frac{r^{-s-2p} \widehat{f}(z + 2p)}{r^{-s-2p} \widehat{f}(z)} = \frac{F(z + 2p)}{F(z)}, \quad \text{Re}z > 2p + 2s. \tag{7}$$

Eq. (7) combined with [13, Lemma 6] gives that there exists a constant c such that

$$r^{-s-2p} \widehat{f}(z) = cF(z), \quad \text{Re}z > 2p + 2s. \tag{8}$$

For a choice of $p = 2$, $s = 1$ and $m = 6$, and again using the identity $\Gamma(z + 1) = z\Gamma(z)$, one can see that

$$F(z) = \frac{4(z - 2)}{z(z - 4)} = 2\left[\frac{1}{z} + \frac{1}{z - 4}\right].$$

Since $\widehat{1}(z) = \frac{1}{z}, r^{-4}(z) = \frac{1}{z-4}$, (4) becomes

$$r^{-5} \widehat{f}(z) = c[\widehat{1}(z) + r^{-4}(z)], \quad \text{Re}z > 6.$$

Now the proceeding equation and Remark 2.3 imply that

$$f(r) = c(r^5 + r),$$

where c is a constant. It is clear that the function f is bounded, so Toeplitz operator $T_{e^{i\theta}f}$ is bounded.

Finally, by taking the constant c to be equal to 1, the radial function $f(r) = r^5 + r$ satisfies

$$\begin{aligned} T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(z^k) &= T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(z^k), \quad \forall k \geq 0, \\ T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\bar{z}^k) &= T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\bar{z}^k), \quad \forall k \geq 3. \end{aligned}$$

However, using Lemma 3.2, it is easy to see that

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\bar{z}) = \frac{9}{20}z^2, \quad T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\bar{z}) = \frac{16}{25}z^2.$$

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\bar{z}^2) = \frac{32}{75}z, \quad T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\bar{z}^2) = \frac{3}{10}z.$$

Therefore the commutator $[T_{e^{2i\theta}r^6}, T_{e^{i\theta}(r^5+r)}]$ has rank two.

Example 4.5 We give an example of Theorem 4.3. Similarly to Example 4.4, there exist $\varphi = \frac{63}{4}r^{-2} - \frac{35}{2} + \frac{15}{4}r^2$ and $\psi = r^6, p = 1, s = 2$ such that

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(z^k) = T_{e^{-2i\theta}r^6}(z^k)T_{e^{i\theta}\varphi}(z^k), \quad k \geq 2,$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\bar{z}^k) = T_{e^{-2i\theta}r^6}(\bar{z}^k)T_{e^{i\theta}\varphi}(\bar{z}^k), \quad k \geq 1.$$

By a direct calculation, we have

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(1) = \frac{192}{35}z, \quad T_{e^{-2i\theta}r^6}(z^k)T_{e^{i\theta}\varphi}(1) = \frac{256}{15}z,$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(z) = \frac{128}{15}, \quad T_{e^{-2i\theta}r^6}(z^k)T_{e^{i\theta}\varphi}(z) = \frac{96}{35},$$

so the rank of $[T_{e^{i\theta}(\frac{63}{4}r^{-2} - \frac{35}{2} + \frac{15}{4}r^2)}, T_{e^{-2i\theta}r^6}]$ is equal to 2.

5. Finite rank semi-commutator

We will discuss the semi-commutators of two Toeplitz operators with quasihomogeneous symbols.

Theorem 5.1 Let $p, s \geq 0$ and at least one of them be nonzero. Let φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{is\theta}\psi}$ are bounded operators. If the semi-commutator $(T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi})$ has a finite rank, then its rank is equal to $p + s - 1$.

Proof Let S denote the semi-commutator $(T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi})$. Using Lemma 3.2, we have

$$S(z^k) = 2(k + P + s + 1)[2(k + s + 1)\widehat{\psi}(2k + s + 2)\widehat{\varphi}(2k + 2s + p + 2) - \widehat{\psi}\widehat{\varphi}(2k + p + s + 2)]z^{k+p+s},$$

$$S(\bar{z}^k) = \begin{cases} 2(k - P - s + 1)[2(k - s + 1)\widehat{\psi}(2k - s + 2)\widehat{\varphi}(2k - 2s - p + 2) - \widehat{\psi}\widehat{\varphi}(2k - p - s + 2)]z^{k-p-s}, & k \geq p + s, \\ 2(P + s - k + 1)[2(k - s + 1)\widehat{\psi}(2k - s + 2)\widehat{\varphi}(p + 2) - \widehat{\psi}\widehat{\varphi}(-p + s + 2)]z^{p+s-k}, & s \leq k \leq s + p - 1, \\ 2(P + s - k + 1)[2(s - k + 1)\widehat{\psi}(s + 2)\widehat{\varphi}(2s - 2k + p + 2) - \widehat{\psi}\widehat{\varphi}(p + s + 2)]z^{p+s-k}, & 0 < k \leq s - 1. \end{cases}$$

If the semi-commutator S has finite rank, using the same arguments as in the proof of Theorem 4.1, we have

$$S(z^k) = 0, \quad \forall k \geq 0, \quad S(\bar{z}^k) = 0, \quad \forall k \geq s + p,$$

which implies that the rank of S is at most equal to $s + p - 1$. The proof is completed. \square

Corollary 5.2 Let $p > 0$, φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\psi}$ and T_φ are bounded operators. If the semi-commutator $(T_\varphi, T_{e^{ip\theta}\psi})$ has finite rank, then its rank is equal to $p - 1$.

Theorem 5.3 Let $p, s \geq 0, s \geq p$ and at least one of them is nonzero. Let φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If the semi-commutator $(T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi})$ has finite rank, then its rank is equal to s .

Proof Let S denote the semi-commutator $(T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi})$. By Lemma 3.2, we have

$$S(z^k) = \begin{cases} 2(k + P - s + 1)[2(k - s + 1)\widehat{\psi}(2k - s + 2)\widehat{\varphi}(2k - 2s + p + 2) - \widehat{\psi\varphi}(2k + p - s + 2)]z^{k+p-s}, & k \geq s, \\ 2(k + P - s + 1)[2(s - k + 1)\widehat{\psi}(s + 2)\widehat{\varphi}(p + 2) - \widehat{\psi\varphi}(2k + p - s + 2)]z^{k+p-s}, & s - p \leq k \leq s - 1, \\ 2(s - p - k + 1)[2(s - k + 1)\widehat{\psi}(s + 2)\widehat{\varphi}(2s - 2k - p + 2) - \widehat{\psi\varphi}(s - p + 2)]\bar{z}^{s-p-k}, & 0 \leq k \leq s - p - 1. \end{cases}$$

$$S(\bar{z}^k) = 2(k - P + s + 1)[2(k + s + 1)\widehat{\psi}(2k + s + 2)\widehat{\varphi}(2k + 2s - p + 2) - \widehat{\psi\varphi}(2k - p + s + 2)]\bar{z}^{k-p+s}.$$

Since S has finite rank, using the same arguments as in the proof of Theorem 4.1, we have

$$S(z^k) = 0, \forall k \geq s, \quad S(\bar{z}^k) = 0, \forall k > 0.$$

From the above two equality, we know that the rank of S is at most s . \square

Theorem 5.4 Let $p, s \geq 0, s \geq p$ and at least one of them is nonzero. Let φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If the semi-commutator $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi})$ has finite rank, then the rank is equal to $s - 1$.

Proof Let S denote the semi-commutator $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi})$. Using Lemma 3.2, we know that

$$S(z^k) = \begin{cases} 2(k - s + p + 1)[2(k + p + 1)\widehat{\varphi}(2k + p + 2)\widehat{\psi}(2k + 2p - s + 2) - \widehat{\psi\varphi}(2k - s + p + 2)]z^{k-s+p}, & k \geq s - p, \\ 2(s - P - k + 1)[2(k + p + 1)\widehat{\varphi}(2k + p + 2)\widehat{\psi}(s + 2) - \widehat{\psi\varphi}(s - p + 2)]\bar{z}^{s-k-p}, & 0 \leq k \leq s - p - 1. \end{cases}$$

$$S(\bar{z}^k) = \begin{cases} 2(k + s - p + 1)[2(k - p + 1)\widehat{\varphi}(2k - p + 2)\widehat{\psi}(2k - 2p + s + 2) - \widehat{\psi\varphi}(2k + s - p + 2)]z^{k+s-p}, & k \geq p, \\ 2(k + s - p + 1)[2(p - k + 1)\widehat{\varphi}(p + 2)\widehat{\psi}(s + 2) - \widehat{\psi\varphi}(2k + s - p + 2)]\bar{z}^{k+s-p}, & 0 < k \leq p - 1. \end{cases}$$

Since S has finite rank, using the method in Theorem 4.1, we obtain

$$S(z^k) = 0, \forall k \geq s - p, \quad S(\bar{z}^k) = 0, \forall k \geq p.$$

These imply that the rank of S is at most $s - 1$. The proof is completed. \square

Remark 5.5 If $p \geq s, (T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi})$ and $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi})$ have finite rank, then the rank of them are $p - 1$.

Example 5.6 We give an example of Theorem 5.1. As we construct Example 4.4, let $\varphi = r, \psi = r^6, p = 1$ and $s = 1$. We have

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(z^k) = T_{e^{2i\theta}r^7}(z^k), \quad k \geq 0,$$

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(\bar{z}^k) = T_{e^{2i\theta}r^7}(\bar{z}^k), \quad k \geq 2.$$

However,

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(z) = \frac{2}{9}z, \quad T_{e^{2i\theta}r^7}(z) = \frac{4}{11}z.$$

Therefore, the rank of $(T_{e^{i\theta}r}, T_{e^{i\theta}r^6}]$ is equal to 1.

Example 5.7 We give an example of Theorem 5.3. As we construct Example 4.4, let $\varphi = r^6$, $\psi = r^{-2}$, $p = 1$ and $s = 2$. We have

$$T_{e^{i\theta}r^6}T_{e^{-2i\theta}r^{-2}}(z^k) = T_{e^{-i\theta}r^4}(z^k), \quad k \geq 2,$$

$$T_{e^{i\theta}r^6}T_{e^{-2i\theta}r^{-2}}(\bar{z}^k) = T_{e^{-i\theta}r^4}(\bar{z}^k), \quad k > 0.$$

However,

$$T_{e^{i\theta}r^6}T_{e^{-2i\theta}r^{-2}}(1) = \frac{12}{11}\bar{z}, \quad T_{e^{-i\theta}r^4}(1) = \frac{4}{7}\bar{z},$$

$$T_{e^{i\theta}r^6}T_{e^{-2i\theta}r^{-2}}(z) = \frac{4}{9}, \quad T_{e^{-i\theta}r^4}(z) = \frac{2}{7}.$$

Therefore, the rank of $(T_{e^{i\theta}r}, T_{e^{i\theta}r^6}]$ is equal to 2.

Acknowledgements We thank the referees for their time and comments.

References

- [1] A. BROWN, P. R. HALMOS. *Algebraic properties of Toeplitz operators*. J. Reine Angew. Math., 1963/1964, 89–102.
- [2] P. AHERN, Ž. ČUČKOVIĆ. *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*. J. Funct. Anal., 2001, **187**(1): 200–210.
- [3] P. AHERN, Ž. ČUČKOVIĆ. *Some examples related to the Brown-Halmos theorem for the Bergman space*. Acta Sci. Math. (Szeged), 2004, **70**(1-2): 373–378.
- [4] B. R. CHOE, H. KOO, Y. J. LEE. *Finite rank Toeplitz products with harmonic symbols*. J. Math. Anal. Appl., 2008, **343**(1): 81–98.
- [5] Ž. ČUČKOVIĆ, I. LOUHICHI. *Finite rank commutators and semicommutators of quasihomogeneous Toeplitz operators*. Complex Anal. Oper. Theory, 2008, **2**(3): 429–439.
- [6] S. AXLER, S.-Y. A. CHANG, D. SARASON. *Product of Toeplitz operators*. Integral Equations Operator Theory, 1978, **1**(3): 285–309.
- [7] Xuanhao DING, Dechao ZHENG. *Finite rank commutator of Toeplitz operators or Hankel operators*. Houston J. Math., 2008, **34**(4): 1099–1119.
- [8] Kunyu GUO, Shunhua SUN, Dechao ZHENG. *Finite rank commutators and semicommutators of Toeplitz operators with harmonic symbols*. Illinois J. Math., 2007, **51**(2): 583–596.
- [9] Bo ZHANG, Yanyue SHI, Yufeng LU. *Algebraic properties of Toeplitz operators on the polydisk*. Abstr. Appl. Anal., 2011, Art. ID 962313, 18 pp.
- [10] B. R. CHOE, Y. J. LEE. *Commuting Toeplitz operators on the Harmonic Bergman space*. Michigan Math. J., 1999, **46**(1): 163–174.
- [11] R. REMMERT. *Classical Topics in Complex Function Theory*. Springer-Verlag, New York, 1998.
- [12] Xingtang DONG, Zehua ZHOU. *Products of Toeplitz operators on the harmonic Bergman space*. Proc. Amer. Math. Soc., 2010, **138**(5): 1765–1773.
- [13] I. LOUHICHI. *Powers and Roots of Toeplitz Operators*. Proc. Amer. Math. Soc., 2007, **135**(5): 1465–1475.