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Concerning a General Source Formula and Its Applications

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Abstract Here presented is a further investigation on a general source formula (GSF) that has been proved capable of deducing more than 30 special formulas for series expansions and summations in the author's recent paper [On a pair of operator series expansions implying a variety of summation formulas. Anal. Theory Appl., 2015, 31(3): 260–282]. It is shown that the pair of series transformation formulas found and utilized by He, Hsu and Shiue [cf. Disc. Math., 2008, 308: 3427–3440] is also deducible from the GSF as consequences. Thus it is found that the GSF actually implies more than 50 special series expansions and summation formulas. Finally, several expository remarks relating to the $(\Sigma\Delta D)$ formula class are given in the closing section.

Keywords Sheffer-type operator; source formula; triplet; $(\Sigma\Delta D)$ class

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1. Introduction and preliminaries

As may be seen, our several papers have been concerned with finding some source formulas that could be used to draw various special formulas for series expansions and summations [1–6]. The basic tools we employed are the symbolic operator calculus, the theory of formal power series (fps) and that of differential operators. What we have obtained and utilized are certain series transformation formulas involving the ordinary difference operators Δ^k (with increment 1 for Δ) and differential operators D^k (with $D \equiv \mathrm{d}/\mathrm{d}t$), wherein $k \in \mathbf{N}$ (set of non-negative integers) and $\Delta^0 = D^0 = 1$ (identity operator). Some fruitful results may be recalled briefly as follows.

Let $\psi(t)$ and $\phi(t)$ be real-valued functions defined on Z (set of integers) and $\mathbb{R}$ (real number field), respectively. Then there holds a formula series expansion of the form

$$\sum_{k=0}^{\infty} \psi(k)\Delta^k\phi(0)\binom{x}{k} = \sum_{k=0}^{\infty} \Delta^k\psi(0)\Delta^k\phi(x-k)\binom{x}{k}. \tag{1.1}$$

Also, there are two expansion formulas involving derivatives

$$\sum_{k=0}^{\infty} \psi(k)\phi^{(k)}(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} \Delta^k\psi(0)\phi^{(k)}(x)\frac{x^k}{k!}, \tag{1.2}$$

$$\sum_{k=0}^{\infty} \psi(k)\phi^{(k)}(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} \psi^{(k)}(0)A_k(x,\phi(x))/k!, \tag{1.3}$$

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wherein $\phi(t)$ in (1.2)–(1.3) and $\psi(t)$ in (1.3) are infinitely differentiable functions (members of C^∞), and $A_k(x,\phi(x))$ is an extension of Euler's fraction defined by

$$A_k(x,\phi(x)) = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \phi^{(j)}(x)x^j, \quad A_0(x,\phi(x)) = \phi(x) \tag{1.4}$$

with $\left\{ \begin{smallmatrix} k \\ j \end{smallmatrix} \right\} = \frac{1}{j!}(\Delta^j t^k)_{t=0} = S(k,j)$ denoting the Stirling numbers of the second kind (in Knuth's notation).

Apparently, when taking $\psi(t) \equiv 1$, formulas (1.1)–(1.3) will be reduced to Newton's interpolation series and Maclaurin's expansion of $\phi(x)$, respectively. Note that (1.1) and (1.2)–(1.3) are the basic results given by [4] and He-Hsu-Shiue's paper [1], respectively. Since there have been already given plenty of examples showing that a variety of special formulas and identities could be deduced from [3,6], (1.1), (1.2) and (1.3) may be called, respectively, the 1st, 2nd and 3rd source formula, or denoted briefly as SF(1), SF(2) and SF(3). Certainly each of these formulas is associated with a given triplet $\{x,\psi,\phi\}$, and all possible special formulas are deduced via suitable special choices of the triplets.

As was mentioned in [3], the pair of operator series expansions given by Theorem 2.1 could be rewritten as a single formula involving a delta operator δ. Also, it has been proved that the SF(1) is deducible from the single formula with $\delta = \Delta$ (cf. loc. cit), so that the so-called single formula [3, (6.1)] may be adopted as a 'general source formula' (GSF). In the later sections (§2–§3) we shall give a utilizable specialization of the GFS, and will show that both SF(2) and SF(3) are included in the specialization as consequences. Thus as a conclusion, one may think that the GSF is really a common source for all the SF(i)'s ($i = 1,2,3$).

2. A useful specialization of GSF

As in [3], $A(t)$, $g(t)$, $f(t)$, $\phi(t)$, etc. always denote the formal power series (fps) or the functions in C^∞ defined in $\mathbb{R}$ or $\mathbb{C}$ (complex number field). All operators are assumed to be acting on the fps or functions of t, unless otherwise stated. As usual, the shift operator E is defined by $E^\alpha f(t) = f(t+\alpha)$ ($\alpha \in \mathbb{R}$ or $\mathbb{C}$). An operator Q is said to be shift-invariant if $QE^\alpha = E^\alpha Q$ for every α. Moreover, a shift-invariant operator Q is called a delta operator whenever $Qt \neq 0$ (a non-zero constant). Obviously, $\Delta, D, \Delta E^\alpha$ and DE^α are the most useful delta operators.

Let us now reformulate the basic result of [3] in a more general form as follows.

Theorem 2.1 *Let $A(t), g(t)$ and $f(t)$ be fps over $\mathbb{R}$ or $\mathbb{C}$ such that $A(0) = 1$, $g(0) = 0$ and $g'(0) = Dg(0) \neq 0$. Let $\phi(t) \in C^\infty$ and let δ be a delta operator. Then there holds formally an operator series expansion formula (so-called GSF) of the form*

$$A(\delta)f(g(\delta))\phi(t) = \sum_{k\geq 0}(p_k(D)f(0))\delta^k\phi(t). \tag{2.1}$$

Herein $p_k(D)$ $(k \in \mathbf{N})$ *are Sheffer-type differential operators given by the expression*

$$p_k(D) = \sum_{j=0}^{k} (\frac{1}{j!} d_{kj}) D^j \tag{2.2}$$

within which $(d_{kj}) = (A(t), g(t))$ *is a Riordan array obtainable via the use of the extracting-coefficient operator* $[t^k]$, *namely*

$$d_{kj} = [t^k] A(t)(g(t))^j, \quad 0 \le j \le k \in \mathbf{N}. \tag{2.3}$$

Moreover, if it is assumed that

$$\theta = \overline{\lim_{k\to\infty}} \left|p_k(D) f(0)\right|^{1/k} > 0. \tag{2.4}$$

Then the expansion formula (2.1) becomes an exact equality at $t = 0$, *provided that*

$$\overline{\lim_{k\to\infty}} \left|\delta^k \phi(0)\right|^{1/k} < 1/\theta. \tag{2.5}$$

Actually, (2.1) is obtained by twice applications of Mullin-Rota's substitution rule [3, §2]. In what follows we will give a utilizable specialization of (2.1).

Corollary 2.2 *By taking* $\delta = D$, $g(t) = t$ *and* $A(t) = e^{xt}$ *in (2.1) with* x *being a given parameter (real of complex), we may get a formal series expansion of the form*

$$f(D)\phi(x+t) = \sum_{k\ge 0} \frac{1}{k!} (x+D)^k f(0) \cdot D^k \phi(t) \tag{2.6}$$

where f *and* ϕ *are fps or functions in* C^∞ *(defined on* $\mathbb{R}$ *or* $\mathbb{C}$*).*

Proof From the given conditions we see that the LHS (left-hand side) of (2.1) gives

$$e^{xD} f(D)\phi(t) = E^x f(D)\phi(t) = f(D)\phi(x+t) = \text{LHS of (2.6)}.$$

Also, in accordance with the RHS (right-hand side) of (2.1) we have to compute $p_k(D)f(0)$. Using (2.2) and (2.3) we easily find

$$\begin{aligned} p_k(D)f(0) &= \sum_{j=0}^{k} \frac{1}{j!} d_{kj} D^j f(0) = \sum_{j=0}^{k} \frac{1}{j!} [t^k](e^{xt} \cdot t^j) D^j f(0) = \sum_{j=0}^{k} \frac{x^{k-j}}{j!(k-j)!} D^j f(0) \\ &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} x^{k-j} D^j f(0) = \frac{1}{k!} (x+D)^k f(0). \end{aligned}$$

Hence the RHS of (2.1) yields the RHS of (2.6). □

Evidently, Taylor's expansion formula is a particular case of (2.6) with $f(t) \equiv 1$. In the next section (§3) we will give an important application of (2.6).

Example 2.3 Recall that the Bernoulli polynomials $B_k(x)$ and Charlier polynomials $C_k^{(\alpha)}(x)$ $(k \in \mathbf{N})$, are Sheffer-type polynomials, so that $B_k(D)$ and $C_k^{(\alpha)}(D)$ give Sheffer-type differential operators. Note that they are generated by $\{A(t) = t/(e^t - 1), g(t) = t\}$ and $\{A(t) = e^{-\alpha t}, g(t) = \log(1+t)\}$, respectively. Accordingly, the GSF(2.1) yields the following two series expansions

$$\delta/(e^\delta - 1) f(\delta)\phi(t) = \sum_{k=0}^{\infty} \frac{1}{k!} B_k(D) f(0) \cdot \delta^k \phi(t), \tag{2.7}$$

$$e^{-\alpha\delta}f(\log(1+\delta))\phi(t)=\sum_{k=0}^{\infty}C_k^{(\alpha)}(D)f(0)\cdot\delta^k\phi(t). \tag{2.8}$$

In particular, taking $\delta=D$ for (2.7) and $\delta=\Delta$ for (2.8), and noticing that $e^D-1=E-1=\Delta$ and $\log(1+\Delta)=D$, we find the following two formal expansions

$$f(D)\phi'(t)=\sum_{k=0}^{\infty}\frac{1}{k!}B_k(D)f(0)\big(\phi^{(k)}(t+1)-\phi^{(k)}(t)\big), \tag{2.9}$$

$$f(D)\phi(t)=\sum_{k=0}^{\infty}C_k^{(\alpha)}(D)f(0)\cdot e^{\alpha\Delta}\Delta^k\phi(t), \tag{2.10}$$

where the operator $1/\Delta$ involved in the LHS of (2.7) has been removed to the RHS, and a similar process has been applied to the equation (2.8).

3. A proof that GSF implies SF(2) and SF(3)

It suffices to show that SF(2) and SF(3) could be deduced from (2.6) with special choices of $f(t)$.

Proposition 3.1 *The formal expansion formulas (1.2) and (1.3), namely* SF(2) *and* SF(3), *are deducible from (2.6) with evaluation at $t=0$ and with the following choices of $f(t)$*

$$\text{(i)}\quad f(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\Delta^k\psi(0)x^kt^k, \tag{3.1}$$

$$\text{(ii)}\quad f(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\psi^k(0)\sum_{j=0}^{k}\left\{ {k \atop j} \right\}x^jt^j, \tag{3.2}$$

respectively, where x is a given parameter.

Proof First, it may be seen that the LHS of (2.6) evaluated at $t=0$ just provides the RHS of (1.2) and that of (1.3) with $f(t)$ being defined by (i) and (ii), respectively. Moreover, it is clear that for (i) we have formal derivatives

$$f^{(j)}(0)=D^jf(0)=x^j\Delta^j\psi(0),\quad j\in\mathbf{N}.$$

Also, for (ii) we have

$$\begin{aligned} f^{(j)}(0)&=\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k)}(0)j!\left\{ {k \atop j} \right\}x^j=x^j\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k)}(0)(\Delta^jt^k)_{t=0}\\ &=x^j\Big(\Delta^j\sum_{k=0}^{\infty}\frac{1}{k!}\psi^{(k)}(0)t^k\Big)_{t=0}=x^j(\Delta^j\psi(t))_{t=0}=x^j\Delta^j\psi(0). \end{aligned}$$

This shows that for both (i) and (ii) we have the same expression

$$(x+D)^kf(0)=\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\cdot x^j\Delta^j\psi(0)=x^k\psi(k).$$

Hence both (1.2) and (1.3) are implied by (2.6) with evaluation of $t=0$. □

Remark 3.2 What is worth mentioning is the fact that SF(2) and SF(3) have been found before, so that the choices of $f(t)$ for $S(2)$ and $S(3)$ appear to be a relatively easier matter. Also, it is known that SF(1) is a special case of (2.1) (viz. GSF) with $A(t) = (1+t)^x$, $g(t) = t/(t+1)$ and

$$f(t) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k \psi(0) t^k. \tag{3.3}$$

Here (3.3) is a discrete analogue of (3.1). This is quite natural, since SF(1) is actually a discrete analogue of SF(2) (see [3, Theorem 4.1]).

4. Embedding technique via choosing triplets

As shown in our previous papers [1,3,4,6], the main technique used for the derivation of most special formulas as examples is to make suitable choices of the triplets involved in the source formulas. Certainly, the unified technique may be called 'embedding technique'. In this section we will present some additional examples and give some little more explanations for a few selected instances exhibited previously [3,4].

Note that both SF(1) and SF(2) involve computations of higher order differences. We now reproduce here a short table of difference formulas for references. (Herein a printing error in page 280 of [3] is corrected).

(i) $\Delta^k a^t = (a-1)^k a^t \ (a \neq 0)$; $(\Delta^k a^t)_0 = (a-1)^k$;

(ii) $\Delta^k \binom{a+t}{n} = \binom{a+t}{n-k} \ (k \le n)$; $\Delta^k \binom{a+t}{n}_0 = \binom{a}{n-k}$;

(iii) $\Delta^k \binom{a-t}{n} = (-1)^k \binom{a-t-k}{n-k} \ (k \le n)$; $\Delta^k \binom{a-t}{n}_0 = (-1)^k \binom{a-k}{n-k}$;

(iv) $\Delta^k (\frac{1}{t+a}) = \frac{(-1)^k k!}{(t+a)(t+a+1)\cdots(t+a+k)}$, $\Delta^k (\frac{1}{t+a})_0 = \frac{(-1)^k}{a} \Big/ \binom{k+a}{k} \ (a \neq 0)$;

(v) $\Delta^k \cos(at+b) = \big(2\sin\frac{a}{2}\big)^k \cos\big(at+b+\frac{k}{2}(a+\pi)\big)$,
$\Delta^k \sin(at+b) = \big(2\sin\frac{a}{2}\big)^k \sin\big(at+b+\frac{k}{2}(a+\pi)\big)$;

(vi) $(\Delta^k t^n)_0 = k! \left\{ {n \atop k} \right\} = k! S(n,k)$;

(vii) $\Delta^k H_t = \Delta^{k-1} \frac{1}{t+1} = \frac{(-1)^{k-1}(k-1)!}{(t+k)_k}$; $(\Delta^k H_t)_0 = \frac{(-1)^{k-1}}{k} \ (t \in \mathbf{N})$;

(viii) $\Delta^k F_t = F_{t-k} \ (t \in \mathbf{N}, t \ge k)$.

Recall that the harmonic numbers H_t are defined by $H_0 = 1$, $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ $(k \ge 1)$; and the Fibonacci numbers F_t defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \in \{2, 3, \ldots\}$.

Example 4.1 What is worth mentioning is that there are 3 classical formulas due to Euler, all deducible from the SF(1) with special choices of the triplet $\{x, \psi, \phi\}$. The first two formulas are known as the transformation formula for the alternating series and the summation formula for the arithmetical-geometric series [3,4]. The third formula is usually called Euler's finite difference theorem which may be expressed by the following equality

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k)=(-1)^n\Delta^n f(0)=\begin{cases}0, & 0\le m\le n-1,\\ (-1)^n n!a_n, & m=n\end{cases} \tag{4.1}$$

where $f(t)$ is a polynomial of degree m, namely

$$f(t)=\sum_{j=0}^{m}a_j t^j,\quad m\in\mathbf{N}. \tag{4.2}$$

As may be observed, (4.1) follows from (1.1) (SF(1)) via embedding with the special triple $\{x=n,\psi(t)=f(t),\phi(t)=\binom{n-t}{n}\}$, since $\phi(t)$ gives

$$\Delta^k\phi(t)=(-1)^k\binom{n-t-k}{n-k},\quad \Delta^k\phi(0)=(-1)^k$$

and

$$\Delta^k\phi(n-k)=(-1)^k\binom{0}{n-k}=(-1)^k\delta_{nk}=\begin{cases}(-1)^n, & k=n,\\ 0, & k<n.\end{cases}$$

Moreover, $\Delta^n f(0)=n!a_n$ just follows from a simple computation when $m=n$.

Example 4.2 It may be of interest to notice that Abel's famous identity

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a(a-kx)^{k-1}(b+kx)^{n-k} \tag{4.3}$$

is implied by Euler's formula (4.1) as a consequence. Clearly, (4.3) is equivalent to the algebraic identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}[a(kx-a)^{k-1}(kx+b)^{n-k}+(-a)^k b^{n-k}]=0 \tag{4.4}$$

where the LHS is of the same form as that of (4.1) with

$$f(k)=a(kx-a)^{k-1}(kx+b)^{n-k}+(-a)^k b^{n-k}. \tag{4.5}$$

Thus in order to prove that (4.1) implies (4.4), it suffices to show that (4.5) can be expressed algebraically as a polynomial in k of degree $(n-1)$, for $0\le k\le n$. First, it is easily seen that the RHS of (4.5) can be expanded into a polynomial in kx of degree $(k-1)+(n-k)=n-1$ with the last term $(-a)^k b^{n-k}$ being cancelled within the expression and $f(0)=0$. Moreover, $f(k)$ can be expressed algebraically in the following form (with $1\le k\le n$)

$$\begin{aligned}f(k)&=a(kx+b)^{n-k}\sum_{0\le j\le(n-1)}\binom{k-1}{j}(kx+b)^{k-1-j}(-a-b)^j+(-a)^k b^{n-k}\\ &=a\sum_{0\le j\le(n-1)}(kx+b)^{n-j-1}\binom{k-1}{j}(-a-b)^j+(-a)^k b^{n-k}\\ &=\text{polynomial in } k \text{ of degree } (n-j-1)+j=n-1,\end{aligned}$$

wherein the term $(-a)^k b^{n-k}$ is already cancelled. Hence (4.1) implies (4.4). □

As known, two classical proofs of Abel's identity have been given by Lucas and Francon (cf. L. Comtet's book 'Advance Combinatoriss', §3.1, p. 128-129).

Example 4.3 In a recently published book by Quaintance and Gould, chapter 7 is entitled 'Melzak's formula', in which several nice combinatorial identities have been derived as applications of the formulas. Also presented in the chapter is an elaborate proof of the formula (cf. loc. cit., p. 79-82). What we have found here is that a simplified form of Melzak's formula could be embedded in the SF(2) (viz. (1.2)), thus leading to a rather short proof.

Let $f(x)$ be a polynomial in x of degree n, and let $y \in \mathbb{C}$. Melzak's formula states

$$f(x+y) = y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k} \tag{4.6}$$

where $y \neq 0, -1, -2, \ldots, -n$. Clearly, we may treat $F(y) = f(x+y)$ as a polynomial in y of degree n with coefficients involving the parameter x. Thus (4.6) may be rewritten as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{F(-k)}{y+k} = \frac{F(y)}{y}\Big/\binom{y+n}{n}. \tag{4.7}$$

As $F(y)$ is a linear combination of monomials $\alpha_m y^m$ $(0 \le m \le n)$, it suffices to verifty (4.7) with taking $F(y) \mapsto y^m$. We only need to show

$$\sum_{k=0}^{n}\frac{(-k)^m}{k+y}(-1)^k\binom{n}{k} = \frac{y^m}{y}\Big/\binom{y+n}{n}. \tag{4.8}$$

This may be embedded in the particular formula SF(2) with $x = 1$, viz. (1.2) with $x = 1$

$$\sum_{k\ge 0}\psi(k)\phi^{(k)}(0)/k! = \sum_{k\ge 0}\Delta^k\psi(0)\phi^{(k)}(1)/k!. \tag{4.9}$$

Indeed, taking $\psi(t) = (-t)^m/(t+y)$ and $\phi(t) = (1-t)^n$, we find that the LHS of (4.9) just gives the LHS of (4.8). Also, we have $\phi^{(k)}(1) = D^k(1-t)^n|_{t=1} = 0$ $(0 \le k < n)$, $\phi^{(n)}(1) = (-1)^n n!$, and we find the RHS of (4.9)$= (-1)^n\Delta^n\psi(0)$. Now we have (noticing that $(m-1) < n$)

$$\begin{aligned}\underset{t}{\Delta^n}\psi(0) = \Delta^n\psi(t)_0 &= (-1)^m\Big\{\underset{t}{\Delta^n}\Big(\frac{t^m-(-y)^m}{t-(-y)}\Big)_0 + \underset{t}{\Delta^n}\Big(\frac{(-y)^m}{t-(-y)}\Big)_0\Big\}\\ &= \underset{t}{\Delta^n}\Big(\frac{y^m}{t+y}\Big)_0 = (-1)^n\frac{y^m}{y}\Big/\binom{n+y}{n}.\end{aligned}$$

Hence the RHS of (4.9) gives the RHS of (4.8). □

Example 4.4 The following formula due to D. A. Zave (with $m \in \mathbf{N}$ and $|x| < 1$)

$$\sum_{k=1}^{\infty}\binom{k+m}{m}(H_{k+m}-H_m)x^k = (1-x)^{-m-1}\log\Big(\frac{1}{1-x}\Big) \tag{4.10}$$

could be obtained from the SF(1) (viz. (1.1)) with the chosen triplet

$$\{x = -m-1, \psi(t) = H_{m+t} - H_m, \phi(t) = (1-x)^t\}.$$

Indeed, using (1.1) one may find that

$$\begin{aligned}
\text{LHS of } (4.10) &= \sum_{k\ge 1}\binom{-m-1}{k}(H_{m+k}-H_m)(\Delta^k(1-x)^t)_0\\
&= \sum_{j\ge 1}\binom{-m-1}{j}\Delta^j(H_{m+t}-H_m)_0(\Delta^j(1-x)^t)_{t=-m-1-j}\\
&= \sum_{j\ge 1}(-1)^j\binom{m+j}{j}\Big(\Delta^{j-1}\frac{1}{m+1+t}\Big)_0((-x)^j(1-x)^t)_{t=-m-1-j}\\
&= \sum_{j\ge 1}\frac{1}{j}\binom{m+j}{j-1}\frac{(-1)^{j-1}}{\binom{m+j}{j-1}}\Big(\frac{x}{1-x}\Big)^j(1-x)^{-m-1}\\
&= (1-x)^{-m-1}\sum_{j\ge 1}\frac{(-1)^{j-1}}{j}\Big(\frac{x}{1-x}\Big)^j = \text{RHS of } (4.10). \quad \Box
\end{aligned}$$

Example 4.5 Recall that the well-known C-numbers first introduced and utilized by Koutras and Charalambides may be defined by the following [5]

$$C(n,k;a,b) = \frac{1}{k!}\Delta^k(at+b)_n\Big|_{t=0} \tag{4.11}$$

wherein $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \ge 1)$ and $(x)_0 = 1$. As shown in [5], C-numbers are particularly useful for obtaining closed sum formulas for combinatorial identities involving $\binom{ak+b}{m}$ as a factor in the summands, wherein $m, k \in \mathbf{N}$ and $a, b \in \mathbf{R}$. Indeed, there are two related general formulas that have been presented and utilized in [5], namely the following

$$\sum_{k=0}^{\infty}\binom{ak+b}{m}f^{(k)}(0)\frac{t^k}{k!} = \frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)f^{(j)}(t)t^j, \tag{4.12}$$

$$\sum_{k=0}^{\infty}\binom{ak+b}{m}\Delta^k g(0)\binom{t}{k} = \frac{1}{m!}\sum_{j=0}^{m}C(m,j;a,b)\Delta^j g(t-j)(t)_j, \tag{4.13}$$

where $f \in C^\infty$ and $g(t)$ is defined on $\mathbf{Z}$. Obviously, (4.12) and (4.13) could be deduced from the SF(2) and SF(1) with the following special triplets, respectively.

$$\Big\{x=t, \psi(t)=\binom{at+b}{m}, \phi(t)=f(t)\Big\},\quad \Big\{x=t, \psi(t)=\binom{at+b}{m}, \phi(t)=g(t)\Big\}.$$

Remark 4.6 It is easily seen that (4.12) is an exact formula for $|t| < \rho$, provided that $f(t)$ has a Maclaurin series expansion for $|t| < \rho$. Moreover, the absolute convergence of the series in (4.13) is ensured by the conditions $|t| < \infty$ and

$$\overline{\lim_{k\to\infty}}\,|\Delta^k g(0)|^{1/k} < 1. \tag{4.14}$$

Also, note that a variety of special formulas and identities deducible from either (4.12) or (4.13) may be found in [5] or elsewhere.

5. $\Sigma\Delta D$ class and some related remarks

In this section all the mathematical terminologies will be used in the ordinary sense in mathematical sciences. Let us give here the following two definitions.

Definition 5.1 *A mathematical formula is said to be deducible from the GSF, if it could be deduced formally from any of the* SF(i) $(i = 1, 2, 3)$ *or from the GSF itself with a special choice of the quintuplet* $\{\delta, A(t), f(t), g(t), \phi(t)\}$.

Definition 5.2 *All the mathematical formulas which are deducible from the GSF are said to form a formula class, so-called* $\Sigma\Delta D$ *class.*

As known from our former papers quoted in the preceding sections, more than 50 special formulas and identities are the members belonging to the $\Sigma\Delta D$ class. What is worth noticing is the fact that the $\Sigma\Delta D$ class includes as special members those well-known classical formulas due, respectively, to Newton, Taylor, Euler, Stirling, Vandermonde, Montmort, Riordan, Carlitz, Li Shanlai, Knuth, Grosswald, Rosenbaum, Stanley, Melzak, Zave, et al.

Remark 5.3 Sometimes, certain members of the $\Sigma\Delta D$ class may have limits when some parameters tend to ∞. For instance, taking the special triplet of the SF(1)

$$\{x = \alpha, \psi(t) = t^n, \phi(t) = (1 + \frac{1}{\alpha})^t\}, \quad \alpha > 1$$

one may get a special formula of the form

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} (\frac{1}{\alpha})^k \cdot k^n = \sum_{j=0}^{n} \binom{\alpha}{j} (\frac{1}{\alpha})^j j! \left\{ \begin{matrix} n \\ j \end{matrix} \right\} (1 + \frac{1}{\alpha})^{\alpha - j} \tag{5.1}$$

which belongs to the $\Sigma\Delta D$ class. Obviously, (5.1) yields the following limit when $\alpha \to \infty$:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} = e \sum_{j=0}^{n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} = e\omega(n). \tag{5.2}$$

This is the well-known Dobinski formula for Bell numbers $\omega(n)$. Thus, if the $\Sigma\Delta D$ class is extended to include all those limits of members as members, then the Dobinski formula is a member of the class. Similarly, observe that

$$\lim_{a \to \infty} a^{-m} C(m, k; a, b) = \frac{1}{k!} \Delta^k (t^m)_{t=0} = \left\{ \begin{matrix} m \\ k \end{matrix} \right\}$$

and it is seen that the limit form of (4.12) (with $a \to \infty$) yields Grunnert's formula

$$(t \frac{\mathrm{d}}{\mathrm{d}t})^m f(t) = \sum_{k=0}^{\infty} k^m f^{(k)}(0) \frac{t^k}{k!} = \sum_{j=0}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} f^{(j)}(t) t^j. \tag{5.3}$$

Consequently, Grunnert's formula is also a member of the extended $\Sigma\Delta D$ class.

Remark 5.4 The wording 'deduced formally' used in the Definition 5.1 may be given a little more explanation. Clearly, in the present paper, the so-called formal derivation used for getting special formulas from source formulas, generally consists of using (i) operations with fps, (ii) symbolic operations with Δ, D and E, (iii) ordinary algebraic computations, (iv) ordinary computational methods in mathematical analysis (including uses of Taylor's series expansion and Newton's interpolation series), (v) operations with infinite series, and exchange of the orders of repeated series summations without considering convergence problems and (vi) mathematical tables including short tables of difference formulas and derivative formulas.

Remark 5.5 Evidently, by adopting the multi-index notational system, it is easy to formulate the GSF and SF(i) ($i = 1, 2, 3$) in multivariate forms. Certainly, such a higher dimensional extension may be worth giving in details, if it could be found really useful in applications. As regards the problem whether it is possible to extend the main results of this paper to the cases of q-analysis should be worthy of investigation.

Remark 5.6 As seen in [3] (§6), we have defined a formula chain via iterations of the GSF. More precisely, a chain of formulas with freedom-degrees (∞^{3m+1}) ($m = 1, 2, 3, \ldots$) could be generated successively by the iteration formulas [3]

$$\sum_{k\geq 0}(p_k^{(m)}(D)f_m(0))\delta^k\phi_m(t) = f_{m+1}(t), \quad m \geq 1$$

with start from A_1, g_1, f_1 and ϕ_1. In this way, each formula of freedom-degree (∞^{3m+1}) could be used as a general source formula to yield a formula class, denoted by $(\Sigma\Delta D)_{(3m+1)}$. Consequently, we may get a sequence of formula classes with increasing freedom-degrees, viz.

$$(\Sigma\Delta D)_{(4)} \subset (\Sigma\Delta D)_{(7)} \subset \cdots \subset (\Sigma\Delta D)_{(3m+1)} \subset \cdots .$$

Here the first one is just the $\Sigma\Delta D$ class treated in this paper, and it may be the most available class of formulas in the Discrete Analysis and Combinatorics.

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