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Some Notes on Inplace Identities for Compositions

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Abstract In this paper we give combinatorial proofs of two recurrence relations for the special class of objects known as inplace compositions. We also obtain new identities for the numbers of inplace 1-2 compositions and palindromic compositions.

Keywords compositions; 1-2 compositions; palindromic compositions; inplace; identity; combinatorial proof

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1. Introduction

In the classical theory of partitions, studying identities has always been an interesting topic. In recent years, a large number of researchers have been studying and many results in this field have been got [1–6].

In particular, Andrews, Hirschhon and Sellers established the following identity.

Theorem 1.1 ([1]) *The number of partitions of n in which each even part occurs with even multiplicity equals the number of partitions of n where no part is congruent to* $2 \pmod 4$.

In 2015, Munagi-Sellers gave some corresponding identities of compositions.

Theorem 1.2 ([7]) *For all $n \geq 1$, the number of compositions of n when each even part occurs inplace with even multiplicity equals the number of compositions of n in which no part is congruent to* $2 \pmod 4$.

Theorem 1.3 ([7]) *For all $n \geq 1$, the number of compositions of $2n$ such that each odd part appears inplace with even multiplicity equals the number of compositions of n where each odd part can be of two kinds.*

Further, the generalizations of the above identities are also obtained.

Theorem 1.4 ([7]) *Let $k \geq 2$ and $l \geq 2$ be fixed integers. For all $n \geq 1$, the number of compositions of n when each part divisible by k occurs inplace with multiplicity a multiple of l equals the number of compositions of n in which no part is congruent to $ik \pmod{lk}$, where $1 \leq i \leq l-1$.*

Theorem 1.5 ([7]) *Let $k \geq 2$ be fixed integer. For all $n \geq 1$, the number of compositions of kn*

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such that each part not divisible by k appears inplace with multiplicity divisible by k equals the number of compositions of n when each part not divisible by k can be of two kinds.

In this paper, we still adopt the term inplace formulated by researchers. A part appears j times inplace in a composition if it appears in j consecutive positions in the composition. For example, in the composition $(2,2,2,2,3,4,4,5,6,6,2,2,3,1,1)$, even parts appear inplace with even multiplicity while odd parts are inplace distinct.

Let $C_{2,2}(n)$ denote the number of compositions of n in which no part is congruent to $2 \pmod 4$, and let $O_2(n)$ denote the number of compositions of n when each odd part with two kinds. Munagi-Sellers [7] designated the second "kind" of odd part with the use of an asterisk.

In this paper, we will show the combinatorial proofs of the recurrence relation of $C_{2,2}(n)$ and $O_2(n)$, respectively. In Section 3, some inplace identities about the compositions with parts being 1 or 2 and the palindromic compositions can be obtained.

Definition 1.6 ([8]) *A palindromic composition of n is a composition that reads the same forward as backward.*

Thus, for example, the palindromic compositions of 4 are as follows:

$$(4),\quad (2,2),\quad (1,2,1),\quad (1,1,1,1).$$

2. Two combinatorial proofs

The generating function of $C_{2,2}(n)$ as $\sum_{n\geq 1} C_{2,2}(n)x^n = \frac{x+x^3+x^4}{1-x-x^3-2x^4}$ is given by Munagi-Sellers in [7]. From the generating function one can easily get the following recurrence relation of $C_{2,2}(n)$.

Theorem 2.1 *Let $C_{2,2}(n)$ denote the number of compositions of n in which no part is congruent to $2 \pmod 4$. Then*

$$\begin{aligned} &C_{2,2}(1)=1,\ C_{2,2}(2)=1,\ C_{2,2}(3)=2,\ C_{2,2}(4)=4,\\ &C_{2,2}(n)=C_{2,2}(n-1)+C_{2,2}(n-3)+2C_{2,2}(n-4),\quad n\geq 5. \end{aligned}$$

Proof Obviously, the relevant composition of 1 is (1), the relevant composition of 2 is $(1,1)$, the relevant compositions of 3 are (3), $(1,1,1)$, and the relevant compositions of 4 are (4), $(1,3)$, $(3,1)$, $(1,1,1,1)$.

When $n\geq 5$, we split the compositions of n in which no part is congruent to $2 \pmod 4$ into three classes:

(A) the part on the right end is 1;

(B) the part on the right end is 3;

(C) the part on the right end is > 3.

Given any composition in class (A), we delete the part 1 on the right end to get the composition of $n-1$ in which no part is congruent to $2 \pmod 4$. And vice versa.

Given any composition in class (B), we delete the part 3 on the right end to produce the composition of $n-3$ in which no part is congruent to 2 (mod 4). And vice versa.

For all compositions in class (C), let d denote the part on the right end of the composition, where $d \geq 4$. If $d = 4$, then deleting $d = 4$, we get the composition of $n-4$ in which no part is congruent to 2 (mod 4). If $d > 4$, then replacing d by $(d-4)$, we can also get the composition of $n-4$ in which no part is congruent to 2 (mod 4). Conversely, for every composition of $n-4$ in which no part is congruent to 2 (mod 4), we have two ways to get the compositions of n in which no part is congruent to 2 (mod 4). One is adding 4 to the right end, the other is appending 4 to the right end.

For example, the composition $(1,1)$ of 2 produces two compositions of 6 as follows:

$$(1,1) \longrightarrow (1,5), \quad (1,1) \longrightarrow (1,1,4).$$

Hence the class (C) has $2C_{2,2}(n-4)$ compositions, and we have

$$C_{2,2}(n) = C_{2,2}(n-1) + C_{2,2}(n-3) + 2C_{2,2}(n-4).$$

Therefore, we completes the proof. □

Further, Munagi-Sellers [7] gave the recurrence relation of the number of compositions of n when each odd part with two kinds:

Theorem 2.2 ([7]) *Let $O_2(n)$ denote the number of compositions of n when each odd part can be of two kinds. Then*

$$O_2(n) = 2(O_2(n-1) + O_2(n-2)), \quad n \geq 3$$

with initial conditions $O_2(1) = 1,\ O_2(2) = 5$.

They obtained the recurrence relation of $O_2(n)$ from the generating function. In this paper, we give the combinatorial proof of it.

Proof We split the compositions of n when each odd part has two kinds into two classes:

(A) the part on the right end is 1 or 1^*;

(B) the part on the right end is t or t^*, where $t > 1$.

Given any composition in class (A), we delete the right part 1 or 1^* to get the composition of $n-1$, with which each odd part has two kinds. Because there are two compositions of n with the same parts except the part 1 or 1^* on the right end, we have an identical composition of $n-1$ from two compositions of n by deleting 1 or 1^* on the right end. Conversely, for every composition with each odd part having two kinds of $n-1$, we append 1 or 1^* to the right end respectively to obtain two compositions of n. Of course the compositions with only even parts are included in this case. So there are $2O_2(n-1)$ compositions in class (A).

Given any composition in class (B), if $t = 2$, then we delete $t = 2$ to get the composition with each odd part having two kinds of $n-2$, and vice versa. As a result, we obtain all compositions with each odd part having two kinds of $n-2$. If $t > 2$, then replace t by $(t-2)$ or replace t^* by $(t-2)^*$ to get composition of $n-2$. Conversely, for every composition with each odd part having two kinds of $n-2$, we add 2 to the part on the right end to obtain the composition with each odd part having two kinds of n, and the right part is more than 2. Since t and $t-2$ have the same parity, we will still get all compositions of $n-2$ with each odd part having two kinds. Of course, the compositions with even parts are included in these cases above. So we have $2O_2(n-2)$ compositions in class (B).

Hence we obtain $O_2(n)=2(O_2(n-1)+O_2(n-2))$.

Clearly, the relevant compositions of 1 are $(1),(1^*)$, and the relevant compositions of 2 are (2), $(1,1)$, $(1,1^*)$, $(1^*,1)$, $(1^*,1^*)$. $\Box$

We use the following example to demonstrate the proof.

Example 2.3 Let $n=3$. Then the corresponding relations about the compositions of 3 with each odd part having two kinds, the compositions of 2 with each odd part having two kinds and the compositions of 1 with each odd part having two kinds are as follows.

$$\begin{aligned}
&(2,1)\longleftrightarrow(2)\longleftrightarrow(2,1^*),\quad (1,1,1)\longleftrightarrow(1,1)\longleftrightarrow(1,1,1^*),\\
&(1,1^*,1)\longleftrightarrow(1,1^*)\longleftrightarrow(1,1^*,1^*),\quad (1^*,1,1)\longleftrightarrow(1^*,1)\longleftrightarrow(1^*,1,1^*),\\
&(1^*,1^*,1)\longleftrightarrow(1^*,1^*)\longleftrightarrow(1^*,1^*,1^*),\\
&(3)\longleftrightarrow(1)\longleftrightarrow(1,2),\quad (3^*)\longleftrightarrow(1^*)\longleftrightarrow(1^*,2).
\end{aligned}$$

3. Several identities for special compositions

We now consider the compositions having parts of size 1 or 2 and the compositions having only odd parts, and referred to here as 1-2 compositions and odd compositions, respectively. We have the following identity.

Theorem 3.1 *For all $n\geq 1$, the number of 1-2 compositions of n when 2 occurs inplace with even multiplicity equals the number of odd compositions of $n+1$ in which no part is congruent to* $3\ (\mathrm{mod}\,4)$.

Proof Given any 1-2 composition of n and 2 occurs inplace with even multiplicity, we first append 1 to the right end, then we adjoin 1 with all 2's to the left to form a new part from right to left. Consequently we obtain the *odd compositions* of $n+1$ and each odd part is congruent to $1\ (\mathrm{mod}\,4)$. This correspondence is one-to-one. $\Box$

Some further study gives the following identity.

Theorem 3.2 *For all $n\geq 1$, the number of 1-2 compositions of n with the first part being 1 when 2 occurs inplace with even multiplicity equals the number of odd compositions of n in which no part is congruent to* $3\ (\mathrm{mod}\,4)$.

Proof The proof is similar to that of Theorem 3.1, except for that we adjoin 1 with all 2's to the right to form a new part from left to right. $\Box$

Similarly, we also get the following result.

Theorem 3.3 *For all $n \geq 1$, the number of 1-2 compositions of n with the last part being 1 when 2 occurs inplace with even multiplicity equals the number of odd compositions of n in which no part is congruent to* $3 \pmod 4$.

In the followings we study palindromic compositions. The palindromic compositions having only odd parts, we call them as odd palindromic compositions while the palindromic compositions having parts of size 1 or 2, we call them as 1-2 palindromic compositions. We have the following result.

Theorem 3.4 *For all $n \geq 1$, the number of 1-2 palindromic compositions of n when 2 occurs inplace with even multiplicity equals the number of odd palindromic compositions of $n+1$ in which no part is congruent to* $3 \pmod 4$.

The proof is similar to that of Theorem 3.1, so we omit it. We offer the following example to illustrate Theorem 3.4.

Example 3.5 Let $n = 8$. Then the first set of compositions contains the following 3 objects:

$$(2,2,2,2), \quad (1,1,1,1,1,1,1,1), \quad (1,1,2,2,1,1).$$

The second set of compositions contains these 3 objects:

$$(9), \quad (1,1,1,1,1,1,1,1,1), \quad (1,1,5,1,1).$$

For palindromic compositions, we also give the following corollaries presented by Munagi-Sellers.

Corollary 3.6 *For all $n \geq 1$, the number of palindromic compositions of n when each even part occurs inplace with even multiplicity equals the number of palindromic compositions of n in which no part is congruent to* $2 \pmod 4$.

Corollary 3.7 *For all $n \geq 1$, the number of palindromic compositions of $2n$ such that each odd part appears inplace with even multiplicity equals the number of palindromic compositions of n where each odd part can be of two kinds.*

And the generalizations of these results mentioned above are as follows.

Corollary 3.8 *Let $k \geq 2$ and $l \geq 2$ be fixed integers. For all $n \geq 1$, the number of palindromic compositions of n when each part divisible by k occurs inplace with multiplicity a multiple of l equals the number of palindromic compositions of n in which no part is congruent to $ik \pmod{lk}$, where $1 \leq i \leq l-1$.*

Corollary 3.9 *Let $k \geq 2$ be fixed integer. For all $n \geq 1$, the number of palindromic compositions of kn such that each part not divisible by k appears inplace with even multiplicity divisible by k equals the number of palindromic compositions of n when each part not divisible by k can be of two kinds.*

We know that Corollary 3.8 is the generalization of Corollary 3.6, and Corollary 3.9 is the generalization of Corollary 3.7.

The proofs of these corollaries are similar to Munagi-Sellers's proofs in [7], we only give an example to show Corollary 3.7.

Example 3.10 Let $n = 3$. Then the first set of compositions contains the following 6 objects:

$$(6),\ (3,3),\ (2,2,2),\ (2,1,1,2),\ (1,1,2,1,1),\ (1,1,1,1,1,1).$$

The second set of compositions contains these 6 objects:

$$(3),\ (3^*),\ (1,1,1),\ (1,1^*,1),\ (1^*,1,1^*),\ (1^*,1^*,1^*).$$

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