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(m,d)-Injective Covers and Gorenstein (m,d)-Flat Modules

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Abstract We consider the conditions under which the class of (m,d)-injective R-modules is (pre)covering. It is shown that every left R-module over a left (m,d)-coherent ring has an (m,d)-injective cover. Moreover, the classes of Gorenstein (m,d)-flat modules and Gorenstein (m,d)-injecitve modules are introduced and studied. For a right (m,d)-coherent ring R, we prove that a left R-module M is Gorenstein (m,d)-flat if and only if M^+ is Gorenstein (m,d)-injective as a right R-module. Some results on Gorenstein flat modules and Gorenstein n-flat modules are generalized.

Keywords (m,d)-injective cover; Gorenstein (m,d)-flat module; Gorenstein (m,d)-injecitve module; strongly Gorenstein (m,d)-flat module

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1. Introduction

Throughout, unless otherwise indicated, R is an associative ring with identity and modules are unitary. For any left R-module M, we use the notation $\mathrm{pd}_R(M)$ to denote the projective dimension of M. The character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ of a module M is denoted by M^+. Let $\mathscr{F}$ be a class of left R-modules and M a left R-module. Following [1], a homomorphism $\phi: F \longrightarrow M$ is said to be an $\mathscr{F}$-precover if $F \in \mathscr{F}$ and the abelian group homomorphism $\mathrm{Hom}(F',\phi): \mathrm{Hom}(F',F) \longrightarrow \mathrm{Hom}(F',M)$ is surjective for each $F' \in \mathscr{F}$. An $\mathscr{F}$-precover is said to be an $\mathscr{F}$-cover if every endomorphism $\varphi: F \longrightarrow F$ such that $\phi\varphi = \phi$ is an isomorphism. Dually we have the definition of an $\mathscr{F}$-(pre)envelope. We note that $\mathscr{F}$-covers ($\mathscr{F}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism. Given a class $\mathscr{X}$ of left R-modules and a complex $\mathbb{Y}$, we say $\mathbb{Y}$ is $\mathrm{Hom}_R(\mathscr{X},-)$-exact if the complex $\mathrm{Hom}_R(X,\mathbb{Y})$ is exact for each $X \in \mathscr{X}$. Dually, the complex $\mathbb{Y}$ is $\mathrm{Hom}_R(-,\mathscr{X})$-exact if $\mathrm{Hom}_R(\mathbb{Y},X)$ is exact for each $X \in \mathscr{X}$, and $\mathbb{Y}$ is $-\otimes_R \mathscr{X}$-exact if $\mathbb{Y} \otimes_R X$ is exact for each $X \in \mathscr{X}$.

Let R be a ring and n a non-negative integer. According to [2], a left R-module M is called n-presented if it has a finite n-presentation, i.e., there is an exact sequence of left R-modules $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ in which every F_i is a finitely generated free left R-module, equivalently projective left R-module. Clearly, every finitely generated projective

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left R-module is n-presented for each n. An R-module is 0-presented (resp., 1-presented) if and only if it is finitely generated (resp., finitely presented), and each m-presented module is n-presented for $m \geq n$. Recall from [2] that a ring R is said to be left n-coherent if every n-presented left R-module is $(n+1)$-presented. It is easy to see that a ring R is left 0-coherent (resp., 1-coherent) if and only if R is a left Noetherian (resp., coherent) ring. Clearly, every n-coherent ring is m-coherent for $m \geq n$.

Let R be a ring, m a positive integer and d a positive integer or $d = \infty$. According to [3], a left R-module M is said to be (m,d)-injective if $\mathrm{Ext}_R^m(N,M) = 0$ for any m-presented left R-module N with $\mathrm{pd}_R(N) \leq d$. A right R-module M is said to be (m,d)-flat if $\mathrm{Tor}_m^R(M,N) = 0$ for any m-presented left R-module N with $\mathrm{pd}_R(N) \leq d$. It is easy to see that the concept of (m,d)-injective modules unifies the two concepts of n-FP-injective modules in [4] and [5]. Note that the concepts of n-FP-injective modules in [4] and [5] are different. In what follows, we denote by $\mathcal{F}_{m,d}$ (resp., $\mathcal{I}_{m,d}$) the class of all (m,d)-flat (resp., (m,d)-injective) left R-modules. It is well-known that a ring R is left Noetherian if and only if every left R-module has an injective (pre)cover [6, Theorem 5.4.1]. Mao and Ding proved that the class of (m,d)-injective R-modules is preenveloping for any ring R (see [3, Theorem 4.4]). So it is natural to ask: Under what conditions on the base ring R the class of (m,d)-injective R-modules is (pre)covering?

In this paper, we continue to study the conditions under which the class of (m,d)-injective modules is precovering. The classes of Gorenstein (m,d)-flat modules and Gorenstein (m,d)-injecitve modules are also introduced and investigated. The paper is organized as follows. Section 2 contains notations and definitions needed for this paper. In Section 3, we show that every left R-module over a left (m,d)-coherent ring has an (m,d)-injective cover. As a corollary, we prove that every left R-module over a left (n,∞)-coherent ring has an n-FP-injective (modules in [4]) cover. Section 4 is a study of Gorenstein (m,d)-flat modules and Gorenstein (m,d)-injecitve modules. For a right (m,d)-coherent ring, we prove that a left R-module M is Gorenstein (m,d)-flat if and only if M^+ is Gorenstein (m,d)-injective as a right R-module. It is shown that the class of Gorenstein (m,d)-flat left R-modules over a right (m,d)-coherent ring is closed under pure submodules.

2. Preliminaries

In this section, we recall some known notions and facts needed in the sequel. An R-module M is called FP-injective [7] in case $\mathrm{Ext}_R^1(P,M) = 0$ for every finite presented R-module P. As a generalization of FP-injective modules, the concept of n-FP-injective modules was introduced in [4]. According to [4], a right R-module M is n-FP-injective if $\mathrm{Ext}_R^n(N,M) = 0$ for all n-presented right R-modules N. Lee introduced and investigated n-coherent rings from a different point of view. Recall from [5] that a ring R is said to be left n-coherent (for integers $n > 0$ or $n = \infty$) if every finitely generated submodule N of a free left R-module with $\mathrm{pd}_R(N) \leq n-1$ is finitely presented. Clearly, all rings are left 1-coherent and the left coherent rings are exactly those which are d-coherent, where d is the left global dimension of R with $0 < d \leq \infty$. More results and their analogues can be found in [1–11].

Definition 2.1 *A left R-module M is said to be Gorenstein flat, if there exists an exact sequence of flat left R-modules*

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Coker}(F_1 \to F_0)$ such that the sequence is $I \otimes_R -$ exact for every injective right R-module I.

A left R-module N is called Gorenstein FP*-injective ([8]), if there exists an exact sequence of injective left R-modules*

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $N \cong \mathrm{Coker}(I_1 \to I_0)$ such that the sequence is $\mathrm{Hom}_R(E,-)$ exact for every FP*-injective left R-module E.*

Lee also studied the class of n-FP-injective modules which is different from that of [4]. Recall from [5] that a left R-module M is said to be n-FP-injective if $\mathrm{Ext}^1_R(N,M)=0$ for all finitely presented right (resp., left) R-modules N with $\mathrm{pd}_R(N) \le n$. A left (resp., right) R-module M is called n-flat if $\mathrm{Tor}^R_1(N,M)=0$ (resp., $\mathrm{Tor}^R_1(M,N)=0$) for all finitely presented right (resp., left) R-modules N with $\mathrm{pd}_R(N) \le n$.

The following notion was introduced and studied in [10], which is a generalization of n-flat modules in [5].

Definition 2.2 *A left R-module M is said to be Gorenstein n-flat, if there exists an exact sequence of n-flat left R-modules*

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

*with $M \cong \mathrm{Coker}(F_1 \to F_0)$ such that the sequence is $I \otimes_R -$ exact for every n-*FP*-injective right R-module I.*

It is easy to see that every n-flat module is Gorenstein n-flat. In general, a Gorenstein n-flat module need not be n-flat by [10, Example 3.3].

Definition 2.3 *A short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called n-pure [12], if the sequence $\mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \longrightarrow 0$ is exact for any n-presented module M. Moreover, a submodule N of M is called n-pure if the sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ is n-pure exact.*

It is clear that an R module A is 1-pure in B if and only if it is pure, and it is 0-pure if and only if the epimorphism $B \longrightarrow C \longrightarrow 0$ is finitely split. Obviously, if A is n-pure in B, then A is m-pure for any $m > n$.

As a generalization of n-coherent rings in [2] and [5], the following concept of (m,d)-coherent rings was introduced and studied by Mao and Ding in [3].

Definition 2.4 *Let m be a positive integer and d a positive integer or $d=\infty$. A ring R is called a left (m,d)-coherent ring in case every m-presented left R-module N with* $\mathrm{pd}_R(N)\le d$ *is $(m+1)$-presented.*

It is clear that a ring R is left coherent if and only if R is a left $(1,\infty)$-coherent ring if and only if R is a left $(1,d)$-coherent ring, where d denotes the left global dimension of R with $0<d\le\infty$. Obviously, (m,d)-coherent rings unify two different concepts of n-coherent rings appearing in [2] and [5].

3. (m,d)-coherent rings and (m,d)-injective covers

In this section, we investigate the existence of (m,d)-injective (pre)cover of a left R-module. It is proved that every left R-module over a left (m,d)-coherent ring has an (m,d)-injective cover.

We begin with the following

Proposition 3.1 *Let R be a left (m,d)-coherent ring. Then every n-pure submodule of a left (m,d)-injective R-module is (m,d)-injective.*

Proof Let $M\in\mathcal{I}_{m,d}$, N an n-pure submodule of M and H an m-presented left R-module with $\mathrm{pd}_R(H)\le d$. Since R is left (m,d)-coherent, H has a finite $(n+m-1)$-presentation

$$F_{n+m-1}\longrightarrow\cdots\longrightarrow F_{m-2}\longrightarrow F_{m-3}\longrightarrow\cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow H\longrightarrow 0\ .$$

Let $K=\mathrm{Ker}(F_{m-2}\longrightarrow F_{m-3})$. It is easy to see that K is n-presented and we have $\mathrm{Ext}^1_R(K,M)\cong\mathrm{Ext}^m_R(H,M)=0$ since M is (m,d)-injective. Moreover, the short exact sequence

$$0\longrightarrow N\longrightarrow M\longrightarrow M/N\longrightarrow 0$$

induces the exact sequence

$$0\longrightarrow \mathrm{Hom}_R(K,N)\longrightarrow \mathrm{Hom}_R(K,M)\longrightarrow \mathrm{Hom}_R(K,M/N)\longrightarrow \mathrm{Ext}^1_R(K,N)\longrightarrow 0\ .$$

Since N is an n-pure submodule of M, the sequence $\mathrm{Hom}_R(K,M)\longrightarrow\mathrm{Hom}_R(K,M/N)\longrightarrow 0$ is exact. This implies that $\mathrm{Ext}^1_R(K,N)=0$, and hence

$$\mathrm{Ext}^m_R(H,N)\cong\mathrm{Ext}^1_R(K,N)=0.$$

Therefore, N is (m,d)-injective. □

Let R be a ring, M a left R-module and L a submodule of M. By [13, Theorem 5], if for each cardinal λ there is a cardinal κ such that the cardinality $|M|\ge\kappa$ and the cardinality $|M/L|\le\lambda$, then L contains a nonzero pure submodule of M.

The following result is straightforward by Proposition 3.1 and [13, Theorem 5].

Lemma 3.2 *Let R be a left (m,d)-coherent ring and M an (m,d)-injective left R-module. If L is a submodule of M and for each cardinal λ there is a cardinal κ such that the cardinality*

$|M| \geq \kappa$ and the cardinality $|M/L| \leq \lambda$, then L contains a nonzero (m,d)-injective submodule of M.

The following result is a generalization of [9, Lemma 2.5].

Lemma 3.3 *Let R be a left (m,d)-coherent ring and M a left R-module with $|M| = \lambda$. Then there is a cardinal κ such that any homomorphism $E \longrightarrow M$ with E an (m,d)-injective module has a factorization $E \longrightarrow E' \longrightarrow M$ such that $|E'| < \kappa$ and E' is (m,d)-injective.*

Proof Let $E \longrightarrow M$ be any homomorphism with E an (m,d)-injective left R-module. By Lemma 3.2, there is a submodule L of E and a cardinal κ such that if $|E| \geq \kappa$ and $|E/L| \leq \lambda$, then L contains a non-zero (m,d)-injective submodule of E. If $|E| < \kappa$, let $E' = E$ and the result follows. Suppose that $|E| \geq \kappa$. Then we can choose a submodule $S \subset E$ such that S is maximal under the conditions that S is (m,d)-injective and $S \subset \mathrm{Ker}(E \longrightarrow M)$. Let $E' = E/S$. It is clear that the homomorphism $E \longrightarrow M$ has a factorization $E \longrightarrow E' \longrightarrow M$. Since R is left (m,d)-coherent and the sequence

$$0 \longrightarrow S \longrightarrow E \longrightarrow E/S \longrightarrow 0$$

is exact, it follows that E/S is an (m,d)-injective module by [3, Theorem 4.4]. To conclude the proof, it suffices to show that $|E^{'}| < \kappa$. Assume that $|E'| \geq \kappa$ and put $K = \mathrm{Ker}(E' \longrightarrow M)$. It is easy to see that $|E^{'}/K| \leq |M| = \lambda$. Again by Lemma 3.2, there is a non-zero (m,d)-injective submodule T/S of E/S contained in K. Therefore, $T \subset \mathrm{Ker}(E \longrightarrow M)$. Since S and T/S are (m,d)-injective, it follows from the exactness of the sequence

$$0 \longrightarrow S \longrightarrow T \longrightarrow T/S \longrightarrow 0$$

that T is (m,d)-injective. This contradicts the choice of S. This implies that the homomorphism $E \longrightarrow M$ has a factorization $E \longrightarrow E' \longrightarrow M$ with $|E'| < \kappa$ and E' an (m,d)-injective module. □

The next corollary is [9, Lemma 2.4].

Lemma 3.4 *Let $\mathscr{A}$ be a class of left R-modules that is closed under direct sums. If $\mathscr{B} \subset \mathscr{A}$, for some set $\mathscr{B}$, is such that any homomorphism $A \longrightarrow M$ with $A \in \mathscr{A}$ can be factored $A \longrightarrow B \longrightarrow M$ for some $B \in \mathscr{B}$, then M has an $\mathscr{A}$-precover.*

Now we give the following main result of this section

Proposition 3.5 *Let R be a left (m,d)-coherent ring. Then every left R-module has an (m,d)-injective precover.*

Proof Let M be any left R-module such that $|M| = \lambda$. By Lemma 3.3, there is a cardinal κ such that any homomorphism $E \longrightarrow M$ with $E \in \mathcal{I}_{m,d}$ has a factorization $E \longrightarrow E' \longrightarrow M$ such that E' is (m,d)-injective and $|E'| < \kappa$. Let $\mathcal{A}$ be any set with $|\mathcal{A}| = \kappa$ and let $\mathcal{B}$ be all (m,d)-injective left R-modules such that $\mathcal{B} \subset \mathcal{A}$ (as sets). Therefore, if we replace $E^{'}$ with an isomorphic copy, then we can assume $E' \subset \mathcal{A}$ (as a set). Now we apply Lemma 3.4, and the result follows. □

It was shown in [9, Corollary 2.7] that every left R-module over a left coherent ring has an FP-injective cover. As a generalization of this result, we have the following

Theorem 3.6 *Let R be a left (m, d)-coherent ring. Then every left R-module has an (m, d)-injective cover.*

Proof Note that every left R-module has an (m, d)-injective precover by Proposition 3.5. Since the class of (m, d)-injective left R-modules is closed under direct limits by [3, Theorem 4.3], the result follows from [6, Corollary 5.2.7]. □

Corollary 3.7 *Let R be a left $(1, n)$-coherent ring. Then every left R-module has an n-*FP*-injective* (*modules in [5]*) *cover.*

Corollary 3.8 *Let R be a left (n, ∞)-coherent ring. Then every left R-module has an n-*FP*-injective* (*modules in [4]*) *cover.*

Corollary 3.9 [9, Corollary 2.7] *Let R be a left $(1, \infty)$-coherent ring. Then every left R-module has an* FP*-injective cover.*

4. Gorenstein (m, d)-flat modules

In this section, we introduce the concepts of Gorenstein (m, d)-flat modules and Gorenstein (m, d)-injective modules, and investigate their properties.

We begin with the following

Definition 4.1 *A left R-module M is said to be Gorenstein (m, d)-flat, if there exists an exact sequence of (m, d)-flat left R-modules*

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Coker}(F_1 \to F_0)$ such that the sequence is $I \otimes_R -$ exact for every (m, d)-injective right R-module I.

A left R-module N is called Gorenstein (m, d)-injective, if there exists an exact sequence of (m, d)-injective left R-modules

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $N \cong \mathrm{Coker}(I_1 \to I_0)$ such that the sequence is $\mathrm{Hom}_R(E, -)$ exact for every (m, d)-injective left R-module E.

We denote by $\mathcal{GF}_{(m,d)}$ (resp., $\mathcal{GI}_{(m,d)}$) the class of all Gorentein (m, d)-flat (resp., Gorentein (m, d)-injective) left R-modules.

Remark 4.2 (i) Every (m, d)-flat R-module is Gorenstein (m, d)-flat.

(ii) A Gorenstein (m,d)-flat module need not be Gorenstein flat. In fact, if R is a commutative domain and M is a Gorenstein $(1,1)$-flat module, then M is not a Gorenstein flat module by [10, Example 3.3].

(iii) The class of Gorenstein $(1,n)$-flat (resp., Gorenstein $(1,d)$-injective) modules are precisely the class of Gorenstein n-flat (resp., Gorenstein n-FP-injective modules in [10]).

Proposition 4.3 *Direct sums of Gorenstein (m,d)-flat modules are still Gorenstein (m,d)-flat.*

Proof The result follows from [3, Proposition 3.6(2)] since tensor product commutes with direct sums. □

Proposition 4.4 *Let M be a Gorenstein (m,d)-flat left R-module. Then $\mathrm{Tor}^R_{\geq 1}(E,M)=0$ for all (m,d)-injective right R-module E. The converse is true when R is right (m,d)-coherent.*

Proof Suppose that M is a Gorenstein (m,d)-flat left R-module. Then there is an exact sequence of (m,d)-flat left R-modules $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ such that the sequence is $E\otimes_R -$ exact, where $E\in \mathcal{I}_{m,d}$ is any (m,d)-injective right R-module. This implies that $\mathrm{Tor}^R_{\geq 1}(E,M)=0$ for all (m,d)-injective right R-module E. The converse is similar to that of [14, Theorem 3.6]. □

By Proposition 4.4, we immediately get the following result.

Corollary 4.5 *Let R be a left (m,d)-coherent ring and $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ an exact sequence of left R-modules. Then*

(i) If A and C are Gorenstein (m,d)-flat, then so is B.

(ii) If B and C are Gorenstein (m,d)-flat, then so is A.

(iii) If A and B are both Gorenstein (m,d)-flat, then C is Gorenstein (m,d)-flat if and only if the sequence $0\longrightarrow E\otimes_R A\longrightarrow E\otimes_R B$ is exact for every (m,d)-injective right R-module E.

Similarly, we have the following

Proposition 4.6 *Let M be a Gorenstein (m,d)-injective left R-module. Then $\mathrm{Ext}^{\geq 1}_R(E,M)=0$ for all (m,d)-injective left R-modules E. The converse is true when R is left (m,d)-coherent.*

Proof Since M is Gorenstein (m,d)-injective, there exists an exact sequence

$$0\longrightarrow M\longrightarrow I^0\longrightarrow I^1\longrightarrow\cdots$$

with each I^i (m,d)-injective such that the sequence is $\mathrm{Hom}_R(E,-)$ exact for every (m,d)-injective left R-module E. Therefore, we have $\mathrm{Ext}^{\geq 1}_R(E,M)=0$. The inverse is similar to that of [8, Theorem 2.4]. □

Proposition 4.7 *Let R be a right (m,d)-coherent ring. Then a left R-module M is Gorenstein (m,d)-flat if and only if M^+ is Gorenstein (m,d)-injective as a right R-module.*

Proof Assume that R is a left (m,d)-coherent ring and let $M\in\mathcal{GF}_{(m,d)}$. Then there exists an exact sequence of (m,d)-flat left R-modules

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $M \cong \mathrm{Coker}(F_1 \to F_0)$ such that the sequence is $I \otimes_R -$ exact for every (m,d)-injective right R-module. Therefore, we have the following exact sequence

$$\cdots \longrightarrow (F^1)^+ \longrightarrow (F^0)^+ \longrightarrow (F_0)^+ \longrightarrow (F_1)^+ \longrightarrow \cdots$$

such that $M^+ \cong \mathrm{Coker}((F_0)^+ \to (F_1)^+)$. Note that all $(F_i)^+$ and $(F^j)^+$ are (m,d)-injective R-modules for all i and j by [3, Lemma 3.5]. To conclude the proof, it suffices to show that the above sequence is $\mathrm{Hom}_R(E,-)$ exact for every (m,d)-injective left R-module E. In fact, we have the isomorphisms

$$\mathrm{Hom}_R\big(E,(F_i)^+\big) \cong \mathrm{Hom}_R\big(F_i \otimes_R E, \mathbb{Q}/\mathbb{Z}\big),$$
$$\mathrm{Hom}_R\big(E,(F^j)^+\big) \cong \mathrm{Hom}_R\big(F^j \otimes_R E, \mathbb{Q}/\mathbb{Z}\big)$$

for all i and j. It follows that M^+ is (m,d)-injective as a right R-module. Conversely, since R is a right (m,d)-coherent ring, the result can be proved similarly as [14, Theorem 3.6]. □

Corollary 4.8 *Let R be a right (m,d)-coherent ring. Consider an exact sequence of left R-modules* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$. *If M_1 and M_2 are Gorenstein (m,d)-flat and* $\mathrm{Tor}_1^R(E,M_3)=0$, *then M_3 is Gorenstein (m,d)-flat.*

Proof Since M_1 and M_2 are Gorenstein (m,d)-flat, $(M_1)^+$ and $(M_2)^+$ are Gorenstein (m,d)-injective modules by Proposition 4.7. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ to the exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$, then we have the following exact sequence

$$0 \longrightarrow (M_3)^+ \longrightarrow (M_2)^+ \longrightarrow (M_1)^+ \longrightarrow 0\ .$$

For each (m,d)-injective left R-module E, we have the following isomorphism

$$\mathrm{Ext}_R^i(E,\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}\Big(\mathrm{Tor}_i^R\big(E,M\big),\mathbb{Q}/\mathbb{Z}\Big)$$

by [15, Theorem 11.54]. It follows from Proposition 4.6 that $(M_3)^+$ is Gorenstein (m,d)-injective. Therefore, M_3 is Gorenstein (m,d)-flat by Proposition 4.7 since R is a right (m,d)-coherent ring. □

Proposition 4.9 *Let R be a right (m,d)-coherent ring. Then the class of Gorenstein (m,d)-flat left R-modules is closed under pure submodules.*

Proof Let M be a Gorenstein (m,d)-flat left R-module and N a pure submodule of M. Then the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ induces the split short exact sequence

$$0 \longrightarrow (M/N)^+ \longrightarrow (M)^+ \longrightarrow (N)^+ \longrightarrow 0\ .$$

Since M is Gorenstein (m,d)-flat, it follows from Proposition 4.7 that $(M)^+$ is a Gorenstein (m,d)-injective module. It is clear that every direct summand of Gorenstein (m,d)-injective module is still Gorenstein (m,d)-injective, and so $(N)^+$ is Gorenstein (m,d)-injective. Therefore, N is Gorenstein (m,d)-flat again by Proposition 4.7. □

Proposition 4.10 *Let R be a commutative (m,d)-coherent ring. Then the following statements are equivalent:*

(i) Every Gorenstein (m,d)-injective R-module is (m,d)-injective.

(ii) Every Gorenstein (m,d)-flat R-module is (m,d)-flat.

Proof (i)⇒(ii). Let M be a Gorenstein (m,d)-flat R-module. Then M^+ is a Gorenstein (m,d)-injecitve R-module by Proposition 4.7. Therefore, M^+ is (m,d)-injective by (1). This implies that M is (m,d)-flat by [3, Lemma 3.5].

(ii)⇒(i). Let M be a Gorenstein (m,d)-injecitve R-module. Then M^+ is a Gorenstein (m,d)-flat module since R is a commutative (m,d)-coherent ring, and so M^+ is (m,d)-flat by (ii). Therefore, M is (m,d)-injective again by [3, Theorem 4.3]. □

Proposition 4.11 *Let R be a commutaive (m,d)-coherent ring. Then the following statements hold:*

(i) An R-module M is Gorenstein (m,d)-flat if and only if $(M^+)^+$ is Gorenstein (m,d)-flat.

(ii) An R-module M is Gorenstein (m,d)-injective if and only if $(M^+)^+$ is Gorenstein (m,d)-injective.

Proof The result follows from [3, Lemma 3.5] and Proposition 4.7. □

We call a left R-module M a strongly Gorenstein (m,d)-flat module, if there exists an exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ with $F \in \mathcal{F}_{m,d}$ such that the sequence

$$0 \longrightarrow E \otimes_R M \longrightarrow E \otimes_R F \longrightarrow E \otimes_R M \longrightarrow 0$$

is exact for each (m,d)-injective right R-module E.

The following proposition gives a characterization of strongly Gorenstein (m,d)-flat modules.

Proposition 4.12 *For a left R-module M, the following statements are equivalent:*

(i) M is a strongly Gorenstein (m,d)-flat module;

(ii) There exists a short exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ with F an (m,d)-flat left R-module such that $\mathrm{Tor}_i^R(M,E)=0$ for any (m,d)-injective right R-module E;

(iii) There exists a short exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ with F an (m,d)-flat left R-module such that $\mathrm{Tor}_i^R(M,E)=0$ for any module E with finite (m,d)-injective dimension;

(iv) There exists an exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ such that the sequence $0 \longrightarrow E \otimes_R M \longrightarrow E \otimes_R F \longrightarrow E \otimes_R M \longrightarrow 0$ is exact for any right R-module E with finite (m,d)-injective dimension, where F is an (m,d)-flat left R-module.

Proposition 4.13 *Every direct sum of strongly Gorenstein (m,d)-flat modules is also strongly*

Gorenstein (m,d)-flat.

Proof The result follows from [3, Proposition 3.6] and the fact that tensor product commutes with sums. □

Similarly, we call a left R-module M a strongly Gorenstein (m,d)-injective module, if there exists an exact sequence $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ with $I \in \mathcal{I}_{m,d}$ such that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(E,M) \longrightarrow \mathrm{Hom}_R(E,I) \longrightarrow \mathrm{Hom}_R(E,M) \longrightarrow 0$$

is exact for each (m,d)-injective left R-module E.

Proposition 4.14 *If M is a strongly Gorenstein (m,d)-flat left R-module, then M^+ is a strongly Gorenstein (m,d)-injective R-module.*

Proof The proof is similar to that of Proposition 4.7. □

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References

[1] E. E. ENOCHS. *Injective and flat covers, envelopes and resolvents*. Israel J. Math., 1981, **39**(3): 189–209.
[2] D. L. COSTA. *Parameterizing families of non-noetherian rings*. Comm. Algebra, 1994, **22**(10): 3997–4011.
[3] Lixin MAO, Nanqing DING. *On a new generalization of coherent rings*. Publ. Math. Debrecen, 2007, **71**(1-2): 67–82.
[4] Jianlong CHEN, Nanqing DING. *On n-Coherent rings*. Comm. Algebra, 1996, **24**(10): 3211–3216.
[5] S. B. LEE. *n-coherent rings*. Comm. Algebra, 2002, **30**(3): 1119–1126.
[6] E. E. ENOCHS, O. M. G. JENDA. *Relative Homological Algebra*. Walter de Gruyter, Berlin, 2000.
[7] B. STENRSTRÖM. *Coherent rings and FP-injective modules*. J. London Math. Soc., 1970, **2**: 323–329.
[8] Lixin MAO, Nanqing DING. *Goresntein FP-injective and Gorenstein flat modules*. J. Algebra Appl., 2008, **7**(4): 491–506.
[9] K. PINZON. *Absolutely pure covers*. Comm. Algebra, 2008, **36**(6): 2186–2194.
[10] C. SELVARAJ, R. UDHAYAKUMAR, A. UMAMAHESWARAN. *Gorenstein n-flat modules and their covers*. Asian-Eur. J. Math., 2014, **7**(3): 1–13.
[11] Liang ZHAO, Yiqiang ZHOU. *D_C-projective dimensions, Foxby equivalence and $\mathcal{SD}_C$-projective modules*. J. Algebra Appl., 2016, **15**(6): 1–23.
[12] Lixin MAO, Nanqing DING. *Relative projective modules and relative injective modules*. Comm. Algebra, 2006, **34**: 2403–2418.
[13] L. BICAN, R. E. BASHIR, E. ENOCHS. *All modules have flat covers*. Bull. London Math. Soc., 2001, **33**(4): 385–390.
[14] H. HOLM. *Gorenstein homological dimension*. J. Pure Appl. Algebra, 2004, **189**(1-3): 167–193.
[15] J. J. ROTMAN. *An Introduction to Homological Algebra*. Academic Press, New York-London, 1979.