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A New Class of Harmonic Multivalent Functions Defined by Subordination

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Abstract In the present paper, we introduce some new subclasses of harmonic multivalent functions defined by generalized Dziok-Srivastava operator. Sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes are obtained.

Keywords harmonic multivalent functions; Dziok-Srivastava operator; subordination; extreme points; distortion bounds

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1. Introduction and preliminaries

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Let H_m ($m \geq 1$) denote the family of functions $f = h + \overline{g}$ that are multivalent harmonic and orientation preserving functions in D with the normalization $h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=m}^{\infty} b_k z^k$ ($|b_m| < 1$). Ahuja and Jahangiri [2,3] introduced and studied certain subclasses of the family H_m.

Denote by H_p the class of p-valent harmonic functions f that are sense preserving in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and f of the form

$$f = h + \overline{g}, \tag{1.1}$$

where

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ \text{ and } \ g(z) = \sum_{k=p+1}^{\infty} b_k z^k. \tag{1.2}$$

Obvious $H_p \subset H_m$.

Also, we denote by $\overline{H}_{(p)}$ the class of p-valent harmonic functions $f \in H_p$ and

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k \ \text{ and } \ g(z) = -\sum_{k=p+1}^{\infty} |b_k| z^k. \tag{1.3}$$

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Let F be fixed multivalent harmonic function given by

$$F=H(z)+\overline{G(z)}=z^p+\sum_{k=p+1}^{\infty}A_kz^k+\overline{\sum_{k=p+1}^{\infty}B_kz^k}. \tag{1.4}$$

We define the Hadamard product (or convolution) of F and f by

$$(F*f)(z):=z^p+\sum_{k=p+1}^{\infty}a_kA_kz^k+\overline{\sum_{k=p+1}^{\infty}b_kB_kz^k}=(f*F)(z). \tag{1.5}$$

For positive real values of α_i $(i=1,\dots,l)$ and β_j $(j=1,\dots,m)$, the generalized hypergeometric function ${}_lF_m$ (with l numerator and m denominator parameters) is defined by

$${}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)(z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_l)_k}{(\beta_1)_k\dots(\beta_m)_k}\cdot\frac{z^k}{k!},$$

where $l\le m+1$; $l,m\in\mathbb{N}_0:=\{0,1,2,\dots\}=\mathbb{N}\cup\{0\}$, and $(\lambda)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n\in\mathbb{N}.\end{cases}$$

Corresponding to the function

$$h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)=z^{-p}{}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)(z),$$

the linear operator $H_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m):H_p\longrightarrow H_p$ is defined by using the following Hadamard product (or convolution):

$$H_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z)=h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)*f(z).$$

For a function f of the form (1.1), we have

$$\begin{aligned}H_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z)=&z^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}a_kz^k+\\&\overline{\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_k\dots(\alpha_l)_k}{k!(\beta_1)_k\dots(\beta_m)_k}b_kz^k}\\:=&H_{p,l,m}[\alpha_1]f(z).\end{aligned} \tag{1.6}$$

The above-defined operator $H_{p,l,m}[\alpha_1]$ $(p=1)$ was introduced by the Dziok-Srivastava operator [4,5]. Using the same methods of [6], we introduce the generalized Dziok-Srivastava operator in $H_{(p)}$ as follows:

$$\begin{aligned}L^{1,\alpha_1}_{\lambda,l,m}f(z)&=(1-\lambda)H_{p,l,m}[\alpha_1]f(z)+\frac{\lambda}{p}z(H_{p,l,m}[\alpha_1]f(z))'\\&:=L^{\alpha_1}_{\lambda,l,m}f(z),\quad\lambda\ge0,\end{aligned}$$

where

$$z(H_{p,l,m}[\alpha_1]f(z))'=z(H_{p,l,m}[\alpha_1]h(z))'-\overline{z(H_{p,l,m}[\alpha_1]g(z))'}.$$

In general,

$$L^{\tau,\alpha_1}_{\lambda,l,m}f(z)=L^{\alpha_1}_{\lambda,l,m}(L^{\tau-1,\alpha_1}_{\lambda,l,m}f(z)),\quad l\le m+1;l,m\in\mathbb{N}_0,\tau\in\mathbb{N}, \tag{1.7}$$

where

$$L^{\tau,\alpha_1}_{\lambda,l,m}f(z) = z^p + \frac{\displaystyle\sum_{k=p+1}^{\infty}\Big(\frac{(1+\frac{k\lambda}{p})(\alpha_1)_k\ldots(\alpha_l)_k}{k!(\beta_1)_k\ldots(\beta_m)_k}\Big)^{\tau}a_k z^k +}{\displaystyle\sum_{k=p+1}^{\infty}\Big(\frac{(1+\frac{k\lambda}{p})(\alpha_1)_k\ldots(\alpha_l)_k}{k!(\beta_1)_k\ldots(\beta_m)_k}\Big)^{\tau}a_k z^k} \tag{1.8}$$

and $\lambda \geq 0, \tau \in \mathbb{N}$.

For $\mu > 0$ and $\tau \in \mathbb{N}$, we introduce the following linear operator $\mathcal{J}^{\mu}_{\tau} : H_p \longrightarrow H_p$, defined by

$$\mathcal{J}^{\mu}_{\tau}f(z) = \mathcal{J}^{\mu}_{\tau}(z) * f(z) = \mathcal{J}^{\mu}_{\tau}(z) * h(z) + \overline{\mathcal{J}^{\mu}_{\tau}(z) * g(z)}, \quad z \in \mathbb{U}, \tag{1.9}$$

where $\mathcal{J}^{\mu}_{\tau}(z)$ is the function defined as follows:

$$L^{\tau,\alpha_1}_{\lambda,l,m}(z) * \mathcal{J}^{\mu}_{\tau}(z) = \frac{z^p}{(1-z)^{\mu}}, \quad \mu > 0, z \in \mathbb{U}, \tag{1.10}$$

and

$$L^{\tau,\alpha_1}_{\lambda,l,m}(z) = z^p + \sum_{k=p+1}^{\infty}\Big(\frac{(1+\frac{k\lambda}{p})(\alpha_1)_k\ldots(\alpha_l)_k}{k!(\beta_1)_k\ldots(\beta_m)_k}\Big)^{\tau} z^k. \tag{1.11}$$

Since

$$\frac{z^p}{(1-z)^{\mu}} = z^p + \sum_{k=1}^{\infty}\frac{(\mu)_k}{k!}z^{k-p}, \quad \mu > 0, z \in \mathbb{U}, \tag{1.12}$$

combining (1.9)–(1.12), we obtain

$$\mathcal{J}^{\mu}_{\tau}(z) = z^p + \sum_{k=p+1}^{\infty}\Big(\frac{k!(\beta_1)_k\ldots(\beta_m)_k}{(1+\frac{k\lambda}{p})(\alpha_1)_k\ldots(\alpha_l)_k}\Big)^{\tau}\frac{(\mu)_k}{k!}z^k, \quad \mu > 0, z \in \mathbb{U}. \tag{1.13}$$

If f is given by (1.1), then we find from (1.9) and (1.13) that

$$\mathcal{J}^{\mu}_{\tau}f(z) = \mathcal{J}^{\mu}_{\tau}h(z) + \overline{\mathcal{J}^{\mu}_{\tau}g(z)} = z^p + \sum_{k=p+1}^{\infty}\Phi^{\mu}_k a_k z^k + \overline{\sum_{k=p+1}^{\infty}\Phi^{\mu}_k b_k z^k}, \tag{1.14}$$

$$\Phi^{\mu}_k = \Big(\frac{k!(\beta_1)_k\ldots(\beta_m)_k}{(1+\frac{k\lambda}{p})(\alpha_1)_k\ldots(\alpha_l)_k}\Big)^{\tau}\frac{(\mu)_k}{k!}, \quad \mu > 0. \tag{1.15}$$

Let f_1 and f_2 be two analytic functions in the open unit disk $\mathbb{U}$. We say that the function f_1 is subordinate to f_2 in $\mathbb{U}$, and write $f_1(z) \prec f_2(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f_1(z) = f_2(\omega(z))$ $(z \in \mathbb{U})$ (see [7]).

By making use of the principle of subordination between analytic functions, we introduce the class $H_p(A, B; \mu, \tau, \alpha, \delta)$.

Definition 1.1 *A function $f(z) \in H_p$ of the form (1.1) is said to be in the class $H_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if*

$$\chi_{\delta,\mu}(f(z)) - \alpha|(\chi_{\delta,\mu}(f(z)) - 1| \prec \frac{1+Az}{1+Bz}, \tag{1.16}$$

where

$$\chi_{\delta,\mu}(f(z)) = (1-\delta)\frac{\mathcal{J}_\tau^\mu f(z)}{z^p} + \frac{\delta}{pz^{p-1}}(\mathcal{J}_\tau^\mu f(z))' \tag{1.17}$$

and $\mathcal{J}_\tau^\mu f(z)$ *is defined by (1.14) and* $p \in \mathbb{N}$*;* $A, B \in \mathbb{R}, A \neq B, |B| \leq 1$*;* $\tau \in \mathbb{N}, \mu > 0, \alpha \geq 0, \delta \geq 0$.

For $\delta = 0$, we obtain the following new subclass:

A function $f \in H_p$ of the form (1.1) is said to be in the class $L_p(A, B; \mu, \tau, \alpha)$ if and only if

$$\frac{\mathcal{J}_\tau^\mu f(z)}{z^p} - \alpha\left|\frac{\mathcal{J}_\tau^\mu f(z)}{z^p} - 1\right| \prec \frac{1+Az}{1+Bz}, \tag{1.18}$$

where $\mathcal{J}_\tau^\mu f(z)$ is defined by (1.14) and $p \in \mathbb{N}$; $A, B \in \mathbb{R}, A \neq B, |B| \leq 1$; $\tau \in \mathbb{N}, \mu > 0, \alpha \geq 0$.

We also let

$$\overline{H}_p(A, B; \mu, \tau, \alpha, \delta) = \overline{H}_p \bigcap H_p(A, B; \mu, \tau, \alpha, \delta)$$

and

$$\overline{L}_p(A, B; \mu, \tau, \alpha) = \overline{H}_p \bigcap L(A, B; \mu, \tau, \alpha).$$

In this paper, we aim to introduce some new subclasses of harmonic multivalent functions defined by generalized Dziok-Srivastava operator and obtain some results including sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes.

2. Main results

Lemma 2.1 ([8]) *Let* $\alpha \geq 0$ *and* $A, B \in \mathbb{R}, A \neq B, |B| \leq 1$*. If* $\omega(z)$ *is an analytic function with* $\omega(0) = 1$*, then we have*

$$\omega(z) - \alpha|\omega(z) - 1| \prec \frac{1+Az}{1+Bz} \Longleftrightarrow \omega(z)(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1+Az}{1+Bz}, \quad \phi \in \mathbb{R}. \tag{2.1}$$

Using Lemma 2.1 and (1.18), we get that $f(z) \in H_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if

$$\chi_{\delta,\mu}(f(z))(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1+Az}{1+Bz}, \tag{2.2}$$

where $\chi_{\delta,\mu}(f(z))$ is given by (1.17).

Theorem 2.2 *Let* $f = h + \overline{g}$ *be such that* h *and* g *are given by (1.2). Also, suppose that* $p \in \mathbb{N}$*,* $A, B \in \mathbb{R}$ *and* $A \neq B, |B| \leq 1$*. If*

$$\sum_{k=p+1}^{\infty} (1 + |B|)(1 + \alpha)(|\xi_k^\mu||a_k| + |\eta_k^\mu||b_k|) \leq |A - B|, \tag{2.3}$$

where

$$\xi_k^\mu = (1 - \delta + \frac{\delta k}{p})\Phi_k^\mu \quad \text{and} \quad \eta_k^\mu = (1 - \delta - \frac{\delta k}{p})\Phi_k^\mu \tag{2.4}$$

and Φ_k^μ *is given by (1.15), then* $f \in H_p(A, B; \mu, \tau, \alpha, \delta)$.

Proof We first show that if the inequality (2.3) holds for the coefficients of $f = h + \overline{g}$, then the required condition (2.2) is satisfied. In view of (2.2), we need to prove that $p(z) \prec \frac{1+Az}{1+Bz}$, where

$$p(z) = \chi_{\delta,\mu}(f(z))(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi}. \tag{2.5}$$

Using the fact that $p(z) \prec \frac{1+Az}{1+Bz} \Longleftrightarrow |1-p(z)| \leq |Bp(z)-A|$, it suffices to show that

$$|1-p(z)| - |Bp(z)-A| \leq 0. \tag{2.6}$$

Therefore, we get

$$|1-p(z)| - |Bp(z)-A| = \left|(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} [\xi_k^{\mu} a_k z^{k-p} + \eta_k^{\mu} b_k z^{-p}\overline{z^k}]\right| -$$

$$\left|B - B(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} [\xi_k^{\mu} a_k z^{k-p} + \eta_k^{\mu} b_k z^{-p}\overline{z^k}] - A\right|$$

$$\leq \left|(1+\alpha) \sum_{k=p+1}^{\infty} [|\xi_k^{\mu}||a_k||z|^{k-p} + |\eta_k^{\mu}||b_k||z|^{k-p}]\right| -$$

$$(|A-B| - |B|(1+\alpha) \sum_{k=p+1}^{\infty} [|\xi_k^{\mu}||a_k||z|^{k-p} + |\eta_k^{\mu}||b_k||z|^{k-p}]$$

$$= \sum_{k=p+1}^{\infty} (1+|B|)(1+\alpha)[|\xi_k^{\mu}||a_k||z|^{k-p} + |\eta_k^{\mu}||b_k||z|^{k-p}] - |A-B|$$

$$\leq \sum_{k=p+1}^{\infty} (1+|B|)(1+\alpha)[|\xi_k^{\mu}||a_k| + |\eta_k^{\mu}||b_k|] - |A-B| \leq 0.$$

By hypothesis the last expression is non-positive. Thus the proof is completed. The coefficient bound (2.3) is sharp for the function

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{|A-B|}{(1+|B|)(1+\alpha)} (\frac{1}{|\xi_k^{\mu}|} X_k z^k + \frac{1}{|\eta_k^{\mu}|} \overline{Y_k z^k}), \tag{2.7}$$

where $\sum_{k=p+1}^{\infty}(|X_k| + |Y_k|) = 1$. □

Corollary 2.3 *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), ξ_k^{μ} and η_k^{μ} are given by (2.4). Also, suppose that $p \in N$ and $A, B \in R$. Then,*

(i) For $-1 \leq B < A \leq 1, B < 0$, if

$$\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(|\xi_k^{\mu}||a_k| + |\eta_k^{\mu}||b_k|) \leq A - B,$$

then $f \in H_p(A, B; \mu, \tau, \alpha, \delta)$.

(ii) For $-1 \leq A < B \leq 1, B > 0$, if

$$\sum_{k=p+1}^{\infty} (1+B)(1+\alpha)(|\xi_k^{\mu}||a_k| + |\eta_k^{\mu}||b_k|) \leq B - A,$$

then $f \in H_p(A, B; \mu, \tau, \alpha, \delta)$.

Corollary 2.4 *Let $f = h + \overline{g}$ be such that h and g are given by (1.2). Also, suppose that $p \in N, A, B \in R$ and $A \neq B, |B| \leq 1$. If*

$$\sum_{k=p+1}^{\infty} (1+|B|)(1+\alpha)|\Phi_k^{\mu}|(|a_k| + |b_k|) \leq |A-B|,$$

where Φ_k^μ *is given by (1.15), then* $f\in L_p(A,B;\mu,\tau,\alpha)$.

Theorem 2.5 *Let* $f=h+\overline{g}$ *be such that* h *and* g *are given by (1.2),* ξ_k^μ *and* η_k^μ *are given by (2.4). Also, suppose that* $p\in\mathbb{N}, A,B\in\mathbb{R}$ *and* $A\neq B, |B|\leq 1, 0\leq\delta<\frac{p}{2p+1}$. *Then*

(i) *For* $-1\leq B<A\leq 1,\ B<0, f\in\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\xi_k^\mu|a_k|+\eta_k^\mu|b_k|)\leq A-B. \tag{2.8}$$

(ii) *For* $-1\leq A<B\leq 1,\ B>0, f\in\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1+B)(1+\alpha)(\xi_k^\mu|a_k|+\eta_k^\mu|b_k|)\leq B-A. \tag{2.9}$$

Proof Since $\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)\subset H_p(A,B;\mu,\tau,\alpha,\delta)$. According to Corollary 2.3, we only need to prove the "only if" part of the theorem.

(i) Let $f\in\overline{H}_p(A,B;\mu,\tau,\alpha,\delta), -1\leq B<A\leq 1, B<0$. Then

$$|\frac{1-p(z)}{Bp(z)-A}|<1, \tag{2.10}$$

where $p(z)$ is defined by (2.5). Clearly, (2.10) is equivalent to

$$\Big|\frac{(1-\alpha e^{-i\phi})\sum_{k=p+1}^{\infty}(\xi_k^\mu|a_k|z^{k-p}+\eta_k^\mu|b_k|z^{-p}\overline{z^k})}{B-B(1-\alpha e^{-i\phi})\sum_{k=p+1}^{\infty}(\xi_k^\mu|a_k|z^{k-p}+\eta_k^\mu|b_k|z^{-p}\overline{z^k})-A}\Big|<1. \tag{2.11}$$

From (2.11), we have

$$\Big\{\frac{(1-\alpha e^{-i\phi})\sum_{k=p+1}^{\infty}\xi_k^\mu|a_k|z^{k-p}+\eta_k^\mu|b_k|z^{-p}\overline{z^k})}{A-B+B(1-\alpha e^{-i\phi})\sum_{k=p+1}^{\infty}\xi_k^\mu|a_k|z^{k-p}+\eta_k^\mu|b_k|z^{-p}\overline{z^k})}\Big\}<1. \tag{2.12}$$

Taking $z=r$ $(0<r<1)$ and $\phi=\pi$, then (2.12) gives

$$\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\xi_k^\mu|a_k|+\eta_k^\mu|b_k|)r^{k+p}\leq A-B. \tag{2.13}$$

Letting $r\to 1$ in (2.13), we will get (2.8).

(ii) Similar to the proof of (2.8), we can prove (2.9). □

Corollary 2.6 *Let* $f=h+\overline{g}$ *be such that* h *and* g *are given by (1.2),* Φ_k^μ *is given by (1.15). Also, suppose that* $p\in N, A,B\in R$ *and* $A\neq B, |B|\leq 1$. *Then*

(i) *For* $-1\leq B<A\leq 1, B<0,\ f\in\overline{L}(A,B;\mu,\tau,\alpha)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)\Phi_k^\mu(|a_k|+|b_k|)\leq A-B.$$

(ii) *For* $-1\leq A<B\leq 1, B>0,\ \ f\in\overline{L}(A,B;\mu,\tau,\alpha)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1+B)(1+\alpha)\Phi_k^\mu(|a_k|+|b_k|)\leq B-A.$$

Theorem 2.7 *Let $f = h + \overline{g}$ be such that h and g are given by (1.3), ξ_k^μ and η_k^μ are given by (2.4). Also, suppose that $\mu > 1, 0 \le \delta < \frac{p}{2p+1}$. Then*

(i) For $-1 \le B < A \le 1,\ B < 0,$ if $f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$r^p - \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1} \le |f(z)| \le r^p + \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1}. \tag{2.14}$$

(ii) For $-1 \le A < B \le 1,\ B > 0,$ if $f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$r^p - \frac{B-A}{(1+B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1} \le |f(z)| \le r^p + \frac{B-A}{(1+B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1}. \tag{2.15}$$

Proof Since $f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$, by using Theorem 2.5, we have

$$(1-B)(1+\alpha)\eta_{p+1}^\mu \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \le \sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\xi_k^\mu |a_k| + \eta_k^\mu |b_k|) \le A - B, \tag{2.16}$$

which implies that

(i) If $-1 \le B < A \le 1$ and $B < 0$, then from (2.16) we obtain

$$\sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \le \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu}. \tag{2.17}$$

On the other hand,

$$\begin{aligned} |f(z)| &\le r^p + \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) r^k \le r^p + r^{p+1} \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \\ &\le r^p + \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1} \end{aligned}$$

and

$$|f(z)| \ge r^p - \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu} r^{p+1}.$$

Hence (2.14) follows. The case for (ii) $-1 \le A < B \le 1$ and $B > 0$ can be proved in the same manner and hence we omit it. □

Corollary 2.8 *Let $f = h + \overline{g}$ be such that h and g are given by (1.3), ξ_k^μ and η_k^μ are given by (2.4). Also, suppose that $\mu > 1, 0 \le \delta < \frac{p}{2p+1}$. Then*

(i) For $-1 \le B < A \le 1,\ B < 0,$ if $f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$\{w : |w| < 1 - \frac{A-B}{(1-B)(1+\alpha)\eta_{p+1}^\mu}\} \subset f(U).$$

(ii) For $-1 \le A < B \le 1,\ B > 0,$ if $f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$\{w : |w| < 1 - \frac{B-A}{(1+B)(1+\alpha)\eta_{p+1}^\mu}\} \subset f(U).$$

Corollary 2.9 *Let $f = h + \overline{g}$ be such that h and g are given by (1.3), Φ_k^μ is given by (1.15). Also, suppose that $|z| = r < 1, \mu > 1$. Then*

(i) For $-1 \le B < A \le 1,\ B < 0,$ if $f \in \overline{L}_p(A, B; \mu, \tau, \alpha)$, then

$$r^p - \frac{A-B}{(1-B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1} \le |f(z)| \le r^p + \frac{A-B}{(1-B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1}.$$

(ii) For $-1 \le A < B \le 1,\ B > 0$, if $f \in \overline{L}_p(A,B;\mu,\tau,\alpha)$, then

$$r^p - \frac{B-A}{(1+B)(1+\alpha)\Phi_{p+1}^{\mu}} r^{p+1} \le |f(z)| \le r^p + \frac{B-A}{(1+B)(1+\alpha)\Phi_{p+1}^{\mu}} r^{p+1}.$$

Theorem 2.10 *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), ξ_k^{μ} and η_k^{μ} are given by (2.4). Also, suppose that $p \in N, A, B \in R$ and $A \ne B, |B| \le 1, 0 \le \delta < \frac{p}{2p+1}$. Then $f \in \mathrm{clco}\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$ if and only if*

$$f(z) = \sum_{k=p}^{\infty} X_k h_k + \sum_{k=p+1}^{\infty} Y_k (h_p + g_k),\ z \in U^*, \tag{2.18}$$

where

$$h_p = z^p,$$

$$h_k = \begin{cases} z^p - \dfrac{A-B}{(1-B)(1+\alpha)\xi_k^{\mu}} z^k, & k \ge p+1, -1 \le B < A \le 1, B < 0, \\ z^p - \dfrac{B-A}{(1+B)(1+\alpha)\xi_k^{\mu}} z^k, & k \ge p+1, -1 \le A < B \le 1, B > 0, \end{cases}$$

$$g_k = \begin{cases} -\dfrac{A-B}{(1-B)(1+\alpha)\eta_k^{\mu}} \overline{z^k}, & k \ge p+1, -1 \le B < A \le 1, B < 0, \\ -\dfrac{B-A}{(1+B)(1+\alpha)\eta_k^{\mu}} \overline{z^k}, & k \ge p+1, -1 \le A < B \le 1, B > 0, \end{cases}$$

and

$$X_p \equiv 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k), \quad X_k \ge 0, Y_k \ge 0.$$

In particular, the extreme points of $\overline{H}_p(A,B;\mu,\tau,\alpha)$ are h_k and g_k.

Proof Let $-1 \le B < A \le 1, B < 0$. We get

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{A-B}{(1-B)(1+\alpha)} \Big(\frac{1}{\xi_k^{\mu}} X_k z^k + \frac{1}{\eta_k^{\mu}} Y_k \overline{z^k}\Big). \tag{2.19}$$

Since $0 \le X_k \le 1\ \ (k = p+1, \dots)$, we obtain

$$\sum_{k=p+1}^{\infty} \Big(\frac{(1-B)(1+\alpha)\xi_k^{\mu}}{A-B} \frac{A-B}{(1-B)(1+\alpha)\xi_k^{\mu}} X_k + \frac{(1-B)(1+\alpha)\eta_k^{\mu}}{A-B} \frac{A-B}{(1-B)(1+\alpha)\eta_k^{\mu}} Y_k\Big)$$
$$= \sum_{k=p+1}^{\infty} (X_k + Y_k) = 1 - X_p \le 1.$$

Consequently, using Theorem 2.5, we have $f \in \overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$.

Conversely, if $f \in \overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$, then

$$|a_k| \le \frac{A-B}{(1-B)(1+\alpha)\xi_k^{\mu}}, \quad |b_k| \le \frac{A-B}{(1-B)(1+\alpha)\eta_k^{\mu}}. \tag{2.20}$$

Putting

$$X_k = \frac{(1-B)(1+\alpha)\xi_k^{\mu}|a_k|}{A-B}, \quad Y_k = \frac{(1-B)(1+\alpha)\eta_k^{\mu}|b_k|}{A-B} \tag{2.21}$$

and $X_p = 1 - \sum_{k=p+1}^{\infty}(X_k + Y_k) \geq 0$, we obtain

$$\begin{aligned} f(z) &= z^p - \sum_{k=p+1}^{\infty} |a_k| z^k - \sum_{k=p+1}^{\infty} |b_k| \overline{z}^k \\ &= (X_p + \sum_{k=p+1}^{\infty} (X_k + Y_k)) z^p - \sum_{k=p+1}^{\infty} \frac{A-B}{(1-B)(1+\alpha)\xi_k^{\mu}} X_k z^k - \\ &\quad \sum_{k=p+1}^{\infty} \frac{A-B}{(1-B)(1+\alpha)\eta_k^{\mu}} Y_k \overline{z}^k \\ &= X_k z^p + \sum_{k=p+1}^{\infty} h_k(z) X_k + \sum_{k=p+1}^{\infty} (z^p + g_k(z)) Y_k \\ &= X_p h_p + \sum_{k=p+1}^{\infty} h_k X_k + \sum_{k=p+1}^{\infty} (h_p + g_k) Y_k \\ &= \sum_{k=p}^{\infty} h_k X_k + \sum_{k=p+1}^{\infty} (h_p + g_k) Y_k. \end{aligned}$$

Thus f can be expressed in the form (2.18). The case for $-1 \leq A < B \leq 1, B > 0$ can be proved in the same manner and hence we omit it. □

Corollary 2.11 *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), Φ_k^{μ} is given by (1.15). Also, suppose that $p \in N, A, B \in R$ and $A \neq B, |B| \leq 1$. Then $f \in \mathrm{clco}\overline{L}_p(A, B; \mu, \tau, \alpha)$ if and only if*

$$f(z) = \sum_{k=p}^{\infty} X_k h_k + \sum_{k=p+1}^{\infty} Y_k (h_p + g_k), \quad z \in U^*,$$

where

$$h_p = z^p,$$

$$h_k = \begin{cases} z^p - \dfrac{A-B}{(1-B)(1+\alpha)\Phi_k^{\mu}} z^k, & k \geq p+1, -1 \leq B < A \leq 1, B < 0, \\ z^p - \dfrac{B-A}{(1+B)(1+\alpha)\Phi_k^{\mu}} z^k, & k \geq p+1, -1 \leq A < B \leq 1, B > 0, \end{cases}$$

$$g_k = \begin{cases} -\dfrac{A-B}{(1-B)(1+\alpha)\Phi_k^{\mu}} \overline{z^k}, & k \geq p+1, -1 \leq B < A \leq 1, B < 0, \\ -\dfrac{B-A}{(1+B)(1+\alpha)\Phi_k^{\mu}} \overline{z^k}, & k \geq p+1, -1 \leq A < B \leq 1, B > 0, \end{cases}$$

and

$$X_p \equiv 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k).$$

In particular, the extreme points of $\overline{L}_p(A, B; \mu, \tau, \alpha)$ are h_k and g_k.

Theorem 2.12 *The class $\overline{H}_p(A, B; \mu, \tau, \alpha, \delta)$ $(0 \leq \delta < \frac{p}{2p+1})$ is closed under convex combinations.*

Proof For $j=1,2$, let the functions f_j given by

$$f_j(z)=z^p-\sum_{k=p+1}^{\infty}|a_{jk}|z^k-\sum_{k=p+1}^{\infty}|b_{jk}|\overline{z}^k, \tag{2.22}$$

be in the class $\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$.

For λ_j, $\sum_{j=1}^{\infty}\lambda_j=1$, the convex combinations can be expressed in the form

$$\sum_{j=1}^{\infty}\lambda_j f_j=z^p-\sum_{k=p+1}^{\infty}(\sum_{j=1}^{\infty}\lambda_j|a_{jk}|)z^k-\sum_{k=p+1}^{\infty}(\sum_{j=1}^{\infty}\lambda_j|b_{jk}|)\overline{z}^k. \tag{2.23}$$

(i) For $-1\le B<A\le 1,\ B<0$, from (2.8), (2.22) and (2.23), we get

$$\begin{aligned}&\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\sum_{j=1}^{\infty}\lambda_j(\xi_k^{\mu}|a_{jk}|+\eta_k^{\mu}|b_{jk}|))\\&=\sum_{j=1}^{\infty}\lambda_j[\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\xi_k^{\mu}|a_{jk}|+\eta_k^{\mu}|b_{jk}|)]\\&\le\sum_{j=1}^{\infty}\lambda_j(A-B)=A-B.\end{aligned}$$

That is, $\sum_{j=1}^{\infty}\lambda_j f_j\in\overline{H}_p(A,B;\mu,\tau,\alpha,\delta)$. The case for (ii) $-1\le A<B\le 1,\ B>0$ can be proved in the same manner and hence we omit it. $\square$

Corollary 2.13 *The class $\overline{L}_p(A,B;\mu,\tau,\alpha,\delta)$ is closed under convex combinations.*

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