Journal of Mathematical Research with Applications
Sept., 2016, Vol. 36, No. 5, pp. 547–560
DOI:10.3770/j.issn:2095-2651.2016.05.006
Http://jmre.dlut.edu.cn

Global Stability of a Multi-Group Delayed Epidemic Model

Bowen REN[1], **Ke CHEN**[2,*], **Dejun FAN**[2]

1. *School of Economics, Liaoning University, Liaoning* 110136*, P. R. China;*
2. *Department of Mathematics, Harbin Institute of Technology, Shandong* 264209*, P. R. China*

Abstract A multi-group epidemic model with a variables separated incidence rate and delays is analyzed. For strongly and non-strongly connected networks, the basic reproductive number R_0 is calculated, respectively. By applying the Lyapunov functionals and the LaSalle invariance principle, we prove the global asymptotic stability of infection-free equilibrium P_0 when $R_0 < 1$ and the endemic equilibrium P^* when $R_0 > 1$.

Keywords globally asymptotically stable; multi-group delayed system; Lyapunov functionals; connectivity

MR(2010) Subject Classification 34D23; 34D05; 34A34

1. Introduction

In recent years, there are many works on the global dynamics of coupled systems. Multi-group models are used widely to describe the transmission dynamics of some infectious diseases in heterogeneous host populations, such as gonorrhea [1], sexually transmitted diseases [2], malaria [3], and cholera [4], etc. In most deterministic epidemic models, the host population is often divided into susceptible, infective and recovered subclasses. For some epidemic diseases, infected individuals can experience incubation before showing symptoms, so an exposed subclass also occurs in the host population. Recently, Feng and Teng [5] proposed a SEIR model with a variables separated incidence rate as follows:

$$
\begin{aligned}
S_k' &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj}\phi_k(S_k)\psi_j(I_j) - d_k^S S_k,\\
E_k' &= \sum_{j=1}^{n} \beta_{kj}\phi_k(S_k)\psi_j(I_j) - (d_k^E + \epsilon_k)E_k,\\
I_k' &= \epsilon_k E_k - (d_k^I + r_k + \alpha_k)I_k + \eta_k R_k,\\
R_k' &= r_k I_k - d_k^R R_k - \eta_k R_k.
\end{aligned}
\tag{1}
$$

Here the matrix $[\beta_{kj}]$ is the contact matrix, where $\beta_{kj} > 0$ represents the transmission coefficient between compartments S_k and I_j. Λ_k represents the constant input in the k-th

Received March 12, 2016; Accepted May 26, 2016
Supported by Weihai Science and Technology Development Plan Project (Grant No. 2013DXGJ06) and the Natural Science Foundation of Shandong Province (Grant No. ZR2015AM018).
* Corresponding author
E-mail address: chennke1997@126.com (Ke CHEN)

group, d_k^S, d_k^E, d_k^I and d_k^R represent death rates of S, E, I and R population in the k-th group, respectively, ϵ_k stands for the rate of becoming infectious after a latent period in the k-th group. r_k is the recovery rate of infectious individuals in the i-th group. α_k represents disease-caused death rate and η_k represents the rate that recovered individuals relapse and regain infectiousness in the k-th group, All parameters are assumed to be nonnegative except Λ_k, d_k^S, d_k^E, which are positive for all k. Nonnegative functions ϕ_k and ψ_j are assumed to be differentiable and have the following properties:

[H1] (nonnegativity) All nonnegative functions ϕ_k and ψ_j only vanish at 0.

[H2] (monotone) ϕ_k and ψ_j are monotonically nondecreasing.

[H3] (concavity) $\frac{\psi_j(I_j)}{I_j}$ are monotonically nonincreasing.

When the transmission network is strongly-connected, Feng and Teng [5] showed that the global dynamics of system (1) is completely determined by the basic reproduction number R_0.

Time delays are inevitable in biological models, which may change the qualitative behavior of a model. For example, an epidemic model with generalized logistic dynamics can have periodic solutions even when the time in the infective stage is constant [6]. Considering that all infectious diseases have so-called latent period, time delays can be introduced to model constant sojourn times in a state. In this paper, we develop the above model with discrete time delays as follows:

$$\begin{aligned}
S_k'(t) &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj}\phi_k(S_k(t))\psi_j(I_j(t-\tau_{jk})) - d_k^S S_k,\\
E_k'(t) &= \sum_{j=1}^{n} \beta_{kj}\phi_k(S_k(t))\psi_j(I_j(t-\tau_{jk})) - (d_k^E+\epsilon_k)E_k(t),\\
I_k'(t) &= \epsilon_k E_k(t) - (d_k^I + r_k + \alpha_k)I_k(t) + \eta_k R_k,\\
R_k'(t) &= r_k I_k(t) - d_k^R R_k(t) - \eta_k R_k.
\end{aligned} \tag{2}$$

Generally speaking, the underlying network of infectious disease is assumed to be strongly connected, which means the disease can be transmitted from one group to another directly or indirectly. Sometimes, strong connectivity does not hold in reality. So in this paper, we studied the global dynamics of system (2) when the underlying network is strong connected or not strong connected.

The paper is organized as follows. In Section 2, the invariant region is presented by analyzing the positivity and boundedness of solutions for system (2). In Section 3, we investigate the global stability of infection-free equilibrium P_0 and positive equilibrium P^* of strongly connected model. In Section 4, the global dynamics of the non-strongly connected model is studied using Lyapunov functionals and the LaSalle invariance principle.

2. Positivity and boundedness

Consider system (2) in the set $X_1 = \prod_{k=1}^{n}(\mathbb{R}_+^2 \times \mathcal{C}_k^+ \times \mathbb{R}_+)$, where

$$\mathcal{C}_k^+ = \mathcal{C}([-\tau_k, 0], \mathbb{R}_+), \quad \tau_k = \max\{\tau_{jk}\},\ 1 \le j \le n.$$

The initial conditions

$$(S_{10}, E_{10}, \varphi_1(\theta), R_{10}, S_{20}, E_{20}, \varphi_2(\theta), R_{20}, \dots, S_{n0}, E_{n0}, \varphi_n(\theta), R_{n0}) \in X_1, \tag{3}$$

satisfy

$$S_{k0} > 0,\ E_{k0} > 0,\ \varphi_k(0) > 0,\ R_{k0} > 0, \quad k = 1, 2, \dots, n. \tag{4}$$

Then we have the following results

Theorem 1.1 *The solutions of system (2) with initial conditions (3) and (4) are positive for all $t > 0$ and ultimately uniformly bounded in X_1.*

Proof We first prove that $S_k(t) > 0$ for all $t \geq 0$. If there exists a $t_{k1} > 0$ such that $S_k(t_{k1}) = 0$ and $S_k(t) > 0$ for $0 \leq t < t_{k1}$. From the first equation of system (2), we have

$$S_k'(t_{k1}) = \Lambda_k > 0.$$

Hence, for a sufficiently small σ_1, $S_k(t) < 0$ when $t \in (t_{k1} - \sigma_1, t_{k1})$. This contradicts $S_k(t) > 0$ for $0 \leq t < t_{k1}$, and hence $S_k(t) > 0$ for all $t \geq 0$.

Next we prove that $E_k(t)$ is positive. Suppose $t_{k2} > 0$ is the first time such that $E_k(t_{k2}) = 0$. From the second equation of system (2), we have

$$E_k'(t_{k2}) = \sum_{j=1}^{n} \beta_{kj} \phi_k(S_k(t_{k2})) \psi_j(I_j(t_{k2} - \tau_{jk})) > 0.$$

Hence $E_k(t) < 0$ for $t \in (t_{k2} - \sigma_2, t_{k2})$ and a sufficiently small σ_2. This contradicts $E_k(t) < 0$ for $0 \leq t < t_{k2}$, and hence $E_k(t) > 0$ for all $t \geq 0$.

Similarly, if there exists a $t_{k3} > 0$ such that $I_k(t_{k3}) = 0$ and $I_k(t_{k3}) > 0$ for $0 \leq t < t_{k3}$. Then from the third equation of system (2), we have

$$I_k'(t_{k3}) = \epsilon_k E_k(t_{k3}) + \eta_k R_k(t_{k3}). \tag{5}$$

The solution of the fourth equation of system (2) with $R_k(0) = R_{k0}$ is given by

$$R_k(t) = \Big(R_{k0} + \int_0^t r_k I_k(\theta) e^{(d_k^R + \eta_k)\theta} \mathrm{d}\theta\Big) e^{-(d_k^R + \eta_k)t}. \tag{6}$$

Substituting $t = t_{k3}$ into Eq. (5) leads to

$$R_k(t_{k3}) = \Big(R_{k0} + \int_0^{t_{k3}} r_k I_k(\theta) e^{(d_k^R + \eta_k)\theta} \mathrm{d}\theta\Big) e^{-(d_k^R + \eta_k)t_{k3}} > 0.$$

From Eq. (5), we know $I_k'(t_{k3}) > 0$. This is also a contradiction, and hence $I_k(t) > 0$ for all $t \geq 0$. Then from Eq. (6), it is not difficult to find the positivity of $R_k(t)$ by the positivity of $I_k(t)$.

This completes the proof of the positivity of the solutions.

Summing up the four equations in system (2), we obtain

$$\begin{aligned}(S_k(t) + E_k(t) + I_k(t) + R_k(t))' =& \Lambda_k - d_k^S S_k(t) - d_k^E E_k(t) - (d_k^I + \alpha) I_k(t) - d_k^R R(t) \\ \leq& \Lambda_k - d_k(S_k(t) + E_k(t) + I_k(t) + R_k(t)),\end{aligned}$$

where $d_k = \min\{d_k^S, d_k^E, d_k^I + \alpha, d_k^R\}$. Then

$$\overline{\lim_{t\to\infty}} (S_k(t) + E_k(t) + I_k(t) + R_k(t))' \le \frac{\Lambda_k}{d_k}.$$

This completes the proof of the boundedness of the solutions. □

Therefore, the attracting region for system (2) is

$$\Gamma_1 = \Big\{(S_1, E_1, \varphi_1(\theta), R_1, S_2, E_2, \varphi_2(\theta), R_2, \dots, S_n, E_n, \varphi_n(\theta), R_n) \in X_1 : \\ 0 \le S_k + E_k + \|\varphi_k\| + R_k \le \frac{\Lambda_k}{d_k},\ k = 1, 2, \dots, n\Big\}.$$

3. Global stability under strong connectivity

Model (2) always has the disease-free equilibrium $P_0 = (S_1^0, 0, 0, 0, S_2^0, 0, 0, 0, \dots, S_n^0, 0, 0, 0)$ with

$$S_k^0 = \frac{\Lambda_k}{d_k^S}. \tag{7}$$

It follows from [7,8] that the next generation matrix for system (2) is

$$M = [a_{ij}]_{n\times n} = [\beta_{ij}\phi_i(S_i^0)\psi_j'(0)l_j]_{n\times n},$$

where

$$l_j = \frac{\epsilon_j(d_j^R + \eta_j)}{[(d_j^R + \eta_j)(d_j^I + \alpha_j) + d_j^R r_j][d_j^E + \epsilon_j]}.$$

And the spectral radius of M is the basic reproduction number $R_0 = \rho(M)$.

Theorem 2.1 *Suppose that the contact matrix $B = (\beta_{ij})_{n\times n}$ is irreducible.*

(i) If $R_0 < 1$, then P_0 is globally asymptotically stable in Γ_1.

(ii) If $R_0 > 1$, then P_0 is unstable and system (2) is uniformly persistent.

(iii) If $R_0 > 1$, then P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}_1$.

Proof (i) We first claim that M is irreducible since B is irreducible. Then when the spectral radius $\rho(M) < 1$, it has a corresponding positive left eigenvector $(v_1, v_2, \dots, v_n)$. Consider a Lyapunov function for system (2):

$$L = \sum_{i=1}^n v_i l_i\Big(E_i + \sum_{j=1}^n \beta_{ij}\phi_i(S_i)\int_{-\tau_{ji}}^0 \psi_j(\varphi_i(\theta))\mathrm{d}\theta\Big) + \sum_{i=1}^n v_i m_i I_i + \sum_{i=1}^n v_i n_i R_i, \tag{8}$$

where

$$m_i = \frac{d_i^R + \eta_i}{(d_i^R + \eta_i)(d_i^I + \alpha_i) + d_i^R r_i},\quad n_i = \frac{\eta_i}{(d_i^R + \eta_i)(d_i^I + \alpha_i) + d_i^R r_i}.$$

Differentiating L, we have

$$L' = \sum_{i=1}^n v_i l_i\Big[\sum_{j=1}^n \beta_{ij}\phi_i(S_i^0)\psi_j'(0)I_j + \sum_{j=1}^n \beta_{ij}\phi_i(S_i^0)I_j\big(\frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0)I_j} - \psi_j'(0)\big) - \\ (d_i^E + \epsilon_i)E_i\Big] + \sum_{i=1}^n v_i m_i\big[\epsilon_i E_i - (d_i^I + r_i + \alpha_i)I_i + \eta_i R_i\big] +$$

$$
\begin{aligned}
&\sum_{i=1}^{n} v_i n_i \big[r_i I_i(t) - d_i^R R_i(t) - \eta_i R_i \big] \\
=&\sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0) I_j(t-\tau_{ji}) \big(\frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0) I_j} - \psi_j'(0) \big) + \\
&\sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0)\psi_j'(0) I_j - \sum_{i=1}^{n} v_i I_i \\
=&\sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0) I_j \big(\frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0) I_j} - \psi_j'(0) \big) + \\
&\sum_{j=1}^{n} I_j \sum_{i=1}^{n} \beta_{ij}\phi_i(S_i^0)\psi_j'(0) v_i l_i - \sum_{i=1}^{n} v_i I_i \\
=&\sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0) I_j \big(\frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0) I_j} - \psi_j'(0) \big) + \sum_{i=1}^{n} (R_0 - 1) v_i I_i \\
<&0, \quad \text{if } R_0 < 1.
\end{aligned} \tag{9}
$$

When $R_0 < 1$, $L' = 0$ implies that $I_i = 0$, then from (5) and (6) we can obtain that $E_i = 0, R_i = 0$, and $S_i = S_i^0$ for all $1 \le i \le n$ and $t \ge 0$. So the largest invariant set in M is the singleton $\{P_0\}$. Next we prove that P_0 is locally stable. For $\varphi \in X_1$, define

$$
a(|\varphi(0)|) = \sum_{i=1}^{n} v_i l_i \Big(E_i + \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i) \int_{-\tau_{ji}}^{0} \psi_j(\varphi_i(\theta)) \mathrm{d}\theta \Big) + \sum_{i=1}^{n} v_i m_i I_i. \tag{10}
$$

Then

$$
a(|\varphi(0)|) \le L(\varphi),
$$

and $a(r) \to \infty$ as $r \to \infty$. Define

$$
b(|\varphi(0)|) = \sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0) I_j (\psi_j'(0) - \frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0) I_j}).
$$

Obviously, $b(r)$ is non-negative and

$$
\begin{aligned}
L' &= \sum_{i=1}^{n} v_i l_i \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^0) I_j \big(\frac{\phi_i(S_i)\psi_j(I_j)}{\phi_i(S_i^0) I_j} - \psi_j'(0) \big) + \sum_{i=1}^{n} (R_0 - 1) v_i I_i \\
&\le -b(|\varphi(0)|).
\end{aligned}
$$

Applying the Corollary 5.3.1 in Hale [9], we obtain that P_0 is globally asymptotically stable when $R_0 < 1$.

(ii) Choose $-L$ as a Lyapunov functional, where L was given in Eq. (8), with the same proof as [9, Theorem 5.3.3] and [10, Theorem 3.1], we know that P_0 is unstable when $R_0 > 1$. Set $X = X_1$ and $E = \Gamma_1$. Then, similarly as in Li et al. [11] and Shu et al. [12], the largest invariant set N on the boundary of $\partial\Gamma_2$ is the singleton $\{P_0\}$. Therefore, the conditions of Theorem 4.3 in Freedman et al. [13] hold, and hence system (2) is uniformly persistent.

(iii) The instability of P_0, uniform persistence of system system (2) when $R_0 > 1$, together with the uniform boundedness of the solutions imply that (2) admits at least one endemic equilibrium in Γ_1 (see [14]). Let $P^* = (S_k^*, E_k^*, I_k^*, R_k^*)$ be an endemic equilibrium, whose components satisfy

$$\begin{aligned}
&\Lambda_i - \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*) - d_i^S S_i^* = 0,\\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*) - (d_i^E + \epsilon_i)E_i^* = 0,\\
&\epsilon_i E_i^* - (d_i^I + r_i + \alpha_i)I_i^* + \eta_i R_i^* = 0,\\
&r_i I_i^* - d_i^R R_i^* - \eta_i R_i^* = 0, \quad i, j = 1, 2, \ldots, n.
\end{aligned} \tag{11}$$

Let

$$\begin{aligned}
D_i = &\int_{S_i^*}^{S_i} \frac{\phi_i(z) - \phi_i(S_i^*)}{\phi_i(z)} \mathrm{d}z + E_i - E_i^* - E_i^* \ln \frac{E_i}{E_i^*} + \\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*) \int_{-\tau_{ji}}^{0} (\frac{\varphi_j(\theta)}{I_j^*} - \ln \frac{\varphi_j(\theta)}{I_j^*}) \mathrm{d}\theta,\\
&D_{n+i} = \varphi_i(\theta) - I_i^* - I_i^* \ln \frac{\varphi_i(\theta)}{I_i^*},\\
&D_{2n+i} = R_i - R_i^* - R_i^* \ln \frac{R_i}{R_i^*}, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{12}$$

For $i = 1, 2, \ldots, n$, differenting D_i, D_{n+i} and D_{2n+i} along the solutions of model (2), we obtain

$$\begin{aligned}
D_i' = &\big(1 - \frac{\phi_i(S_i^*)}{\phi_i(S_i)}\big)\Big[\sum_{j=1}^{n} \beta_{ij} \big(\phi_i(S_i^*)\psi_j(I_j^*) - \phi_i(S_i(t))\psi_j(I_j(t-\tau_{ji}))\big) - \\
&d_i^S(S_i - S_i^*)\Big] + \big(1 - \frac{E_i^*}{E_i}\big)[\beta_{kj}\phi_i(S_i(t))\psi_j(I_j(t-\tau_{ji})) - (d_i^E + \epsilon_i)E_i] + \\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*} - \ln \frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln \frac{I_j(t-\tau_{ji})}{I_j^*}\big)\\
\le &\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(1 - \frac{\phi_i(S_i^*)}{\phi_i(S_i)}\big)\big(1 - \frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))}{\phi_i(S_i^*)\psi_j(I_j^*)}\big) + \\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))}{\phi_i(S_i^*)\psi_j(I_j^*)} - \frac{E_i}{E_i^*}\big)\big(1 - \frac{E_i^*}{E_i}\big) + \\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*} - \ln \frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln \frac{I_j(t-\tau_{ji})}{I_j^*}\big)\\
= &\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(2 - \frac{E_i}{E_i^*} - \frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))E_i^*}{\phi_i(S_i^*)\psi_j(I_j^*)E_i} + \\
&\frac{\psi_j(I_j(t-\tau_{ji}))}{\psi_j(I_j^*)} - \frac{\phi_i(S_i^*)}{\phi_i(S_i)}\big) + \\
&\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*} - \ln \frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln \frac{I_j(t-\tau_{ji})}{I_j^*}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\ln\frac{\psi_j(I_j^*)}{\psi_j(I_j(t-\tau_{ji}))}+\ln\frac{E_i}{E_i^*}-\frac{E_i}{E_i^*}+\frac{\psi_j(I_j(t-\tau_{ji}))}{\psi_j(I_j^*)}\big)+\\
&\quad\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_j(t-\tau_{ji})}{I_j^*}+\ln\frac{I_j(t-\tau_{ji})}{I_j^*}\big)\\
&\le \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big[\big(1-\frac{\psi_j(I_j^*)I_j(t-\tau_{ji})}{\psi_j(I_j(t-\tau_{ji}))I_j^*}\big)\big(\frac{\psi_j(I_j(t-\tau_{ji}))}{\psi_j(I_j^*)}-1\big)+\\
&\quad\frac{I_j(t-\tau_{ji})}{I_j^*}-\ln\frac{I_j(t-\tau_{ji})}{I_j^*}-\frac{E_i}{E_i^*}+\ln\frac{E_i}{E_i^*}\big]+\\
&\quad\sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_j(t-\tau_{ji})}{I_j^*}+\ln\frac{I_j(t-\tau_{ji})}{I_j^*}\big)\\
&\le \sum_{j=1}^{n} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{E_i}{E_i^*}+\ln\frac{E_i}{E_i^*}\big)\\
&=:\sum_{j=1}^{n} a_{i,n+j}G_{i,n+j},
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
D'_{n+i} =&(1-\frac{I_i^*}{I_i})\{\epsilon_i E_i-(d_k^I+r_i+\alpha_i)I_i+\eta_i R_i\}\\
=&\epsilon_i E_i^*(\frac{E_i}{E_i^*}-\frac{I_i}{I_i^*})(1-\frac{I_i^*}{I_i})+\eta_i R_i^*(\frac{R_i}{R_i^*}-\frac{I_i}{I_i^*})(1-\frac{I_i^*}{I_i})\\
\le&\epsilon_i E_i^*(\frac{E_i}{E_i^*}-\frac{I_i}{I_i^*}-\ln\frac{E_i}{E_i^*}+\ln\frac{I_i}{I_i^*})+\\
&\eta_i R_i^*(\frac{R_i}{R_i^*}-\frac{I_i}{I_i^*}-\ln\frac{R_i}{R_i^*}+\ln\frac{I_i}{I_i^*})\\
=:&a_{n+i,i}G_{n+i,i}+a_{n+i,2n+i}G_{n+i,2n+i},\\
D'_{2n+i} =&(1-\frac{R_i^*}{R_i})\{r_i I_i-d_i^R R_i-\eta_k R_i\}\\
=&r_i I_i^*(\frac{I_i}{I_i^*}-\frac{R_i}{R_i^*})(1-\frac{R_i^*}{R_i})\\
\le&r_i I_i^*(\frac{I_i}{I_i^*}-\frac{R_i}{R_i^*}-\ln\frac{I_i}{I_i^*}+\ln\frac{R_i}{R_i^*})\\
=:&a_{2n+i,n+i}G_{2n+i,n+i}.
\end{aligned}
$$

Define a weighted digraph $(\mathcal{G}, A)$ here with $A=[a_{i,j}]$, and let c_i be the cofactor of the i-th diagonal element of the Laplacian matrix of A. The out-degree $d^+(i)$ is the number of arcs whose initial vertex is i. Since $d^+(2n+i)=1$ and $d^+(i)=1$ hold for each i, by [15, Theorem 3.3], we obtain:

$$
c_{n+i}a_{n+i,2n+i}=c_{2n+i}a_{2n+i,n+i},\quad c_{n+i}a_{n+i,i}=\sum_{j=1}^{n} c_i a_{i,n+j}.
$$

Thus

$$
D=\sum_{i=1}^{n} c_i D_i+\sum_{i=1}^{n} c_{n+i}D_{n+i}+\sum_{i=1}^{n} c_{n+i}a_{n+i,2n+i}\frac{D_{2n+i}}{a_{2n+i,n+i}}. \tag{14}
$$

Since $G_{n+i,2n+i}+G_{2n+i,n+i}=0$ and

$$G_{i,n+j}+G_{n+i,i}=\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}+\ln\frac{I_i}{I_i^*}, \tag{15}$$

it follows that

$$\begin{aligned}
D' \le & \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{i,n+j} G_{i,n+j} + \sum_{i=1}^{n} c_{n+i}(a_{n+i,i}G_{n+i,i}+a_{n+i,2n+i}G_{n+i,2n+i}) + \\
& \sum_{i=1}^{n} c_{n+i}a_{n+i,2n+i}\frac{D_{2n+i}}{a_{2n+i,n+i}} \\
\le & \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{i,n+j} G_{i,n+j} + \sum_{i=1}^{n} c_{n+i}a_{n+i,i}G_{n+i,i} + \\
& c_{n+i}a_{n+i,2n+i}G_{n+i,2n+i} + \sum_{i=1}^{n} c_{n+i}a_{n+i,2n+i}G_{2n+i,n+i} \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{i,n+j} G_{i,n+j} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{i,n+j} G_{n+i,i} \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{i,n+j}\Big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}+\ln\frac{I_i}{I_i^*}\Big).
\end{aligned} \tag{16}$$

Let $\tilde{c}_i$, $i=1,2,\ldots,n$, be given as in [10, Proposition 2.1] with $(\mathcal{G},\tilde{A})$, where the entry of the $n\times n$ matrix $\tilde{A}=[\tilde{a}_{ij}]$ is defined as $\tilde{a}_{ij}=a_{i,n+j}$. Let

$$\tilde{c}_{n+i}=\sum_{j=1}^{n}\tilde{c}_i\frac{a_{i,n+j}}{a_{n+i,j}},\quad \tilde{c}_{2n+i}=\tilde{c}_{n+i}\frac{a_{n+i,2n+i}}{a_{2n+i,n+i}}.$$

Now, we claim that

$$\tilde{D}=\sum_{i=1}^{n}\tilde{c}_iD_i+\sum_{i=1}^{n}\tilde{c}_{n+i}D_{n+i}+\sum_{i=1}^{n}\tilde{c}_{2n+i}D_{2n+i},$$

is a Lyapunov function for system (2). In fact, replacing all c_i by $\tilde{c}_{n+i}$ in the calculation of Eq. (14) yields

$$\tilde{D}'\le\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}+\ln\frac{I_i}{I_i^*}\Big).$$

Furthermore, by [15, Theorem 3.2], we can obtain that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_i}{I_i^*}-\ln\frac{I_i}{I_i^*}\Big)=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}\Big). \tag{17}$$

Then

$$\begin{aligned}
\tilde{D}' \le & \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}+\ln\frac{I_i}{I_i^*}\Big) \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_i}{I_i^*}-\ln\frac{I_i}{I_i^*}\Big)-\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_i\tilde{a}_{i,n+j}\Big(\frac{I_j}{I_j^*}-\ln\frac{I_j}{I_j^*}\Big)=0.
\end{aligned} \tag{18}$$

Additionally, Eq. (13) implies that $\tilde{D}' = 0$ if and only if $P = P^*$. Hence P^* is globally attractive. Next we prove P_0 is locally stable. For $\varphi \in X_1$, define

$$a(|\varphi(0)|) = \sum_{i=1}^{n} \tilde{c}_i D_i + \sum_{i=1}^{n} \tilde{c}_{n+i} D_{n+i}.$$

Then $a(|\varphi(0)|) \le L(\phi)$, and $a(r) \to \infty$ as $r \to \infty$. Define $b(|\phi(0)|) = 0$. It is obvious that $b(r)$ is non-negative and

$$D' \le -b(|\phi(0)|).$$

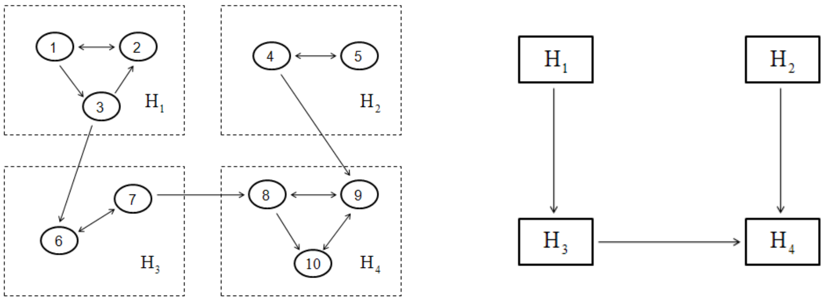
From Hale [9], P^* is globally asymptotically stable in $\mathring{\Gamma}_1$. $\square$

4. Global stability under non-strong connectivity

In the previous section, we have investigated the global stability of system (2) when the interaction network is strongly connected. In this section, we will study system (2) when the underlying network is not strongly connected through the method given by Du and Li in [16]. We first introduce some useful definitions and theorems given in [16].

4.1. Preliminary

Du and Li [16] described two concepts: strongly connected components and the condensed graph, which play an important role in the global dynamics of coupled systems. The following Figure 1 is a sketch map given in [16] to help understanding the above two concepts.



(a) Connected components H_i (b) The condensed digraph $\mathcal{H}$

Figure 1 Connected components H_i and the condensed digraph $\mathcal{H}$ of a digraph $\mathcal{G}$

Du [16] also defined a partial order relation as follows. Let $V(\mathcal{G}) := \{1, 2, \ldots, n\}$ be the vertex set of digraph $\mathcal{G}$, and define a partial order $\preceq$ between two elements in $V(\mathcal{G})$: for i and j, $i \preceq j$ if there exists an oriented path from i to j and $i \sim j$ if $i \preceq j$ and $j \preceq i$. They claimed that the relation $\sim$ is an equivalence relation. Theory of discrete mathematics shows that equivalence relation $\sim$ on the set $V(\mathcal{G})$ determines a partition of $V(\mathcal{G})$, that is the quotient set $V(\mathcal{G})/\sim$ whose elements denoted by H_i $(i = 1, 2, \ldots, l)$ consisting of n_i elements of $V(\mathcal{G})$, where $\sum_{i=1}^{l} n_i = n$ and each H_i is a strongly connected component of $\mathcal{G}$. For example in Figure 1 (a), the strongly connected components of digraph $\mathcal{G}$ are H_1, H_2, H_3, H_4. Define $\mathcal{H}$ as the condensed graph of $\mathcal{G}$ by collapsing each H_i as a single vertex, thus $V(\mathcal{H}) = \{H_i : i = 1, 2, \ldots, l\}$. Define a canonical partial order $\prec$ between two elements in $V(\mathcal{H})$: for H_i and H_j, $H_i \prec H_j$ if there exists $i' \in H_i$ and $j' \in H_j$, such that $i' \preceq j'$. Du and Li [16] also showed that $\prec$ is a strict partial order, and there exist minimal and maximal elements in $V(\mathcal{H})$ with respect to the strict partial order $\prec$. For example in Figure 1 (b), the minimal elements are H_1 and H_2, and the maximal element is H_4. For any $G \subseteq V(\mathcal{G})$, define the subsystems of system (2) as follows:

(i) The G-subsystem:

$$\begin{aligned}
S_i'(t) &= \Lambda_i - \sum_{j\in V(\mathcal{G})} \beta_{ij}\phi_i(S_i(t))\psi_j(I_j(t-\tau_{ji})) - d_i^S S_i,\\
E_i'(t) &= \sum_{j\in V(\mathcal{G})} \beta_{ji}\phi_i(S_i(t))\psi_j(I_j(t-\tau_{ji})) - (d_i^E + \epsilon_i)E_i(t),\\
I_i'(t) &= \epsilon_i E_i(t) - (d_i^I + r_i + \alpha_i)I_i(t) + \eta_i R_i,\\
R_i'(t) &= r_i I_i(t) - d_i^R R_i(t) - \eta_i R_i, \quad i \in G.
\end{aligned}$$

(ii) The reduced G-subsystem:

$$\begin{aligned}
S_i'(t) &= \Lambda_i - \sum_{j\in G} \beta_{ij}\phi_k(S_i)\psi_j(I_j(t-\tau_{ji})) - d_i^S S_i,\\
E_i'(t) &= \sum_{j\in G} \beta_{ij}\phi_i(S_i(t))\psi_j(I_j(t-\tau_{ji})) - (d_i^E + \epsilon_i)E_i,\\
I_i'(t) &= \epsilon_i E_i - (d_i^I + r_i + \alpha_i)I_i + \eta_i R_i,\\
R_i'(t) &= r_i I_i - d_i^R R_i - \eta_i R_i, \quad i \in G.
\end{aligned} \tag{19}$$

(iii) The restricted system on H at $\mathbf{c}$:

$$\begin{aligned}
S_i'(t) =&\Lambda_i - \sum_{j\in H} \beta_{ij}\phi_i(S_i)\psi_j(I_j(t-\tau_{ji})) - d_i^S S_i - \sum_{k\in V(\mathcal{G})\setminus H} \beta_{ik}\phi_i(S_i(t))\psi_k(c_k),\\
E_i'(t) =& \sum_{j\in H} \beta_{ij}\phi_i(S_i)\psi_j(I_j(t-\tau_{ji})) + \sum_{k\in V(\mathcal{G})\setminus H} \beta_{ik}\phi_i(S_i(t))\psi_k(c_k) - (d_i^E + \epsilon_i)E_i(t),\\
I_i'(t) =&\epsilon_i E_i - (d_i^I + r_i + \alpha_i)I_i + \eta_i R_i,\\
R_i'(t) =&r_k I_i - d_i^R R_i - \eta_i R_i, \quad i \in G.
\end{aligned} \tag{20}$$

For $H \in V(\mathcal{H})$ is a strongly connected component and $\mathbf{c} = (c_1, c_2, \ldots, c_n) \geq 0$, where $c_i \in \mathbb{R}_+^4$, $1 \leq i \leq n$.

Du and Li [16] made five additional assumptions (A_1–A_5) on the couple system, and here we list some of them which will be used in the following:

(A_3) For $H \in V(\mathcal{H})$ and $\mathbf{c} \geq 0$, the restricted system (20) on H at $\mathbf{c}$ has a nonnegative equilibrium that attracts all positive solutions.

(A_4) For $1 \leq i \leq n$, the vertex system has at most one boundary equilibrium.

(A_5) For $H \in V(\mathcal{H})$, if the reduced H-subsystem (19) has a positive equilibrium, then system (19) is uniformly persistent.

Du and Li [16] also made assumptions (F_1–F_6) on the incidence function $f_{ij}(S_i, I_j)$. In our model (2), $f_{ij}(S_i, I_j) = \phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))$, it is easy to verify that assumptions (F_1–F_6) are satisfied.

Let $\mathcal{P}$ be the set of all equilibria and define a mapping $\pi : \mathcal{P} \to (0,1)^{|V(\mathcal{H})|}$

$$\pi : u^* \to \tilde{u}^* = (\tilde{u}^*_H)_{H\in V(\mathcal{H})},$$

and

$$\tilde{u}^*_H = \begin{cases} 0, & \text{if } Pu^*_i = 0, \text{ for } i \in H, \\ 1, & \text{if } Pu^*_i > 0, \text{ for } i \in H, \end{cases}$$

for any $u^* \in \mathcal{P}$, where $|V(\mathcal{H})|$ is the order of set $V(\mathcal{H})$. For the map π, Du and Li [16] presented three propositions (Propositions 2.6-2.8 in [16]), we rewrite them in the following for further application.

Corollary 4.1 *For $u^* \in \mathcal{P}$, if $H \prec H'$, then $\tilde{u}^*_H < \tilde{u}^*_{H'}$.*

Corollary 4.2 *An equilibrium $u^* \in \mathcal{P}$ is positive if and only if $\tilde{u}^*_H = 1$ at all minimal elements $H \in V(\mathcal{H})$.*

Corollary 4.3 *Suppose that (A_3) and (A_4) are satisfied. Then the following holds.*

(a) For $H \in V(\mathcal{H})$ and $\mathbf{c} \geq 0$, the positive or boundary equilibrium of (19) on H at $\mathbf{c}$ is unique.

(b) The map π is one-to-one.

Du and Li [16] also defined an evaluation function $E : \mathcal{P} \to \mathbb{R}_+$

$$E(u^*) = \sum_{H\in V(\mathcal{H})} \pi(u^*)_H,$$

and Theorem 2.9 in [16] was used to identify the global attracting equilibrium, we rewrite it here.

Lemma 4.4 *Suppose that (A_3) and (A_5) are satisfied. Then*

(a) All positive solutions of system (2) converge to a maximizer of function E.

(b) If in addition (A_4) is satisfied, then the maximizer of function E is unique.

Du and Li [16] showed the structure of the set of equilibrium without the strong connectivity assumption on $\mathcal{G}$. On each strongly connected component H_i, the solutions of system (2) tend to synchronize. Applying the evaluation function E and Theorem 2.9 in [16], they showed that the unique maximizer P^* of E corresponds to a unique equilibrium of (2), either positive or mixed, that attracts all positive solutions.

4.2. Global stability of system (2)

This subsection uses the same notations as in Section 4.1. Without loss of generality, we assume that the digraph $\mathcal{G}$ of n vertices generated by (2) is connected but not strongly connected.

Suppose that there are $m(\leq n)$ vertexes in strongly connected component H and renumber its elements S_i, E_i, I_i, R_i from 1 to m. Let $q_k = \sum_{i\in V(\mathcal{G})\setminus H} \beta_{ki}\psi_i(c_i)$ and rewrite (19) as

$$\begin{aligned} S_k'(t) &= \Lambda_k - \sum_{j\in H} \beta_{kj}\phi_k(S_k(t))\psi_j(I_j(t-\tau_{kj})) - d_k^S S_k - q_k\phi_k(S_k(t)),\\ E_k'(t) &= \sum_{j\in H} \beta_{kj}\phi_k(S_k(t))\psi_j(I_j(t-\tau_{jk})) + q_k\phi_k(S_k(t)) - (d_k^E+\epsilon_k)E_k(t),\\ I_k'(t) &= \epsilon_k E_k(t) - (d_k^I + r_k + \alpha_k)I_k(t) + \eta_k R_k,\\ R_k'(t) &= r_k I_k(t) - d_k^R R_k(t) - \eta_k R_k, \end{aligned} \tag{21}$$

where $k = 1, 2, \ldots, m$. The phase space of system (21) is chosen as

$$X_2 = \prod_{k=1}^m (\mathbb{R}_+^2 \times \mathcal{C}_k^+ \times \mathbb{R}_+),$$

where $\mathcal{C}_k^+$ is defined in Section 2, and the invariant region of (20) is

$$\Gamma_2 = \Big\{(S_1, E_1, \varphi_1(\theta), R_1, S_2, E_2, \varphi_2(\theta), R_2, \ldots, S_m, E_m, \varphi_m(\theta), R_m) \in X_2 : \\ 0 \le S_k + E_k + \|\varphi_k\| + R_k \le \frac{\Lambda_k}{d_k},\ k = 1, 2, \ldots, m\Big\}.$$

The basic reproductive number for each H_i is

$$R_{0,H_i} = \rho\left([\beta_{ij}\phi_i(S_i^0)\psi_j'(0)l_j]\right)_{i,j\in V(H_i)},$$

where

$$l_j = \frac{\epsilon_j(d_j^R + \eta_j)}{[(d_j^R+\eta_j)(d_j^I+\alpha_j) + d_j^R R_j][d_j^E+\epsilon_j]}, \tag{22}$$

and [16] shows that the basic reproductive number for the whole network (2) is

$$R_0 = \max\{R_{0,H} : H \in V(\mathcal{H})\}. \tag{23}$$

For our model, (A_4) and (A_5) hold obviously, in the following we only need to prove that (A_3) holds. In fact, a stronger conclusion can be obtained as follows.

Theorem 4.5 *Suppose that $B = (\beta_{kj})$ is irreducible. Then system (21) has a unique endemic equilibrium which is globally asymptotically stable with respect to $\mathring{\Gamma}_2$.*

Proof Let $P^* = (S_1^*, E_1^*, I_1^*, R_1^*, S_2^*, E_2^*, I_2^*, R_2^*, \ldots, S_m^*, E_m^*, I_m^*, R_m^*)$ be an equilibrium of system (21). Consider a Lyapunov functional $V : X_2 \to \mathbb{R}$

$$L = \sum_{i=1}^m c_i D_i + \sum_{i=1}^m c_{m+i} D_{m+i} + \sum_{i=1}^m c_{m+i} a_{m+i,2m+i} \frac{D_{2m+i}}{a_{2m+i,m+i}}. \tag{24}$$

Let

$$D_i = \int_{S_i^*}^{S_i} \frac{\phi_i(z) - \phi_i(S_i^*)}{\phi_i(z)}\mathrm{d}z + E_i - E_i^* - E_i^* \ln\frac{E_i}{E_i^*} + \\ \sum_{j=1}^m \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)\int_{-\tau_{ji}}^0 (\frac{I_j(t+\theta)}{I_j^*} - \ln\frac{I_j(t+\theta)}{I_j^*})\mathrm{d}\theta,$$

$$
\begin{aligned}
D_{m+i} =& I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}, \\
D_{2m+i} =& R_i - R_i^* - R_i^* \ln \frac{R_i}{R_i^*}, \quad i = 1, 2, \dots, m.
\end{aligned} \tag{25}
$$

Then

$$
\begin{aligned}
D_i' =& (1 - \frac{\phi_i(S_i^*)}{\phi_i(S_i)})\Big[\sum_{j=1}^{m} \beta_{ij} \left(\phi_i(S_i^*)\psi_j(I_j^*) - \phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))\right) - \\
& d_i^S(S_i - S_i^*) - q_i\left(\phi_i(S_i(t)) - \phi_i(S_i^*)\right)\Big] + \\
& (1 - \frac{E_i^*}{E_i})\left[\beta_{ij}\phi_i(S_i)\psi_j(I_j(t-\tau_{ji})) + q_i\phi_i(S_i(t)) - (d_i^E + \epsilon_i)E_i\right] + \\
& \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln\frac{I_j(t-\tau_{ji})}{I_j^*}) \\
\le & \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(1 - \frac{\phi_i(S_i^*)}{\phi_i(S_i)})(1 - \frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))}{\phi_i(S_i^*)\psi_j(I_j^*)}) + \\
& \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))}{\phi_i(S_i^*)\psi_j(I_j^*)} - \frac{E_i}{E_i^*})(1 - \frac{E_i^*}{E_i}) + \\
& \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln\frac{I_j(t-\tau_{ji})}{I_j^*}) + \\
& q_i\phi_i(S_i^*)(3 - \frac{\phi_i(S_i^*)}{\phi_i(S_i)} - \frac{E_i^*\phi_i(S_i)}{E_i\phi_i(S_i^*)} - \frac{E_i}{E_i^*}) \\
\le & \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(2 - \frac{E_i}{E_i^*} - \frac{\phi_i(S_i)\psi_j(I_j(t-\tau_{ji}))E_i^*}{\phi_i(S_i^*)\psi_j(I_j^*)E_i} + \frac{\psi_j(I_j(t-\tau_{ij}))}{\psi_j(I_j^*)} - \\
& \frac{\phi_i(S_i^*)}{\phi_i(S_i)}) + \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln\frac{I_j(t-\tau_{ji})}{I_j^*}) \\
\le & \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\ln\frac{\psi_j(I_j^*)}{\psi_j(I_j(t-\tau_{ji}))} + \ln\frac{E_i}{E_i^*} - \frac{E_i}{E_i^*} + \frac{\psi_j(I_j(t-\tau_{ji}))}{\psi_j(I_j^*)}) + \\
& \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln\frac{I_j(t-\tau_{ji})}{I_j^*}) \\
\le & \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)[(1 - \frac{\psi_j(I_j^*)I_j(t-\tau_{ji})}{\psi_j(I_j(t-\tau_{ji}))I_j^*})(\frac{\psi_j(I_j(t-\tau_{ji}))}{\psi_j(I_j^*)} - 1) + \frac{I_j}{I_j^*} - \\
& \ln\frac{I_j(t-\tau_{ji})}{I_j^*} - \frac{E_i}{E_i^*} + \ln\frac{E_i}{E_i^*}] + \\
& \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{I_j(t-\tau_{ji})}{I_j^*} + \ln\frac{I_j(t-\tau_{ji})}{I_j^*}) \\
\le & \sum_{j=1}^{m} \beta_{ij}\phi_i(S_i^*)\psi_j(I_j^*)(\frac{I_j}{I_j^*} - \ln\frac{I_j}{I_j^*} - \frac{E_i}{E_i^*} + \ln\frac{E_i}{E_i^*})
\end{aligned}
$$

$$=: \sum_{j=1}^{m} a_{i,m+j} G_{i,m+j}. \tag{26}$$

Similarly, as in the calculation in Section 3, we can obtain that $L' \leq 0$, and equality holds if and only if $P = P^*$. Then we complete the proof of assumption (A_3). As in Section 3, suitable $a(r)$ and $b(r)$ can be found to prove the local stability. Hence P^* is globally asymptotically stable in $\overset{\circ}{\Gamma}_2$. □

According to Lemma 4.4, we obtain the following results:

Theorem 4.6 *All positive solutions of system (2) converge to the unique maximizer P^* of function E.*

Theorem 4.7 *Let P^* be the nonnegative globally asymptotically stable equilibrium of system (2). Then P^* is a positive equilibrium if and only if $R_{0,H} > 1$ for all minimal elements $H \in V(\mathcal{H})$.*

Theorem 4.8 *A positive equilibrium P^* exists if and only if $R_{0,H} > 1$ for all minimal elements $H \in V(\mathcal{H})$. In this case, P^* is unique and attracts all positive solutions.*

References

[1] I. K. C. LEUNG, K. GOPALSAMY. *Dynamics of continuous and discrete time SIV models of Gonorrhea transmission.* Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, 2012, **19**(3): 351–375.

[2] C. C. CHAVEZ, Wenzhang HUANG, Jia LI. *Competitive exclusion and coexistence of multiple strains in an SIS STD model.* SIAM J. Appl. Math., 1999, **59**(5): 1790–1811.

[3] Daozhou GAO, Shigui RUAN. *A multi-patch malaria model with logistic growth populations.* SIAM J. Appl. Math., 2012, **72**(3): 819–841.

[4] Mingtao LI, Guiqun SUN, Juan ZHANG, et al. *Global dynamic behavior of a multigroup cholera model with indirect transmission.* Discrete Dyn. Nat. Soc. 2013, Art. ID 703826, 11 pp.

[5] Xiaomei FENG, Zhidong TENG, Fengqin ZHANG. *Global dynamics of a general class of multi-group epidemic models with latency and relapse.* Math. Biosci. Eng., 2015, **12**(1): 99–115.

[6] H. W. HETHCOTE, P. V. D. DRIESSCHE. *Two SIS epidemiologic models with delays.* J. Math. Biol., 2000, **40**(1): 3–26.

[7] Wendi WANG, Xiaoqiang ZHAO. *A nonlocal and time-delayed reaction-diffusion model of dengue transmission.* SIAM J. Appl. Math., 2011, **71**(1): 147–168.

[8] H. R. THIEME. *Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity.* SIAM J. Appl. Math., 2009, **70**(1): 188–211.

[9] J. K. HALE, S. M. V. LUNEL. *Introduction to functional differential equations.* Springer-Verlag, New York, 1993.

[10] M. Y. LI, Zhisheng SHUAI, Chuncheng WANG. *Global stability of multi-group epidemic models with distributed delays.* J. Math. Anal. Appl., 2010, **361**(1): 38–47.

[11] M. Y. LI, J. R. GRAEF, Liancheng WANG, et al. *Global dynamics of a SEIR model with varying total population size.* Math. Biosci., 1999, **160**(2): 191–213.

[12] Hongying SHU, Dejun FAN, Junjie WEI. *Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission.* Nonlinear Anal. Real World Appl., 2012, **13**(4): 1581–1592.

[13] H. I. FREEDMAN, Shigui RUAN, Moxun TANG. *Uniform persistence and flows near a closed positively invariant set.* J. Dynam. Differential Equations, 1994, **6**(4): 583–600.

[14] H. L. SMITH, P. WALTMAN. *The Theory of the Chemostat: Dynamics of Microbial Competition.* Cambridge University Press, 1995.

[15] Zhisheng SHUAI, P. V. D. DRIESSCHE. *Global stability of infectious disease models using Lyapunov functions.* SIAM J. Appl. Math., 2013, **73**(4): 1513–1532.

[16] Peng DU, M. Y. LI. *Impact of network connectivity on the synchronization and global dynamics of coupled systems of differential equations.* Phys. D, 2014, **286**: 32–42.