

Real-Valued Functions and Some Covering Properties

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Abstract In the paper [properties defined with semi-continuous functions and some related spaces', *Houston J. Math.*, 2015, **41**(3): 1097–1106] properties $(UL)_m^{wl}$, $(UL)_m^K$ and $(UL)_m$ were defined and it was shown that spaces having these properties coincide with countably paracompact spaces, countably mesocompact spaces and countably metacompact spaces, respectively. In this paper, we continue with the study on the relationship between properties defined with real-valued functions and some covering properties. Some characterizations of countably compact spaces and pseudo-compact spaces in terms of real-valued functions are obtained.

Keywords real-valued functions; countably compact spaces; pseudo-compact spaces; insertion theorems

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1. Introduction

Throughout, a space always means a topological space. The set of all positive integers is denoted by \mathbb{N} . Let X be a space, we use $\mathbf{0}$ to denote the constant function on X with value 0.

A sequence $\{f_n : n \in \mathbb{N}\}$ of real valued functions on a space X is called weakly locally uniformly convergent to f on X if for each $x \in X$ and $\varepsilon > 0$, there exists an open neighborhood U of x and $m \in \mathbb{N}$ such that $|f_n(y) - f(y)| < \varepsilon$ for all $n \geq m$ and $y \in U$. $\{f_n : n \in \mathbb{N}\}$ is called locally uniformly convergent to f on X if for each $x \in X$, there exists an open neighborhood U of x such that $\{f_n : n \in \mathbb{N}\}$ converges uniformly to f on U .

For a sequence $\{f_n : n \in \mathbb{N}\}$ of real valued functions on X , we use the symbol $f_n \xrightarrow{P} f$ (resp., $f_n \xrightarrow{U} f$; $f_n \xrightarrow{LU} f$; $f_n \xrightarrow{WLU} f$) to denote that $\{f_n : n \in \mathbb{N}\}$ pointwise (resp., uniformly; locally uniformly; weakly locally uniformly) converges to f on X .

The following implications are clear.

$$f_n \xrightarrow{U} f \Rightarrow f_n \xrightarrow{LU} f \Rightarrow f_n \xrightarrow{WLU} f \Rightarrow f_n \xrightarrow{P} f.$$

A real-valued function f on a space X is called lower (resp., upper) semi-continuous if for any real number r , the set $\{x \in X : f(x) > r\}$ (resp., $\{x \in X : f(x) < r\}$) is open.

We write $C(X)$ (resp., $L(X)$, $U(X)$, $R(X)$) for the set of all continuous (resp., lower semi-continuous, upper semi-continuous, real-valued) functions from X into the unit interval $[0, 1]$.

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Also, $UL^+(X) = \{ \langle g, h \rangle \in U(X) \times L(X) : g < h \}$ and $L^+(X) = \{ h \in L(X) : h > 0 \}$.

In [2], Ohta and Sakai introduced the property $(USC)_m$: a space X has property $(USC)_m$ if for every decreasing sequence $\{f_n \in U(X) : n \in \mathbb{N}\}$ of functions such that $f_n \xrightarrow{P} \mathbf{0}$, there exists a sequence $\{g_n \in C(X) : n \in \mathbb{N}\}$ of functions such that $f_n \leq g_n$ for each $n \in \mathbb{N}$ and $g_n \xrightarrow{P} \mathbf{0}$. It was shown that spaces having property $(USC)_m$ are equivalent to cb -spaces, where a space X is called a cb -space [3] if for each locally bounded function f on X there is a continuous function g on X such that $|f| \leq g$. In [1], properties $(UL)_m^{wl}$, $(UL)_m^K$ and $(UL)_m$ were introduced as generalizations of property $(USC)_m$ and it was proved that spaces with these properties coincide with countably paracompact spaces, countably mesocompact spaces and countably metacompact spaces, respectively. In the same paper, to keep a uniform appearance, property $(USC)_m$ was renamed as $(UC)_m$. Moreover, properties $(UC)_m^{wl}$ and $(UC)_m^K$ were introduced and it was shown that they are both equivalent to property $(UC)_m$ (i.e., $(USC)_m$).

We unify these properties to give them a general form as follows.

Definition 1.1 A space X is said to have property ${}^\alpha(AB)_m^\beta$ if for every decreasing sequence $\{f_n \in A : n \in \mathbb{N}\}$ of functions such that $f_n \xrightarrow{\alpha} \mathbf{0}$, there exists a sequence $\{g_n \in B : n \in \mathbb{N}\}$ of functions such that $f_n \leq g_n$ for each $n \in \mathbb{N}$ and $g_n \xrightarrow{\beta} \mathbf{0}$.

In the above definition, $A, B \in \{C(X), L(X), U(X), R(X)\}$ and $\alpha, \beta \in \{p, u, wl, lu\}$ where p (resp., u, wl, lu) denotes pointwise (resp., uniform, weakly locally uniform, locally uniform) convergence.

A natural question is that if a space X has property ${}^p(UC)_m^u$ or other properties, then which space X is. In Section 2, we shall show that a space X is countably compact if and only if it has one of the following properties: ${}^p(UC)_m^u$, ${}^p(UL)_m^u$, ${}^p(UR)_m^u$, ${}^{wl}(UC)_m^u$, ${}^{wl}(UL)_m^u$, ${}^{wl}(UR)_m^u$. With this result, an insertion theorem of countably compact spaces is obtained. In Section 3, we investigate the relationship between semi-continuous functions and pseudo-compact spaces. Some similar results as that in Section 2 are obtained.

Let X be a space and $A \subset X$. We use χ_A to denote the characteristic function of A .

2. On countably compact spaces

In this section, we show that spaces having property ${}^p(UC)_m^u$ coincide with countably compact spaces. With this result, an insertion theorem of countably compact spaces is obtained. To begin, we need the following lemma.

Lemma 2.1 ([4]) A space X is countably compact if and only if every decreasing sequence of nonempty closed subsets of X has nonempty intersection.

With Lemma 2.1, we can get a somewhat direct result first.

Proposition 2.2 A space X is countably compact if and only if for every decreasing sequence $\{f_n \in U(X) : n \in \mathbb{N}\}$, if $f_n \xrightarrow{P} \mathbf{0}$, then $f_n \xrightarrow{U} \mathbf{0}$.

Proof Suppose that X is countably compact and let $\{f_n \in U(X) : n \in \mathbb{N}\}$ be decreasing with

$f_n \xrightarrow{P} \mathbf{0}$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, put $F_n^\varepsilon = \{x \in X : f_n(x) \geq \varepsilon\}$, then $\{F_n^\varepsilon : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X . From $f_n \xrightarrow{P} \mathbf{0}$ it follows that $\bigcap_{n \in \mathbb{N}} F_n^\varepsilon = \emptyset$. By Lemma 2.1, there is $m \in \mathbb{N}$ such that $F_n^\varepsilon = \emptyset$ for all $n \geq m$. Hence, for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $f_n(x) < \varepsilon$ for all $x \in X$ and $n \geq m$ which implies that $f_n \xrightarrow{U} \mathbf{0}$.

Conversely, let $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence of nonempty closed subsets of X . For each $n \in \mathbb{N}$, let $f_n = \chi_{F_n}$. Then $\{f_n \in U(X) : n \in \mathbb{N}\}$ is decreasing. If $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$, then $f_n \xrightarrow{P} \mathbf{0}$. By the condition, $f_n \xrightarrow{U} \mathbf{0}$. Thus there exists $m \in \mathbb{N}$ such that $f_n(x) < 1$ for each $x \in X$ and $n \geq m$, which implies that $F_n = \emptyset$ for each $n \geq m$, a contradiction. By Lemma 2.1, X is countably compact. \square

The following lemma follows readily from Lemma 2.1.

Lemma 2.3 *For a space X , the following are equivalent:*

- (1) X is countably compact;
- (2) For every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of X with empty intersection, there exists a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of clopen subsets of X such that $F_n \subset A_n$ for each $n \in \mathbb{N}$ and $A_m = \emptyset$ for some $m \in \mathbb{N}$;
- (3) For every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of X with empty intersection, there exists a decreasing sequence $\{B_n : n \in \mathbb{N}\}$ of subsets of X such that $F_n \subset B_n$ for each $n \in \mathbb{N}$ and $B_m = \emptyset$ for some $m \in \mathbb{N}$.

Theorem 2.4 *For a space X , the following are equivalent:*

- (a) X is countably compact;
- (b) X has property ${}^P(UC)_m^u$;
- (c) X has property ${}^P(UL)_m^u$;
- (d) X has property ${}^P(UR)_m^u$.

Proof (a) \Rightarrow (b). Suppose that X is countably compact and let $\{f_n \in U(X) : n \in \mathbb{N}\}$ be decreasing with $f_n \xrightarrow{P} \mathbf{0}$. For each $n, k \in \mathbb{N}$, put $F_{nk} = \{x \in X : f_n(x) \geq 1/2^k\}$. Then for each $k \in \mathbb{N}$, $\{F_{nk} : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X with empty intersection. By Lemma 2.3, for each $k \in \mathbb{N}$, there exists a sequence $\{A_{nk} : n \in \mathbb{N}\}$ of clopen subsets of X such that $F_{nk} \subset A_{nk}$ for all $n \in \mathbb{N}$ and there exists $m_k \in \mathbb{N}$ such that $A_{nk} = \emptyset$ for all $n \geq m_k$. For each $n \in \mathbb{N}$, let

$$g_n = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{A_{nk}}.$$

Then $g_n \in C(X)$. For each $n \in \mathbb{N}$ and $x \in X$, if $f_n(x) = 0$, then $f_n(x) \leq g_n(x)$; if $f_n(x) > 0$, then $1/2^k \leq f_n(x) \leq 1/2^{k-1}$ for some $k \in \mathbb{N}$. Hence $x \in F_{nm} \subset A_{nm}$ for all $m \geq k$, so we have

$$g_n(x) \geq \sum_{m=k}^{\infty} \frac{1}{2^m} \chi_{A_{nm}}(x) = \sum_{m=k}^{\infty} \frac{1}{2^m} = \frac{1}{2^{k-1}} \geq f_n(x).$$

It remains to show that $g_n \xrightarrow{U} \mathbf{0}$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $1/2^k < \varepsilon$. Let $m = \max\{m_i : i \leq k\}$. Then for each $i \leq k$ and $n \geq m$, $A_{ni} = \emptyset$. Thus for each $n \geq m$ and

$x \in X$,

$$g_n(x) = \sum_{i=k+1}^{\infty} \frac{1}{2^i} \chi_{A_{ni}}(x) \leq \sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k} < \varepsilon.$$

Consequently, $g_n \xrightarrow{U} \mathbf{0}$.

(b) \Rightarrow (c) and (c) \Rightarrow (d) are clear.

(d) \Rightarrow (a). Let $\{f_n \in U(X) : n \in \mathbb{N}\}$ be decreasing and $f_n \xrightarrow{P} \mathbf{0}$. By (c), there exists a sequence $\{g_n \in R(X) : n \in \mathbb{N}\}$ of functions on X such that $f_n \leq g_n$ for each $n \in \mathbb{N}$ and $g_n \xrightarrow{U} \mathbf{0}$. It follows that $f_n \xrightarrow{U} \mathbf{0}$. By Proposition 2.2, X is countably compact. \square

Remark 2.5 With a similar argument as the proof of Theorem 2.4, one readily sees that spaces having property ${}^{wl}(UC)_m^u$ or ${}^{wl}(UL)_m^u$ or ${}^{wl}(UR)_m^u$ also coincide with countably compact spaces.

Question 2.6 If a spaces X has property ${}^p(UL)_m^{lu}$ or ${}^p(UC)_m^{lu}$, then which space X is?

With Theorem 2.4, we can give an insertion theorem of countably compact spaces as follows.

Theorem 2.7 X is countably compact if and only if there is a mapping $\phi : L^+(X) \rightarrow C(X)$ such that for each $h \in L^+(X)$, $\phi(h) < h$ and $\inf\{\phi(h)(x) : x \in X\} > 0$.

Proof Let X be a countably compact space. For each $h \in L^+(X)$ and $n \in \mathbb{N}$, let $F(n, h) = \{x \in X : h(x) \leq 1/2^{n-1}\}$. Then $\{F(n, h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X with empty intersection. Let $f(n, h) = \chi_{F(n, h)}$. Then $\{f(n, h) \in U(X) : n \in \mathbb{N}\}$ and $f(n, h) \xrightarrow{P} \mathbf{0}$. By Theorem 2.4, there exists $\{g(n, h) \in C(X) : n \in \mathbb{N}\}$ such that $f(n, h) \leq g(n, h)$ for each $n \in \mathbb{N}$ and $g(n, h) \xrightarrow{U} \mathbf{0}$. Let

$$\phi(h) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} g(n, h).$$

Then $\phi(h) \in C(X)$. Let $x \in X$. From $\bigcap_{n \in \mathbb{N}} F(n, h) = \emptyset$ it follows that $x \notin F(m, h)$ for some $m \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \notin F(n, h)\}$. Then $h(x) > 1/2^{k-1}$ and $x \notin F(n, h)$ for all $n \geq k$ while $x \in F(n, h)$ for all $n < k$. Thus

$$\phi(h)(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} g(n, h)(x) \leq 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} f(n, h)(x) = \sum_{n=1}^{k-1} \frac{1}{2^n} = \frac{1}{2^{k-1}} < h(x)$$

Now, Since $g(n, h) \xrightarrow{U} \mathbf{0}$, there exists $m \in \mathbb{N}$ such that $g(n, h)(x) < 1/2$ for all $n \geq m$ and $x \in X$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} g(n, h)(x) = \sum_{n=1}^{m-1} \frac{1}{2^n} g(n, h)(x) + \sum_{n=m}^{\infty} \frac{1}{2^n} g(n, h)(x) < 1 - \frac{1}{2^{m-1}} + \frac{1}{2^m} = 1 - \frac{1}{2^m}.$$

Consequently, $\phi(h)(x) > 1/2^m$ which implies that $\inf\{\phi(h)(x) : x \in X\} > 0$.

To prove the sufficiency, let $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence of closed subsets of X with empty intersection. Put

$$h = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{F_n},$$

then $h \in L^+(X)$. For each $n \in \mathbb{N}$, let $B_n = \{x \in X : \phi(h)(x) < 1/2^n\}$. For a fixed $n \in \mathbb{N}$, let $x \in F_n$ and $k = \max\{n \in \mathbb{N} : x \in F_n\}$. Then $n \leq k$ and thus

$$\phi(h)(x) < h(x) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{F_n}(x) = 1 - \sum_{n=1}^k \frac{1}{2^n} = \frac{1}{2^k} \leq \frac{1}{2^n}.$$

This implies that $x \in B_n$ and thus $F_n \subset B_n$. By (b), there exists $m \in \mathbb{N}$ such that $\phi(h)(x) > 1/2^m$ for each $x \in X$. Thus $B_m = \emptyset$. By Lemma 2.3 (3), X is countably compact. \square

3. On pseudo-compact spaces

In this section, we investigate which property characterizes a pseudo-compact space. Some analogous results as that in section 2 are obtained. All spaces are assumed to be Tychonoff.

Lemma 3.1 ([4]) *A space X is pseudo-compact if and only if for each decreasing sequence $\{W_n : n \in \mathbb{N}\}$ of nonempty open subsets of X , $\bigcap_{n \in \mathbb{N}} \overline{W_n} \neq \emptyset$.*

In [5], pseudo-compactness was characterized with the convergence of sequence of continuous functions. The following proposition gives an analogous characterization of pseudo-compactness with semi-continuous functions.

Proposition 3.2 *For a space X , the following are equivalent:*

- (i) X is pseudo-compact;
- (ii) For every decreasing sequence $\{f_n \in L(X) : n \in \mathbb{N}\}$, if $f_n \xrightarrow{WLU} \mathbf{0}$, then $f_n \xrightarrow{U} \mathbf{0}$;
- (iii) For every decreasing sequence $\{f_n \in L(X) : n \in \mathbb{N}\}$, if $f_n \xrightarrow{LU} \mathbf{0}$, then $f_n \xrightarrow{U} \mathbf{0}$.

Proof (i) \Rightarrow (ii). Suppose that X is pseudo-compact and $\{f_n \in L(X) : n \in \mathbb{N}\}$ is decreasing and $f_n \xrightarrow{WLU} \mathbf{0}$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, let $U_n^\varepsilon = \{x \in X : f_n(x) > \varepsilon\}$. Then $\{U_n^\varepsilon : n \in \mathbb{N}\}$ is a decreasing sequence of open subsets of X . From $f_n \xrightarrow{WLU} \mathbf{0}$ it follows that $\bigcap_{n \in \mathbb{N}} \overline{U_n^\varepsilon} = \emptyset$. By Lemma 3.1, there is $m \in \mathbb{N}$ such that $U_n^\varepsilon = \emptyset$ for all $n \geq m$. Thus for each $x \in X$ and $n \geq m$, $f_n(x) \leq \varepsilon$ which implies that $f_n \xrightarrow{U} \mathbf{0}$.

(ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i). Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing sequence of nonempty open subsets of X . Assume that $\bigcap_{n \in \mathbb{N}} \overline{U_n} = \emptyset$. For each $n \in \mathbb{N}$, let $f_n = \chi_{U_n}$, then $\{f_n \in L(X) : n \in \mathbb{N}\}$ is decreasing and converges locally uniformly to $\mathbf{0}$. By the condition, $f_n \xrightarrow{U} \mathbf{0}$. Then there exists $m \in \mathbb{N}$ such that $f_n(x) < 1$ for each $x \in X$ and $n \geq m$. This implies that $U_n = \emptyset$ for all $n \geq m$, a contradiction. By Lemma 3.1, X is pseudo-compact. \square

Lemma 3.3 *For a space X , the following are equivalent:*

- (1) X is pseudo-compact;
- (2) For every locally finite decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X , there exists a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of clopen subsets of X such that $U_n \subset A_n$ for each $n \in \mathbb{N}$ and $A_m = \emptyset$ for some $m \in \mathbb{N}$;
- (3) For every locally finite decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X , there exists a decreasing sequence $\{B_n : n \in \mathbb{N}\}$ of subsets of X such that $U_n \subset B_n$ for each $n \in \mathbb{N}$

and $B_m = \emptyset$ for some $m \in \mathbb{N}$.

Proof (1) \Rightarrow (2). Suppose that X is pseudo-compact and $\{U_n : n \in \mathbb{N}\}$ is a locally finite decreasing sequence of open subsets of X . Then $\bigcap_{n \in \mathbb{N}} \overline{U_n} = \emptyset$. By Lemma 3.1, there exists $m \in \mathbb{N}$ such that $U_n = \emptyset$ for all $n \geq m$. For each $n < m$, let $A_n = X$ and let $A_n = \emptyset$ while $n \geq m$. Then $\{A_n : n \in \mathbb{N}\}$ is the desired sequence of clopen sets.

(2) \Rightarrow (3) and (3) \Rightarrow (1) are clear. \square

Theorem 3.4 For a space X , the following are equivalent:

- (a) X is pseudo-compact;
- (b) X has property ${}^{wl}(LC)_m^u$;
- (c) X has property ${}^{wl}(LU)_m^u$;
- (d) X has property ${}^{wl}(LR)_m^u$.

Proof (a) \Rightarrow (b). Suppose that X is pseudo-compact and let $\{f_n \in L(X) : n \in \mathbb{N}\}$ be decreasing with $f_n \xrightarrow{WLU} \mathbf{0}$. For each $n, k \in \mathbb{N}$, put $U_{nk} = \{x \in X : f_n(x) > 1/2^k\}$. Then for each $k \in \mathbb{N}$, $\{U_{nk} : n \in \mathbb{N}\}$ is a locally finite decreasing sequence of open subsets of X . By Lemma 3.3, for each $k \in \mathbb{N}$, there exists a sequence $\{A_{nk} : n \in \mathbb{N}\}$ of clopen subsets of X such that $U_{nk} \subset A_{nk}$ for all $n \in \mathbb{N}$ and there exists $m_k \in \mathbb{N}$ such that $A_{nk} = \emptyset$ for all $n \geq m_k$. For each $n \in \mathbb{N}$, let

$$g_n = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi_{A_{nk}}.$$

Then $g_n \in C(X)$. With a similar argument as that in the proof of Theorem 2.4, one can show that $f_n \leq g_n$ for each $n \in \mathbb{N}$ and $g_n \xrightarrow{U} \mathbf{0}$.

(b) \Rightarrow (c) and (c) \Rightarrow (d) are clear.

(d) \Rightarrow (a). Using Proposition 3.2. \square

Remark 3.5 By Proposition 3.2 and Theorem 3.4, it is clear that spaces having property ${}^{lu}(LC)_m^u$ or ${}^{lu}(LU)_m^u$ or ${}^{lu}(LR)_m^u$ also coincide with pseudo-compact spaces.

Remark 3.6 It was shown that [1] a space X is countably paracompact if and only if for every decreasing sequence $\{f_n \in U(X) : n \in \mathbb{N}\}$ of functions such that $f_n \xrightarrow{P} \mathbf{0}$, there exists a sequence $\{g_n \in L(X) : n \in \mathbb{N}\}$ of functions such that $f_n \leq g_n$ for each $n \in \mathbb{N}$ and $g_n \xrightarrow{WLU} \mathbf{0}$. That is, X has property ${}^P(UL)_m^{wl}$. Without loss of generality, we can assume that $\{g_n : n \in \mathbb{N}\}$ is decreasing. So, by Theorems 2.4 and 3.4, we have the following known result: a pseudo-compact countably paracompact space is countably compact.

In the following theorem, $ULLB(X) = \{f \in U(X) : \text{for each } x \in X, \text{ there is an open neigh-borhood } U \text{ of } x \text{ such that } \inf\{f(y) : y \in U\} > 0\}$.

Theorem 3.7 For a space X , the following are equivalent:

- (a) X is pseudo-compact;
- (b) There is a mapping $\phi : ULLB(X) \rightarrow C(X)$ such that for each $h \in ULLB(X)$, $\phi(h) < h$ and $\inf\{\phi(h)(x) : x \in X\} > 0$;

(c) There is a mapping $\phi : ULLB(X) \rightarrow R(X)$ such that for each $h \in ULLB(X)$, $\phi(h) < h$ and $\inf\{\phi(h)(x) : x \in X\} > 0$.

Proof (a) \Rightarrow (b). Let X be a pseudo-compact space. For each $h \in ULLB(X)$ and $n \in \mathbb{N}$, let $U(n, h) = \{x \in X : h(x) < 1/2^{n-1}\}$. Then $\{U(n, h) : n \in \mathbb{N}\}$ is a locally finite decreasing sequence of open subsets of X . Let $f(n, h) = \chi_{U(n, h)}$. Then $\{f(n, h) \in L(X) : n \in \mathbb{N}\}$ is decreasing and $f_n \xrightarrow{WLU} \mathbf{0}$. By Theorem 3.4, there exists $\{g(n, h) \in C(X) : n \in \mathbb{N}\}$ such that $f(n, h) \leq g(n, h)$ for each $n \in \mathbb{N}$ and $g(n, h) \xrightarrow{U} \mathbf{0}$. Let

$$\phi(h) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} g(n, h).$$

Then $\phi(h) \in C(X)$. That $\phi(h) < h$ and $\inf\{\phi(h)(x) : x \in X\} > 0$ can be shown with a similar argument as that in the proof of Theorem 2.7.

(b) \Rightarrow (c) is clear.

(c) \Rightarrow (a). Let ϕ be the map in (c) and $\{U_n : n \in \mathbb{N}\}$ a locally finite decreasing sequence of open subsets of X . Put

$$h = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{U_n},$$

then $h \in ULLB(X)$. For each $n \in \mathbb{N}$, let $B_n = \{x \in X : \phi(h)(x) < 1/2^n\}$. Then $U_n \subset B_n$. By (c), $B_m = \emptyset$ for some $m \in \mathbb{N}$. By Lemma 3.3 (3), X is pseudo-compact. \square

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