On the Parallel and Totally Umbilical Hypersurfaces of the Euclidean Ellipsoid

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Abstract We give a complete classification of parallel hypersurfaces in the Euclidean ellipsoid $Q^{n+1}(c,d)$. Moreover, we prove that a hypersurface in $Q^{n+1}(c,d)$ is totally umbilical, if and only if it is parallel.

Keywords   totally umbilical; parallel hypersurface; ellipsoid

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1. Introduction

Parallel and totally umbilical submanifolds are natural generalizations of totally geodesic submanifolds. It is meaningful to study these classes of submanifolds, as they often provide nice examples. In space forms, the parallel submanifolds are classified completely. The parallel hypersurface in $\mathbb{M}^n$ is either totally umbilical, or an open part of a standard product $\mathbb{S}^k \times \mathbb{M}^{n-k}$ of a k-dimensional sphere and the $n-k$ dimensional space form. The case of parallel submanifolds are classified by Ferus [1] in the Euclidean space, and by Backes-Reckziegel [2], Takeuchi [3] independently in the hyperbolic space. The classification of parallel submanifolds in the sphere follows from the Euclidean case.

Apart from space forms, there are also many studies on the parallel submanifolds in a Riemannian manifold. Classification results of parallel submanifolds are obtained for some special cases, for example, in products of space forms [4,5]; in the complex projective space $\mathbb{CP}^n$ under Lagrangian condition [6]; in simply connected rank one symmetric spaces (see, e.g., the discussion in Chapter 9 of [9]).

In this paper, we study the parallel hypersurfaces and the totally umbilical ones in the following ellipsoid

$$Q^{n+1}(c,d) = \left\{ (x,y) \in \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2} : \frac{|x|^2}{c^2} + \frac{y^2}{d^2} = 1 \right\}, \tag{1.1}$$

where c,d are fixed positive constants. As we will see, $Q^{n+1}(c,d)$ with $c \neq d$, as a hypersurface of the Euclidean space $\mathbb{R}^{n+2}$, has two distinct principal curvatures, one of which is of multiplicity n. So from the viewpoint of totally umbilical Euclidean hypersurface, the ellipsoid $Q^{n+1}(c,d)$ is a natural generalization of the Euclidean sphere. An interesting question is to determine all

the parallel hypersurfaces in $Q^{n+1}(c, d)$. Since the parallel hypersurfaces in spheres are clear, we may as well assume that $c \neq d$ in the following.

To state our classification theorem, let us see the following parallel hypersurfaces in $Q^{n+1}(c, d)$ first.

Example 1.1 Consider the hypersphere $\mathbb{S}^n(a) \times \{b\}$ $(0 < a \leq c)$ in $Q^{n+1}(c, d)$ given by

$$i : \mathbb{S}^n(a) \hookrightarrow Q^{n+1}(c, d), \quad x \mapsto (x, b), \tag{1.2}$$

where $\mathbb{S}^n(a)$ denotes the sphere in $\mathbb{R}^{n+1}$ centered at the origin of radius a, and b is a constant satisfying $\frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$. By a simple calculation, one can show that $\mathbb{S}^n(a) \times \{b\}$ is totally umbilical in $Q^{n+1}(c, d)$, with constant mean curvature $\frac{bc^2}{a\sqrt{a^2 d^2 + b^2 c^2}}$. Clearly, $\mathbb{S}^n(a) \times \{b\}$ is a parallel hypersurface in $Q^{n+1}(c, d)$.

Example 1.2 Consider the hyper-ellipsoid $Q^n(c, d)$ in $Q^{n+1}(c, d)$ given by

$$Q^n(c, d) = \left\{ (0, x_2, \ldots, x_n, y) \in \mathbb{R}^{n+2} : \sum_{i=2}^{n} \frac{|x_i|^2}{c^2} + \frac{y^2}{d^2} = 1 \right\}. \tag{1.3}$$

It is easy to see that $Q^n(c, d)$ is totally geodesic in $Q^{n+1}(c, d)$.

Remark 1.3 When intersecting the hyperplane

$$\left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = r \right\}, \quad |r| < c$$

with the ellipsoid $Q^{n+1}(c, d)$, we get a hypersurface of two distinct principal curvatures, which are not constants.

Now we can state our main result as below.

Theorem 1.4 *Let M be a hypersurface of the ellipsoid $Q^{n+1}(c, d)$. Then the following statements are equivalent:*

(i) M is parallel;

(ii) M is totally umbilical;

(iii) M is locally either the hypersphere $\mathbb{S}^n(a) \times \{b\}$ in Example 1.1, or the totally geodesic hyper-ellipsoid $Q^n(c, d)$ in Example 1.2.

Remark 1.5 The parallel hypersurfaces in the ellipsoid $Q^n(c, d)$ must be totally umbilical. This fact is quite different from the case of spheres, whose parallel hypersurfaces include Clifford tori as non-trivial examples.

We see that both hypersurfaces given in Examples 1.1 and 1.2 are totally umbilical and parallel. Hence, to give a proof of Theorem 1.4, it is sufficient to show that (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(i), as we will do in Sections 3 and 4.

2. Preliminaries

Let $f : M^n \hookrightarrow N^{n+1}$ be an isometric immersion of Riemannian manifolds with Levi-Civita connections ∇ and $\bar{\nabla}$, respectively. Denote by N the unit normal vector field of M, and let

X, Y, Z and W be arbitrary tangent vector fields on M. We define the shape operator S by $SX = -\bar{\nabla}_X N$, and the second fundamental form h by $h(X, Y) = \langle SX, Y \rangle$. The formula of Gauss is that

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)N. \tag{2.1}$$

Moreover, the equations of Gauss and Codazzi are given respectively by

$$\langle \bar{R}(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle + h(X, W)h(Y, Z) - h(X, Z)h(Y, W), \tag{2.2}$$

$$\langle \bar{R}(X, Y)Z, N \rangle = (\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z), \tag{2.3}$$

where $\bar{R}$ and R are the Riemannian curvature tensor of N and M, respectively. Here, for a hypersurface, the covariant derivative of h is defined by

$$(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.4}$$

We call that M is parallel in N if $\nabla h = 0$, that M is totally umbilical if h is a scalar multiple of the metric at every point. The totally umbilical submanifolds in space forms are completely understood. Apart from space forms, classifications are also obtained for the totally umbilical submanifolds in Riemannian product $\mathbb{S}^n \times \mathbb{R}$ (see [4,8]), for the totally umbilical hypersurface in $\mathbb{H}^n \times \mathbb{R}$ (see [9]) and in three-dimensional Thurston geometries of non-constant curvature as well as in the Berger spheres [10].

Let $\mathbb{R}^{n+2}$ be the $n + 2$-dimensional Euclidean space. We define the following Euclidean ellipsoid

$$Q^{n+1}(c, d) = \left\{ (x, y) \in \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2} : \frac{|x|^2}{c^2} + \frac{y^2}{d^2} = 1 \right\}, \tag{2.5}$$

where c, d are positive constants. Using the coordinates $\{x_1, \ldots, x_{n+1}, y\}$ of $\mathbb{R}^{n+2}$, we see that a vector field $W = \sum X^i \frac{\partial}{\partial x_i} + Y \frac{\partial}{\partial y} =: X + Y$ is tangent to $Q^{n+1}(c, d)$ at a point $(x_1, \ldots, x_{n+1}, y) \in Q^{n+1}(c, d)$, if and only if it satisfies

$$\sum \frac{X^i x_i}{c^2} + \frac{Yy}{d^2} = 0. \tag{2.6}$$

Then the vector field

$$\eta^Q = \left(\frac{x_1}{c^2}, \ldots, \frac{x_{n+1}}{c^2}, \frac{y}{d^2} \right) \tag{2.7}$$

gives a unit normal vector field $\eta = \frac{\eta^Q}{|\eta^Q|}$ of $Q^{n+1}(c, d)$ in $\mathbb{R}^{n+2}$.

Let $\bar{\nabla}$ be the induced connection on $Q^{n+1}(c, d)$, and B the second fundamental form w.r.t. the normal vector field η. A straightforward calculation yields [11]:

$$B(W_1, W_2) = -\frac{1}{|\eta^Q|} \left(\frac{1}{c^2} \langle X_1, X_2 \rangle + \frac{1}{d^2} Y_1 Y_2 \right), \tag{2.8}$$

where $W_i = (X_i, Y_i)$, $i = 1, 2$, are tangent vector fields on $Q^{n+1}(c, d)$. Then the Gauss formula (2.1) gives [11]

$$\bar{\nabla}_{W_1} W_2 = D_{W_1} W_2 + \frac{1}{|\eta^Q|} \left(\frac{1}{c^2} \langle X_1, X_2 \rangle + \frac{1}{d^2} Y_1 Y_2 \right) \eta. \tag{2.9}$$

To obtain the expression of the Riemannian curvature tensor $\bar{R}$ of $Q^{n+1}(c,d)$, it is convenient for us to consider the following parameterization

$$Q^{n+1}(c,d) = \left\{(c\cos\theta\cdot\tilde{x}, d\sin\theta) : \tilde{x}\in\mathbb{S}^n(1), \theta\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}]\right\}. \tag{2.10}$$

Correspondingly, we re-write the normal vector fields η^Q and η as

$$\eta^Q = (\frac{\cos\theta}{c}\tilde{x}, \frac{\sin\theta}{d}), \quad \eta = \frac{cd}{\sqrt{c^2\sin^2\theta + d^2\cos^2\theta}}(\frac{\cos\theta}{c}\tilde{x}, \frac{\sin\theta}{d}). \tag{2.11}$$

Let $\tilde{e}_i$, $i = 1,\dots,n$, be local tangent vector fields on $\mathbb{S}^n(1)$, and orthonormal w.r.t. the Euclidean metric $\langle\cdot,\cdot\rangle$. We define the following vector fields

$$e_i = (\tilde{e}_i, 0), \ E = \frac{\partial}{\partial\theta}/|\frac{\partial}{\partial\theta}| = \frac{1}{\sqrt{c^2\sin^2\theta + d^2\cos^2\theta}}(-c\sin\theta\cdot\tilde{x}, d\cos\theta), \tag{2.12}$$

to obtain a local orthonormal basis $\{e_1,\dots,e_n,E\}$ on $Q^{n+1}(c,d)$. From (2.8), we derive that

Lemma 2.1 *The second fundamental form B satisfies*

$$B(e_i, e_j) = -\frac{1}{c^2|\eta^Q|}\delta_{ij}, \ B(e_j, E) = 0, \ 1 \le i, j \le n,$$

$$B(E, E) = -\frac{1}{(c^2\sin^2\theta + d^2\cos^2\theta)|\eta^Q|} = -\frac{1}{c^2 d^2|\eta^Q|^3}, \tag{2.13}$$

where $|\eta^Q| = \frac{\sqrt{c^2\sin^2\theta + d^2\cos^2\theta}}{cd}$.

Lemma 2.1 shows that $Q^{n+1}(c,d)$, as a Euclidean hypersurface, has two distinct principal curvatures, one of which is of multiplicity n.

Let $E^\perp$ be the orthogonal complement of the principal direction E in $TQ^{n+1}(c,d)$. For any tangent vector w on $Q^{n+1}(c,d)$, we project it to $E^\perp$ and E, respectively, obtaining the decomposition $w = w_{\mathbb{S}} + w_E$. Applying the following equation of Gauss for $Q^{n+1}(c,d)$

$$\bar{R}(x,y,z,w) = B(x,w)B(y,z) - B(x,z)B(y,w), \tag{2.14}$$

where $x, y, z, w \in TQ^{n+1}(c,d)$, we derive from Lemma 2.1 that

$$\bar{R}(x,y,z,w) = \frac{1}{c^4|\eta^Q|^2}\Big\{\langle x_{\mathbb{S}}, w_{\mathbb{S}}\rangle\langle y_{\mathbb{S}}, z_{\mathbb{S}}\rangle - \langle x_{\mathbb{S}}, z_{\mathbb{S}}\rangle\langle y_{\mathbb{S}}, w_{\mathbb{S}}\rangle +$$
$$\frac{1}{d^2|\eta^Q|^2}\big[\langle x_{\mathbb{S}}, w_{\mathbb{S}}\rangle\langle y_E, z_E\rangle + \langle x_E, w_E\rangle\langle y_{\mathbb{S}}, z_{\mathbb{S}}\rangle -$$
$$\langle x_{\mathbb{S}}, z_{\mathbb{S}}\rangle\langle y_E, w_E\rangle - \langle x_E, z_E\rangle\langle x_{\mathbb{S}}, w_{\mathbb{S}}\rangle\big]\Big\}. \tag{2.15}$$

3. Parallel hypersurfaces

In this section, we will prove that (i)$\Rightarrow$(iii) in Theorem 1.4. Let M be a hypersurface in the ellipsoid $Q^{n+1}(c,d)$ and N be the unit normal vector field. We define an angle function φ on M by $\cos\varphi = \langle N, E\rangle$, where E is the vector field given by (2.12). For the angle function φ on a parallel hypersurface in $Q^{n+1}(c,d)$, we have that

Lemma 3.1 *If M is parallel, then $\cos\varphi = 0$ or $\cos\varphi = 1$.*

Proof We consider at a point $p \in M$, where $\cos \varphi(p) \neq 1$. In this case, the projection T on TM of E has no zeros in some neighborhood of p. Moreover, we have the following decomposition

$$E = T + \cos \varphi \, N. \tag{3.1}$$

Let X be a unit tangent vector field on M, which is orthogonal to T. Clearly, X belongs to $E^\perp$. We denote by h and ∇ the second fundamental form and induced connection on M, respectively. Since M is parallel, we simplify the equation of Codazzi on M

$$(\nabla h)(X, T, X) - (\nabla h)(T, X, X) = \bar{R}(N, X, T, X), \tag{3.2}$$

to get

$$0 = \bar{R}(N, X, T, X) = \sin^2 \varphi \cos \varphi \frac{1}{c^4 |\eta^Q|^2} (1 - \frac{1}{d^2 |\eta^Q|^2}), \tag{3.3}$$

where in the last equality we have used (2.15). Noting that $c \neq d$, we derive from (3.3) that $\cos \varphi = 0$ at p. By a continuity argument, we conclude $\cos \varphi = 0$ or $\cos \varphi = 1$ on M. $\square$

Now we can give a

Proof of Theorem 1.4 ((i)$\Rightarrow$(iii)) By Lemma 3.1, two cases may arise: $\cos \varphi = 1$ and $\cos \varphi = 0$.

Case 1 $\cos \varphi = 1$. In this case, the vector field E is normal to M. We consider the parameterization (2.10) of $Q^{n+1}(c, d)$, and note that the integral curve of the vector field $\frac{\partial}{\partial \theta} = \sqrt{c^2 \sin^2 \theta + d^2 \cos^2 \theta} E$ on $Q^{n+1}(c, d)$ is

$$\gamma(\theta) = \big(c \cos(\theta - \theta_0) \cdot \tilde{x}, d \sin(\theta - \theta_0)\big), \quad \theta \in (-\pi/2, \pi/2), \tag{3.4}$$

where $\tilde{x} \in \mathbb{S}^n(1) \hookrightarrow \mathbb{R}^{n+1}$, and θ_0 is some initial value. Then we get that M is contained in a slice $\theta_1 \times \mathbb{S}^n(1)$ given by

$$\big\{(c \cos \theta_1 \cdot \tilde{x}, d \sin \theta_1) | \tilde{x} : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}\big\}, \tag{3.5}$$

for some constant $\theta_1 \in (-\pi/2, \pi/2)$. Setting $a = c|\cos \theta_1|$, we see M is locally the hypersurface $\mathbb{S}^n(a) \times \{b\}$ in Example 1.1.

Case 2 $\cos \varphi = 0$. In this case, E is tangent to M. We first consider near a point $p \in M$ with $\cos \theta(p) \neq 0$. Let $E_1, \ldots, E_{n-1} \in E^\perp$ be orthonormal tangent vector fields of M. Then $\{E_1, \ldots, E_{n-1}, E, N\}$ is a local orthonormal basis of $Q^{n+1}(c, d)$ along M. We calculate that

$$D_{E_j} E = \frac{1}{\sqrt{c^2 \sin^2 \theta + d^2 \cos^2}} D_{E_j}(-c \sin \theta \cdot \tilde{x}, \ d \cos \theta) = \frac{-\tan \theta}{\sqrt{c^2 \sin^2 \theta + d^2 \cos^2 \theta}} E_j, \tag{3.6}$$

for $1 \leq j \leq n-1$. By use of (3.6) and (2.12), respectively, the second fundamental form h satisfies

$$\begin{aligned}
h(E_j, E) &= \langle N, D_{E_j} E \rangle = \frac{-\tan \theta}{\sqrt{c^2 \sin^2 \theta + d^2 \cos^2 \theta}} \langle E_j, N \rangle = 0, \\
h(E, E) &= -\frac{1}{c^2 \sin^2 \theta + d^2 \cos^2 \theta} \langle (c \cos \theta \tilde{x}, d \sin \theta), N \rangle = 0.
\end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we further have

$$(\nabla_{E_i} h)(E_j, E) = -h(\nabla_{E_i} E, E_j) - h(\nabla_{E_i} E_j, E)$$

$$= \frac{\tan\theta}{\sqrt{c^2\sin^2\theta + d^2\cos^2\theta}} h(E_i, E_j), \tag{3.8}$$

for $1 \le i, j \le n-1$. From the assumption $\nabla h = 0$ and (3.7), we conclude M is totally geodesic.

To determine the expression of immersion M in $Q^{n+1}(c,d)$, we observe from (3.6) that the distribution $D = \mathrm{Span}\{E_1, \ldots, E_{n-1}\}$ is integrable, and we denote its integral manifold by $\tilde{M}$. Let us consider the integral curve of the tangent vector field $\frac{\partial}{\partial\theta} = \sqrt{c^2\sin^2\theta + d^2\cos^2\theta} E$, which is

$$\gamma(\theta) = \big(c\cos(\theta - \theta_0) \cdot \tilde{x}, d\sin(\theta - \theta_0)\big), \quad \theta \in (-\pi/2, \pi/2), \tag{3.9}$$

where θ_0 is a constant determined by initial value, and $\tilde{x} \in \tilde{M} \hookrightarrow \mathbb{S}^n(1)$. Up to a re-parameterization of θ, M is then locally given by

$$\big\{(c\cos\theta \cdot \tilde{x}, d\sin\theta)| \tilde{x} : \tilde{M} \hookrightarrow \mathbb{S}^n(1), \theta \in (-\pi/2, \pi/2)\big\}. \tag{3.10}$$

From the fact that M is totally geodesic, we see the immersion

$$c\cos\theta \cdot \tilde{x} : \tilde{M} \hookrightarrow \mathbb{S}^n(c\cos\theta),$$

is also totally geodesic for a fixed $\theta \ne \pm\pi/2$. This in turn shows that M is locally isometric to

$$\big\{(0, c\cos\theta \cdot \bar{x}, d\sin\theta)| \bar{x} : \mathbb{S}^{n-1}(1) \hookrightarrow \mathbb{R}^n, \theta \in (-\pi/2, \pi/2)\big\}. \tag{3.11}$$

The hypersurface (3.11) can be extended smoothly to the points where $\cos\theta = 0$, hence M is exactly the hyper-ellipsoid $Q^n(c,d)$. $\square$

4. Totally umbilical hyperfurfaces

In this section, we prove the remaining part of Theorem 1.4, i.e., (ii)$\Rightarrow$(i). Let M be a totally umbilical hypersurface with its normal vector field N. We first observe the following

Lemma 4.1 *The angle function φ on M is a constant.*

Proof We assume that the shape operator satisfies $S = \lambda\,\mathrm{id}$, and consider near a point $p \in M$, where $\cos\varphi(p) \ne 1$. In this case, we have the decomposition (3.1). Taking the covariant derivative of (3.1), we get for any $X \in TM$ that

$$\bar{\nabla}_X E = \bar{\nabla}_X T + X[\cos\varphi]N + \cos\varphi\bar{\nabla}_X N. \tag{4.1}$$

Applying (3.6) and the Gauss formula (2.1), we derive from (4.1)

$$\nabla_Y T = [\lambda\cos\varphi - \frac{\tan\theta}{cd|\eta^Q|}]Y, \quad Y[\cos\varphi] = 0, \quad \forall Y \in T^\perp. \tag{4.2}$$

On the other hand, we can write T as $T = \sin^2\varphi E + \sin\varphi\cos\varphi e$, where e is of unit length and orthogonal to E. Then by (2.12) we have

$$D_T E = T(\frac{1}{cd|\eta^Q|})\frac{\partial}{\partial\theta} + \frac{\sin^2\varphi}{c^2 d^2|\eta^Q|^2}(-c\cos\theta \cdot \tilde{x}, -d\sin\theta) - \frac{\sin\varphi\cos\varphi\tan\theta}{cd|\eta^Q|}e. \tag{4.3}$$

Note that $\frac{\partial}{\partial\theta}$ and $(-c\cos\theta \cdot \tilde{x}, -d\sin\theta)$ belong to the distribution spanned by E and η, we see

from (4.3) that

$$\bar{\nabla}_T E = -\frac{\sin\varphi\cos\varphi\tan\theta}{cd|\eta^Q|}e = -\frac{\sin\varphi\cos\varphi\tan\theta}{cd|\eta^Q|}(-\sin\varphi\ N + \cot\varphi T). \tag{4.4}$$

Substituting X by T in (4.1) and using the Gauss formula (2.1), we obtain by (4.4)

$$\nabla_T T = 0, \quad T[\cos\varphi] = 0, \tag{4.5}$$

and

$$\cos\varphi(\lambda - \frac{\cos\varphi\tan\theta}{cd|\eta^Q|}) = 0. \tag{4.6}$$

Since $\sin\varphi \neq 0$ in a neighborhood of p, we obtain from the second equations in (4.2) and in (4.5) that $\nabla\varphi = 0$ locally. This shows φ is a constant near p, and hence a constant on M by its continuity. $\square$

In Section 3, we have proved that the constant angle hypersurface in $Q^{n+1}(c,d)$ with the condition $\cos\varphi = 1$ is the hypersphere $\mathbb{S}^n(a) \times \{b\}$. So in the following, we assume that $\cos\varphi \neq 1$, and prove that

Lemma 4.2 *The principle curvature λ of M is a constant.*

Proof We let $e_1 = T/\sin\varphi$, and extend it to get a local orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ on M. If $\cos\varphi = 0$, then we see from the second equation in (3.7) that $\lambda = 0$. So, in view of (4.6) and Lemma 4.1, it suffices to show that

$$\lambda = \frac{\cos\varphi\tan\theta}{cd|\eta^Q|} \tag{4.7}$$

is a constant when $\cos\varphi \neq 0$. For this purpose, we apply the following equations of Codazzi

$$\begin{aligned}(\nabla_{e_i}h)(e_2, e_2) - (\nabla_{e_2}h)(e_2, e_i) &= \langle \bar{R}(e_i, e_2)e_2, N\rangle, \\ (\nabla_{e_2}h)(e_3, e_3) - (\nabla_{e_3}h)(e_2, e_3) &= \langle \bar{R}(e_2, e_3)e_3, N\rangle. \end{aligned} \tag{4.8}$$

Since M is totally umbilical, we get by (2.15) and (4.8) that

$$e_i(\lambda) = 0, \quad i = 2, \ldots, n. \tag{4.9}$$

On the other hand, from the equation of Codazzi

$$(\nabla_{e_1}h)(e_2, e_2) - (\nabla_{e_2}h)(e_2, e_1) = \langle \tilde{R}(e_1, e_2)e_2, N\rangle, \tag{4.10}$$

we obtain by use of (2.15)

$$e_1(\lambda) = -\sin\varphi\cos\varphi\frac{1}{c^4|\eta^Q|^2}(1 - \frac{1}{d^2|\eta^Q|^2}). \tag{4.11}$$

We observe from (4.2) that the distribution $T^\perp$ is integrable. Indeed, for any $X, Y \in T^\perp$, we have that

$$\langle [X, Y], T\rangle = -\langle X, \nabla_Y T\rangle + \langle Y, \nabla_X T\rangle = 0. \tag{4.12}$$

This means that we can choose local coordinates $t, v_1, \ldots, v_{n-1}$, such that $e_1 = \partial_t$, and $\partial_t \perp \partial_{v_i}$. Then the relation (4.9) shows that λ depends only on t. Hence we can rewrite (4.11) as

$$\lambda' = -\sin\varphi\cos\varphi\frac{1}{c^4|\eta^Q|^2}(1 - \frac{1}{d^2|\eta^Q|^2}). \tag{4.13}$$

Using (4.2) and (4.5), we calculate that

$$\langle R(\partial_{v_i}, \partial_t)\partial_t, \partial_{v_j}\rangle = -\langle \nabla_{\partial_t}\nabla_{\partial_{v_i}}\partial_t, \partial_{v_j}\rangle$$
$$= -\frac{1}{\sin^2\varphi}\Big[(\lambda\cos\varphi - \frac{\tan\theta}{cd|\eta^Q|})' + (\lambda\cos\varphi - \frac{\tan\theta}{cd|\eta^Q|})^2\Big]\langle\partial_{v_i}, \partial_{v_j}\rangle. \quad (4.14)$$

Note that the equation (4.7) yields

$$\lambda\cos\varphi - \frac{\tan\theta}{cd|\eta^Q|} = -\sin^2\varphi\frac{\tan\theta}{cd|\eta^Q|} = -\lambda\sin^2\varphi\cos\varphi. \quad (4.15)$$

This together with (4.14) gives

$$\langle R(\partial_{v_i}, \partial_t)\partial_t, \partial_{v_j}\rangle = \big[\lambda'\cos\varphi - \lambda^2\sin^2\varphi\cos^2\varphi\big]\langle\partial_{v_i}, \partial_{v_j}\rangle. \quad (4.16)$$

On the other hand, we have by the equation of Gauss (2.2) that

$$\langle R(\partial_{v_i}, \partial_t)\partial_t, \partial_{v_j}\rangle = \Big[\frac{1}{c^4|\eta^Q|^2}\big(\frac{\sin^2\varphi}{d^2|\eta^Q|^2} + \cos^2\varphi\big) + \lambda^2\Big]\langle\partial_{v_i}, \partial_{v_j}\rangle. \quad (4.17)$$

Putting (4.16) and (4.17) together, and using (4.7) and (4.13), we obtain

$$\frac{\sin\varphi\cos^2\varphi}{c^3d^3|\eta^Q|^3} - \frac{\sin^2\varphi}{c^2d^2|\eta^Q|^2} - \frac{\sin\varphi\cos^2\varphi}{c^3d|\eta^Q|} - \frac{\cos^2\varphi}{c^2} - \frac{(1+\sin^2\varphi\cos^2\varphi)\cos^2\varphi\tan^2\theta}{d^2} = 0. \quad (4.18)$$

As φ is a constant by Lemma 4.1, we see from (4.18) that θ can only be some constant, and hence so is λ by (4.7). $\square$

Now the proof of Theorem 1.4 ((ii)$\Rightarrow$(i)) follows immediately from Lemmas 4.1 and 4.2.

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