Journal of Mathematical Research with Applications
Sept., 2016, Vol. 36, No. 5, pp. 583–607
DOI:10.3770/j.issn:2095-2651.2016.05.010
Http://jmre.dlut.edu.cn

Recursive Schemes for Scattered Data Interpolation via Bivariate Continued Fractions

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Abstract In the paper, firstly, based on new non-tensor-product-typed partially inverse divided differences algorithms in a recursive form, scattered data interpolating schemes are constructed via bivariate continued fractions with odd and even nodes, respectively. And equivalent identities are also obtained between interpolated functions and bivariate continued fractions. Secondly, by means of three-term recurrence relations for continued fractions, the characterization theorem is presented to study on the degrees of the numerators and denominators of the interpolating continued fractions. Thirdly, some numerical examples show it feasible for the novel recursive schemes. Meanwhile, compared with the degrees of the numerators and denominators of bivariate Thiele-typed interpolating continued fractions, those of the new bivariate interpolating continued fractions are much low, respectively, due to the reduction of redundant interpolating nodes. Finally, the operation count for the rational function interpolation is smaller than that for radial basis function interpolation.

Keywords Scattered data interpolation; bivariate continued fraction; three-term recurrence relation; characterization theorem; radial basis function

MR(2010) Subject Classification 65D05; 41A20

1. Introduction

Scattered data interpolation via bivariate continued fractions belongs to multivariate approximation, which is nowadays an increasingly active research area [1]. The field is both fascinating and intellectually stimulating since a lot of classical univariate theory cannot be straightforwardly generalized to the multivariate one. As a result, new tools have had to be, and must continue to be developed, such as radial basis functions [2], multivariate splines based on the Conformality of Smoothing Cofactor Method [3] that is applied for the construction of spline quasi-interpolation [4–7], rational approximation [8–10], etc.

To be mentioned, besides these new developments are the results on continued fraction interpolation and expansion. Cuyt and Verdonk constructed branched Thiele continued fractions

Received January 10, 2016; Accepted March 18, 2016
Supported by the Special Funds Tianyuan for the National Natural Science Foundation of China (Grant No. 11426086), the Fundamental Research Funds for the Central Universities (Grant No. 2016B08714) and the Natural Science Foundation of Jiangsu Province for the Youth (Grant No. BK20160853).
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rational interpolation in [11,12]. By introducing blending differences which look partially like multivariate divided differences and partially like multivariate inverse ones, Tan constructed the schemes for blending rational interpolation in Newton-Thiele type [13] and Thiele-Newton type [14]. Moreover, without the restriction of interpolation over rectangular domain, Wang and Qian have determined three-term recurrence relations for branched continued fractions, and constructed the modified branched continued fractions interpolation over pyramid-typed grids in $\mathbf{R}^3$ with the algorithm of partial inverse differences in tensor-product-like manner in [15]. Also, they have presented a novel continued fractions interpolation scheme over arbitrary ortho-triples [16] by using a new symmetrical algorithm of partial inverse differences in $\mathbf{R}^2$.

To the best of our knowledge, however, there are few papers before the present on surveying scattered data interpolation via bivariate continued fractions. Hence, by considering the arbitrary distribution of the interpolating nodes, we are interested in the construction of non-tensor-product-typed bivariate continued fraction interpolating scheme.

A brief outline of this article is as follows. In Section 2, by considering two types of scattered data distribution, we investigate new non-tensor-product-typed partial divided differences algorithms in a recursive form, and construct scattered data interpolation based on bivariate continued fractions with odd and even nodes, respectively. Also we obtain an equivalent identity between the interpolated function and the bivariate continued fraction. Then in Section 3, by the well-known three-term recurrence relations for continued fractions, we investigate the characterization theorem to disclose the degrees of the numerators and denominators of the interpolating continued fractions. Then, in Section 4, we compute the explicit representation of the interpolating continued fractions with some numerical examples, and plot the figures of these rational interpolating functions and the corresponding error functions. Numerical examples show it valid for the recursive scheme. We also compare the degrees of the numerators and denominators of bivariate Thiele-typed interpolating continued fractions with those of the new bivariate interpolating continued fractions, respectively. Finally, section 5 makes an analysis of the complexity of the bivariate continued fraction interpolation and the well-known radial basis function interpolation.

2. Bivariate recursive continued fraction interpolating algorithm

We shall propose a new recursive algorithm for bivariate inverse divided difference in the form of non-tensor product. Based upon the algorithm, we shall work out the interpolating coefficients of continued fractions recursively, and then complete the construction of scattered data interpolating schemes in the case of odd and even nodes, respectively.

Firstly, suppose that the $2n+1$ distinct points are contained in the node collection $\Pi_{2n+1} = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{2n}, y_{2n})\}$. And we choose the interpolating nodes to guarantee the s-mooth computation in the paper, for instance, $x_i \neq x_j$, $y_i \neq y_j$, for $i \neq j$, which means that the interpolating nodes do not lie in the horizontal or vertical line. Then we shall consider the bivariate continued fraction as

$$\begin{aligned} R_{2n+1}(x,y) =& c_0 + \frac{x-x_0 \quad |}{| \quad c_1} + \frac{(y-y_0)(x-x_1) \ |}{| \qquad c_2} + \frac{(y-y_1)(x-x_2) \ |}{| \qquad c_3} + \\ & \cdots \frac{(y-y_{2n-3})(x-x_{2n-2}) \ |}{| \qquad c_{2n-1}} + \frac{(y-y_{2n-2})(x-x_{2n-1}) \ |}{| \qquad c_{2n}} \\ \equiv & c_0 + \frac{x-x_0 \quad |}{| \ c_1} + K_{i=0}^{2n-2} \frac{(y-y_i)(x-x_{i+1}) \ |}{| \qquad c_{i+2}}. \end{aligned} \tag{2.1}$$

Secondly, we shall define the representation of the bivariate continued fraction $R_{2n+2}(x,y)$ by comparing with $R_{2n+1}(x,y)$, where the $2n+2$ distinct points are contained in the node collection $\Pi_{2n+2} = \{(x_0,y_0),(x_1,y_1),\ldots,(x_{2n+1},y_{2n+1})\}$, where $x_i \neq x_j$, $y_i \neq y_j$, for $i \neq j$.

$$R_{2n+2}(x,y) = R_{2n+1}(x,y) + \frac{(y-y_{2n-1})(x-x_{2n}) \ |}{| \qquad c_{2n+1}}, \tag{2.2}$$

where $R_{2n+1}(x,y)$ is defined as in (2.1).

Now we shall illustrate that the bivariate polynomials R_{2n+1} in (2.1) and R_{2n+2} in (2.2) can be put into application in scattered data interpolation over the node collection Π_{2n+1} and Π_{2n+2}, respectively. So we shall make some preparation for it by developing new non-tensor-product-type algorithms of bivariate partially inverse divided differences. To save space, the bivariate partially inverse divided difference $\phi[x_0,\ldots,x_k; y_0,\ldots,y_k]$, i.e., $\phi_{0\ldots,k}$ may be used to denote $\phi[x_0,x_1,\ldots,x_{k-1},x_k; y_0,y_1,\ldots,y_{k-1},y_k]$, and additional letters may be written explicitly only when the subscripts are not consecutive. And we denote the vertical coordinates at the interpolating nodes $P_i(x_i,y_i)$ by $f(x_i,y_i) \equiv f_i$ for $i=0,1,\ldots,2n,2n+1$, where $f(x,y)$ is defined as the interpolated function.

Definition 2.1 *For arbitrary node form Π_{2n+2}, where $x_i \neq x_j$, $y_i \neq y_j$, for $i \neq j$, we define the following bivariate non-tensor-product-typed partially inverse divided differences.*

$$\phi_i \equiv \phi[x_i;\ y_i] = f_i, \quad i = 0,1,\ldots,2n,2n+1. \tag{2.3}$$

$$\phi_{01} \equiv \phi[x_0,x_1;\ y_0,y_1] = \frac{x_1-x_0}{f_1-f_0}. \tag{2.4}$$

$$\phi_{012} \equiv \phi[x_0,x_1,x_2;\ y_0,y_1,y_2] = \frac{(y_2-y_0)(x_2-x_1)}{\phi[x_0,x_2;\ y_0,y_2]-\phi[x_0,x_1;\ y_0,y_1]}, \tag{2.5}$$

where $\phi[x_0,x_1;\ y_0,y_1]$ is defined as (2.4), and

$$\phi_{02} \equiv \phi[x_0,x_2;\ y_0,y_2] = \frac{x_2-x_0}{f_2-f_0}.$$

$$\phi_{0\cdots 3} \equiv \phi[x_0,\ldots,x_3;\ y_0,\ldots,y_3] = \frac{(y_3-y_1)(x_3-x_2)}{\phi[x_0,x_1,x_3;\ y_0,y_1,y_3]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]}, \tag{2.6}$$

where $\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]$ is defined as (2.5), and

$$\phi_{013} \equiv \phi[x_0,x_1,x_3;\ y_0,y_1,y_3] = \frac{(y_3-y_0)(x_3-x_1)}{\phi[x_0,x_3;\ y_0,y_3]-\phi[x_0,x_1;\ y_0,y_1]},$$

$$\phi_{03} \equiv \phi[x_0,x_3;\ y_0,y_3] = \frac{x_3-x_0}{f_3-f_0}.$$

$$\begin{aligned}\phi_{0\cdots 4} &\equiv \phi[x_0,\dots,x_4;\ y_0,\dots,y_4]\\ &= \frac{(y_4-y_2)(x_4-x_3)}{\phi[x_0,x_1,x_2,x_4;\ y_0,y_1,y_2,y_4]-\phi[x_0,\dots,x_3;\ y_0,\dots,y_3]},\end{aligned}\tag{2.7}$$

where $\phi[x_0,\dots,x_3;\ y_0,\dots,y_3]$ *is defined as (2.6), and*

$$\begin{aligned}\phi_{0124} &\equiv \phi[x_0,x_1,x_2,x_4;\ y_0,y_1,y_2,y_4]\\ &= \frac{(y_4-y_1)(x_4-x_2)}{\phi[x_0,x_1,x_4;\ y_0,y_1,y_4]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]},\end{aligned}$$

$$\phi_{014} \equiv \phi[x_0,x_1,x_4;\ y_0,y_1,y_4] = \frac{(y_4-y_0)(x_4-x_1)}{\phi[x_0,x_4;\ y_0,y_4]-\phi[x_0,x_1;\ y_0,y_1]},$$

$$\phi_{04} \equiv \phi[x_0,x_4;\ y_0,y_4] = \frac{x_4-x_0}{f_4-f_0}.$$

$$\cdots$$

$$\begin{aligned}\phi_{0\cdots,2n} &\equiv \phi[x_0,\dots,x_{2n};\ y_0,\dots,y_{2n}]\\ &= \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1})}{\phi[x_0,\dots,x_{2n-2},x_{2n};\ y_0,\dots,y_{2n-2},y_{2n}]-\phi[x_0,\dots,x_{2n-1};\ y_0,\dots,y_{2n-1}]},\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}\phi_{0\cdots,2n-2,2n} &\equiv \phi[x_0,\dots,x_{2n-2},x_{2n};\ y_0,\dots,y_{2n-2},y_{2n}]\\ &= \frac{(y_{2n}-y_{2n-3})(x_{2n}-x_{2n-2})}{\phi[x_0,\dots,x_{2n-3},x_{2n};\ y_0,\dots,y_{2n-3},y_{2n}]-\phi[x_0,\dots,x_{2n-2};\ y_0,\dots,y_{2n-2}]},\\ &\cdots,\end{aligned}$$

$$\begin{aligned}\phi_{0\cdots 4,2n} &\equiv \phi[x_0,\dots,x_4,x_{2n};\ y_0,\dots,y_4,y_{2n}]\\ &= \frac{(y_{2n}-y_3)(x_{2n}-x_4)}{\phi[x_0,\dots,x_3,x_{2n};\ y_0,\dots,y_3,y_{2n}]-\phi[x_0,\dots,x_4;\ y_0,\dots,y_4]},\end{aligned}$$

$$\begin{aligned}\phi_{0\cdots 3,2n} &\equiv \phi[x_0,\dots,x_3,x_{2n};\ y_0,\dots,y_3,y_{2n}]\\ &= \frac{(y_{2n}-y_2)(x_{2n}-x_3)}{\phi[x_0,x_1,x_2,x_{2n};\ y_0,y_1,y_2,y_{2n}]-\phi[x_0,\dots,x_3;\ y_0,\dots,y_3]},\end{aligned}$$

$$\begin{aligned}\phi_{012,2n} &\equiv \phi[x_0,x_1,x_2,x_{2n};\ y_0,y_1,y_2,y_{2n}]\\ &= \frac{(y_{2n}-y_1)(x_{2n}-x_2)}{\phi[x_0,x_1,x_{2n};\ y_0,y_1,y_{2n}]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]},\end{aligned}$$

$$\phi_{01,2n} \equiv \phi[x_0,x_1,x_{2n};\ y_0,y_1,y_{2n}] = \frac{(y_{2n}-y_0)(x_{2n}-x_1)}{\phi[x_0,x_{2n};\ y_0,y_{2n}]-\phi[x_0,x_1;\ y_0,y_1]},$$

$$\phi_{0,2n} \equiv \phi[x_0,x_{2n};\ y_0,y_{2n}] = \frac{x_{2n}-x_0}{f_{2n}-f_0}.$$

Up to

$$\begin{aligned}\phi_{0\cdots,2n+1} &\equiv \phi[x_0,\dots,x_{2n+1};\ y_0,\dots,y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n})}{\phi[x_0,\dots,x_{2n-1},x_{2n+1};\ y_0,\dots,y_{2n-1},y_{2n+1}]-\phi[x_0,\dots,x_{2n};\ y_0,\dots,y_{2n}]},\end{aligned}\tag{2.9}$$

where $\phi[x_0,\ldots,x_{2n};\ y_0,\ldots,y_{2n}]$ *is defined as (2.8), and*

$$\begin{aligned}\phi_{0\cdots,2n-1,2n+1} &\equiv \phi[x_0,\ldots,x_{2n-1},x_{2n+1};\ y_0,\ldots,y_{2n-1},y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_{2n-2})(x_{2n+1}-x_{2n-1})}{\phi[x_0,\ldots,x_{2n-2},x_{2n+1};\ y_0,\ldots,y_{2n-2},y_{2n+1}]-\phi[x_0,\ldots,x_{2n-1};\ y_0,\ldots,y_{2n-1}]},\end{aligned}$$

$$\cdots,$$

$$\begin{aligned}\phi_{0\cdots4,2n+1} &\equiv \phi[x_0,\ldots,x_4,x_{2n+1};\ y_0,\ldots,y_4,y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_3)(x_{2n+1}-x_4)}{\phi[x_0,\ldots,x_3,x_{2n+1};\ y_0,\ldots,y_3,y_{2n+1}]-\phi[x_0,\ldots,x_4;\ y_0,\ldots,y_4]},\end{aligned}$$

$$\begin{aligned}\phi_{0\cdots3,2n+1} &\equiv \phi[x_0,\ldots,x_3,x_{2n+1};\ y_0,\ldots,y_3,y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_2)(x_{2n+1}-x_3)}{\phi[x_0,x_1,x_2,x_{2n+1};\ y_0,y_1,y_2,y_{2n+1}]-\phi[x_0,\ldots,x_3;\ y_0,\ldots,y_3]},\end{aligned}$$

$$\begin{aligned}\phi_{012,2n+1} &\equiv \phi[x_0,x_1,x_2,x_{2n+1};\ y_0,y_1,y_2,y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_1)(x_{2n+1}-x_2)}{\phi[x_0,x_1,x_{2n+1};\ y_0,y_1,y_{2n+1}]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]},\end{aligned}$$

$$\begin{aligned}\phi_{01,2n+1} &\equiv \phi[x_0,x_1,x_{2n+1};\ y_0,y_1,y_{2n+1}]\\ &= \frac{(y_{2n+1}-y_0)(x_{2n+1}-x_1)}{\phi[x_0,x_{2n+1};\ y_0,y_{2n+1}]-\phi[x_0,x_1;\ y_0,y_1]},\end{aligned}$$

$$\phi_{0,2n+1} \equiv \phi[x_0,x_{2n+1};\ y_0,y_{2n+1}] = \frac{x_{2n+1}-x_0}{f_{2n+1}-f_0}.$$

Let us summarize the results in Definition 2.1 into the corresponding algorithms for the computation of the partially inverse divided differences with odd and even interpolating nodes, respectively, which are also shown in Table 1.

Algorithm 2.1

1. *Initialization:* $\phi_i = f(x_i,y_i)$, *where* $i=0,1,\ldots,2n$.
2. *Recursive case: By means of Definition 2.1, calculate*

$$\begin{aligned}&\phi_{0,i}\,(i=1,2,\ldots,2n)\to\phi_{01,i}\,(i=2,\ldots,2n)\to\phi_{012,i}\,(i=3,\ldots,2n)\\ &\qquad\to\phi_{0123,i}\,(i=4,\ldots,2n)\to\cdots\to\phi_{0\cdots,2n-2,i}\,(i=2n-1,2n)\end{aligned}$$

3. *Result:* $\phi_{0\cdots,2n}$

Algorithm 2.2

1. *Initialization:* $\phi_i = f(x_i,y_i)$, *where* $i=0,1,\ldots,2n+1$.
2. *Recursive case: By means of Definition 2.1, calculate*

$$\begin{aligned}&\phi_{0,i}\,(i=1,2,\ldots,2n+1)\to\phi_{01,i}\,(i=2,\ldots,2n+1)\to\phi_{012,i}\,(i=3,\ldots,2n+1)\\ &\qquad\to\phi_{0123,i}\,(i=4,\ldots,2n+1)\to\cdots\to\phi_{0\cdots,2n-1,i}\,(i=2n,2n+1)\end{aligned}$$

3. *Result:* $\phi_{0\cdots,2n+1}$

ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	$\cdots$	ϕ_{2n}	ϕ_{2n+1}
	ϕ_{01}	ϕ_{02}	ϕ_{03}	ϕ_{04}	ϕ_{05}	$\cdots$	$\phi_{0,2n}$	$\phi_{0,2n+1}$
		ϕ_{012}	ϕ_{013}	ϕ_{014}	ϕ_{015}	$\cdots$	$\phi_{01,2n}$	$\phi_{01,2n+1}$
			ϕ_{0123}	ϕ_{0124}	ϕ_{0125}	$\cdots$	$\phi_{012,2n}$	$\phi_{012,2n+1}$
				$\phi_{01\cdots 4}$	ϕ_{01235}	$\cdots$	$\phi_{0123,2n}$	$\phi_{0123,2n+1}$
					$\phi_{0\cdots 5}$	$\cdots$	$\phi_{0\cdots 4,2n}$	$\phi_{0\cdots 4,2n+1}$
						$\ddots$	$\vdots$	$\vdots$
							$\phi_{0\cdots,2n}$	$\phi_{0\cdots,2n-1,2n+1}$
								$\phi_{0\cdots,2n+1}$

Table 1 Recursive computation of bivariate non-tensor-product-typed partially inverse divided differences

Remark 2.2 In the case of threes of the interpolating nodes lying in the horizontal or vertical lines, the bivariate interpolating continued fractions have been considered over the orthor-triples in [16].

Now we shall establish bivariate continued fraction interpolation over the node collection Π_{2n+1} and Π_{2n+2}, respectively, and computation of the interpolating coefficients of them is based upon Algorithms 2.1 and 2.2.

Theorem 2.3 *(i) For each* $(x_i, y_i) \in \Pi_{2n+1}$, *the bivariate interpolating continued fraction* $R_{2n+1}(x,y)$ *defined in (2.1) satisfies*

$$R_{2n+1}(x_i, y_i) = f(x_i, y_i) = f_i, \quad i = 0, 1, \dots, 2n, \tag{2.10}$$

where the interpolating coefficients

$$c_i = \phi[x_0, \dots, x_i;\ y_0, \dots, y_i], \quad i = 0, 1, \dots, 2n. \tag{2.11}$$

(ii) The bivariate interpolating continued fraction $R_{2n+2}(x,y)$ *in the form of (2.2) interpolates over scattered data* Π_{2n+2}, *that is, for* $i = 0, 1, \dots, 2n+1$,

$$R_{2n+2}(x_i, y_i) = f(x_i, y_i) = f_i, \tag{2.12}$$

where the interpolating coefficients

$$c_i = \phi[x_0, \dots, x_i;\ y_0, \dots, y_i], \quad i = 0, 1, \dots, 2n+1. \tag{2.13}$$

Proof We shall perform the proof of Theorem 2.3 for n by induction.

For $n = 1$, $\Pi_2 = \{(x_0, y_0), (x_1, y_1)\}$, $R_2(x,y) = c_0 + \frac{x-x_0}{c_1}$, which satisfies $R_2(x_i, y_i) = f_i$, $i = 0, 1$. Thus, it follows that

$$c_0 = f_0, \quad c_1 = \frac{x_1 - x_0}{f_1 - f_0} \equiv \phi[x_0, x_1;\ y_0, y_1].$$

Moreover, for $n = 2$, $\Pi_3 = \{(x_i, y_i), i = 0, 1, 2\}$,

$$R_3(x,y) = c_0 + \frac{x - x_0 \ |}{|\ \ c_1} + \frac{(y - y_0)(x - x_1) \ |}{|\ \ c_2}.$$

By considering the interpolation at the node (x_2, y_2), we can obtain from the above formula $R_3(x, y)$,

$$c_1 + \frac{(y_2 - y_0)(x_2 - x_1)}{c_2} = \frac{x_2 - x_0}{f_2 - f_0} \equiv \phi[x_0, x_2;\ y_0, y_2],$$
$$\Rightarrow c_2 = \frac{(y_2 - y_0)(x_2 - x_1)}{\phi[x_0, x_2;\ y_0, y_2] - \phi[x_0, x_1;\ y_0, y_1]} \equiv \phi[x_0, x_1, x_2;\ y_0, y_1, y_2].$$

For $n = 3,\ \Pi_4 = \{(x_i, y_i), i = 0, 1, 2, 3\}$,

$$R_4(x, y) = c_0 + \frac{x - x_0\ \big|}{\big|\ c_1} + \frac{(y - y_0)(x - x_1)\ \big|}{\big|\ c_2} + \frac{(y - y_1)(x - x_2)\ \big|}{\big|\ c_3}.$$

By substituting the interpolating node (x_3, y_3) into $R_4(x, y)$, we have

$$c_1 + \frac{(y_3 - y_0)(x_3 - x_1)\big|}{\big|\ c_2} + \frac{(y_3 - y_1)(x_3 - x_2)\big|}{\big|\ c_3} = \frac{x_3 - x_0}{f_3 - f_0} \equiv \phi[x_0, x_3;\ y_0, y_3],$$
$$\Rightarrow c_2 + \frac{(y_3 - y_1)(x_3 - x_2)}{c_3} = \frac{(y_3 - y_0)(x_3 - x_1)}{\phi[x_0, x_3;\ y_0, y_3] - \phi[x_0, x_1;\ y_0, y_1]}$$
$$\equiv \phi[x_0, x_1, x_3;\ y_0, y_1, y_3]$$
$$\Rightarrow c_3 = \frac{(y_3 - y_1)(x_3 - x_2)}{\phi[x_0, x_1, x_3;\ y_0, y_1, y_3] - \phi[x_0, x_1, x_2;\ y_0, y_1, y_2]}$$
$$\equiv \phi[x_0, \ldots, x_3;\ y_0, \ldots, y_3].$$

For $n = 4,\ \Pi_5 = \{(x_i, y_i), i = 0, \ldots, 4\}$,

$$R_5(x, y) = R_4(x, y) + \frac{(y - y_2)(x - x_3)\ \big|}{\big|\ c_4}.$$

By substituting the interpolating node (x_4, y_4) into $R_5(x, y)$, we have

$$c_1 + \frac{(y_4 - y_0)(x_4 - x_1)\big|}{\big|\ c_2} + \frac{(y_4 - y_1)(x_4 - x_2)\big|}{\big|\ c_3} + \frac{(y_4 - y_2)(x_4 - x_3)\big|}{\big|\ c_4}$$
$$= \frac{x_4 - x_0}{f_4 - f_0} \equiv \phi[x_0, x_4;\ y_0, y_4],$$
$$\Rightarrow c_2 + \frac{(y_4 - y_1)(x_4 - x_2)\big|}{\big|\ c_3} + \frac{(y_4 - y_2)(x_4 - x_3)\big|}{\big|\ c_4}$$
$$= \frac{(y_4 - y_0)(x_4 - x_1)}{\phi[x_0, x_4;\ y_0, y_4] - \phi[x_0, x_1;\ y_0, y_1]} \equiv \phi[x_0, x_1, x_4;\ y_0, y_1, y_4]$$
$$\Rightarrow c_3 + \frac{(y_4 - y_2)(x_4 - x_3)}{c_4} = \frac{(y_4 - y_1)(x_4 - x_2)}{\phi[x_0, x_1, x_4;\ y_0, y_1, y_4] - \phi[x_0, x_1, x_2;\ y_0, y_1, y_2]}$$
$$\equiv \phi[x_0, x_1, x_2, x_4;\ y_0, y_1, y_2, y_4]$$
$$\Rightarrow c_4 = \frac{(y_4 - y_2)(x_4 - x_3)}{\phi[x_0, x_1, x_2, x_4;\ y_0, y_1, y_2, y_4] - \phi[x_0, \ldots, x_3;\ y_0, \ldots, y_3]}$$
$$\equiv \phi[x_0, \ldots, x_4;\ y_0, \ldots, y_4].$$

Hence, we show it valid for the case of $n = 1, 2, 3, 4, 5$ in Theorem 2.3.

Furthermore, we assume the results hold for $\Pi_i,\ i = 1, \ldots, 2n$. To be more precise, we denote the corresponding interpolating coefficients by $c_i^{'}$s in Algorithm 2.1 and 2.2. Then on the one hand, it follows in the case of $i = 2n+1$,

$$R_{2n+1}(x,y) = c_0 + \frac{x-x_0 \mid}{\mid c_1} + K_{i=0}^{2n-2} \frac{(y-y_i)(x-x_{i+1}) \mid}{\mid c_{i+2}}.$$

By considering the interpolation at the node (x_{2n}, y_{2n}), we have from $R_{2n+1}(x,y)$

$$\begin{aligned}
& c_1 + \frac{(y_{2n}-y_0)(x_{2n}-x_1) \mid}{\mid c_2} + \cdots + \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1}) \mid}{\mid c_{2n}} \\
& \frac{x_{2n}-x_0}{f_{2n}-f_0} \equiv \phi[x_0, x_{2n};\ \ y_0, y_{2n}] \\
\Rightarrow & c_2 + \frac{(y_{2n}-y_1)(x_{2n}-x_2) \mid}{\mid c_3} + \cdots + \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1}) \mid}{\mid c_{2n}} \\
& = \frac{(y_{2n}-y_0)(x_{2n}-x_1)}{\phi[x_0, x_{2n};\ y_0, y_{2n}] - \phi[x_0, x_1;\ y_0, y_1]} \equiv \phi[x_0, x_1, x_{2n};\ y_0, y_1, y_{2n}] \\
\Rightarrow & c_3 + \frac{(y_{2n}-y_2)(x_{2n}-x_3) \mid}{\mid c_4} + \cdots + \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1}) \mid}{\mid c_{2n}} \\
& = \frac{(y_{2n}-y_1)(x_{2n}-x_2)}{\phi[x_0, x_1, x_{2n};\ y_0, y_1, y_{2n}] - \phi[x_0, x_1, x_2;\ y_0, y_1, y_2]} \\
& \equiv \phi[x_0, x_1, x_2, x_{2n};\ y_0, y_1, y_2, y_{2n}] \\
\Rightarrow & c_4 + \frac{(y_{2n}-y_3)(x_{2n}-x_4) \mid}{\mid c_5} + \cdots + \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1}) \mid}{\mid c_{2n}} \\
& = \frac{(y_{2n}-y_2)(x_{2n}-x_3)}{\phi[x_0, x_1, x_2, x_{2n};\ y_0, y_1, y_2, y_{2n}] - \phi[x_0, \ldots, x_3;\ y_0, \ldots, y_3]} \\
& \equiv \phi[x_0, \ldots, x_3, x_{2n};\ y_0, \ldots, y_3, y_{2n}] \\
\Rightarrow & \cdots \\
\Rightarrow & c_{2n-1} + \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1}) \mid}{\mid c_{2n}} \\
& = \frac{(y_{2n}-y_{2n-3})(x_{2n}-x_{2n-2})}{\phi[x_0, \ldots, x_{2n-3}, x_{2n};\ y_0, \ldots, y_{2n-3}, y_{2n}] - \phi[x_0, \ldots, x_{2n-2};\ y_0, \ldots, y_{2n-2}]} \\
& \equiv \phi[x_0, \ldots, x_{2n-2}, x_{2n};\ y_0, \ldots, y_{2n-2}, y_{2n}] \\
\Rightarrow & c_{2n} \\
& = \frac{(y_{2n}-y_{2n-2})(x_{2n}-x_{2n-1})}{\phi[x_0, \ldots, x_{2n-2}, x_{2n};\ y_0, \ldots, y_{2n-2}, y_{2n}] - \phi[x_0, \ldots, x_{2n-1};\ y_0, \ldots, y_{2n-1}]} \\
& \equiv \phi[x_0, \ldots, x_{2n};\ \ y_0, \ldots, y_{2n}].
\end{aligned}$$

On the other hand, we deduce for $i = 2n+2$ similarly that

$$R_{2n+2}(x,y) = c_0 + \frac{x-x_0 \mid}{\mid c_1} + K_{i=0}^{2n-1} \frac{(y-y_i)(x-x_{i+1}) \mid}{\mid c_{i+2}}.$$

By considering the interpolation at the node (x_{2n+1}, y_{2n+}), we obtain from $R_{2n+2}(x,y)$

$$\begin{aligned}
& c_1 + \frac{(y_{2n+1}-y_0)(x_{2n+1}-x_1) \mid}{\mid c_2} + \cdots + \frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n}) \mid}{\mid c_{2n+1}} \\
& = \frac{x_{2n+1}-x_0}{f_{2n+1}-f_0} \equiv \phi[x_0, x_{2n+1};\ y_0, y_{2n+1}] \\
\Rightarrow & c_2 + \frac{(y_{2n+1}-y_1)(x_{2n+1}-x_2) \mid}{\mid c_3} + \cdots + \frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n}) \mid}{\mid c_{2n+1}}
\end{aligned}$$

$$= \frac{(y_{2n+1}-y_0)(x_{2n+1}-x_1)}{\phi[x_0,x_{2n+1};\ y_0,y_{2n+1}]-\phi[x_0,x_1;\ y_0,y_1]} \equiv \phi[x_0,x_1,x_{2n+1};\ y_0,y_1,y_{2n+1}]$$

$$\Rightarrow c_3+\frac{(y_{2n+1}-y_2)(x_{2n+1}-x_3)\ |}{|\quad c_4}+\cdots+\frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n})\ |}{|\quad c_{2n+1}}$$

$$= \frac{(y_{2n+1}-y_1)(x_{2n+1}-x_2)}{\phi[x_0,x_1,x_{2n+1};\ y_0,y_1,y_{2n+1}]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]}$$

$$\equiv \phi[x_0,x_1,x_2,x_{2n+1};\ y_0,y_1,y_2,y_{2n+1}]$$

$$\Rightarrow c_4+\frac{(y_{2n+1}-y_3)(x_{2n+1}-x_4)\ |}{|\quad c_5}+\cdots+\frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n})\ |}{|\quad c_{2n+1}}$$

$$= \frac{(y_{2n+1}-y_2)(x_{2n+1}-x_3)}{\phi[x_0,x_1,x_2,x_{2n+1};\ y_0,y_1,y_2,y_{2n+1}]-\phi[x_0,\ldots,x_3;\ y_0,\ldots,y_3]}$$

$$\equiv \phi[x_0,\ldots,x_3,x_{2n};\ y_0,\ldots,y_3,y_{2n}]$$

$$\Rightarrow\cdots$$

$$\Rightarrow c_{2n}+\frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n})\ |}{|\quad c_{2n+1}}$$

$$= \frac{(y_{2n+1}-y_{2n-2})(x_{2n+1}-x_{2n-1})}{\phi[x_0,\ldots,x_{2n-2},x_{2n+1};\ y_0,\ldots,y_{2n-2},y_{2n+1}]-\phi[x_0,\ldots,x_{2n-1};\ y_0,\ldots,y_{2n-1}]}$$

$$\equiv \phi[x_0,\ldots,x_{2n-1},x_{2n+1};\quad y_0,\ldots,y_{2n-1},y_{2n+1}]$$

$$\Rightarrow c_{2n+1}$$

$$= \frac{(y_{2n+1}-y_{2n-1})(x_{2n+1}-x_{2n})}{\phi[x_0,\ldots,x_{2n-1},x_{2n+1};\ y_0,\ldots,y_{2n-1},y_{2n+1}]-\phi[x_0,\ldots,x_{2n};\ y_0,\ldots,y_{2n}]}$$

$$\equiv \phi[x_0,\ldots,x_{2n+1};\quad y_0,\ldots,y_{2n+1}].$$

As a result, we show it valid for (2.10) up to (2.13) by induction. □

Since we establish the bivariate continued fraction interpolation, we shall make a further study on the equivalent identities between the interpolated functions and the interpolating continued fractions. Before dealing with it, we shall define some bivariate non-tensor-product-typed inverse divided differences with respect to x and y by means of Definition 2.1.

Definition 2.4 *For arbitrary node form Π_{2n+2}, where $x_i \neq x_j$, $y_i \neq y_j$, for $i \neq j$, and $x \neq x_i$, $y \neq y_i$, we define some bivariate non-tensor-product-typed inverse divided differences with respect to x and y.*

(1)

$$\phi[x_0,x;\ y_0,y]=\frac{x-x_0}{f(x,y)-f_0}.$$

(2)

$$\phi[x_0,x_1,x;\ y_0,y_1,y]=\frac{(y-y_0)(x-x_1)}{\phi[x_0,x;\ y_0,y]-\phi[x_0,x_1;\ y_0,y_1]},$$

(3)

$$\phi[x_0,x_1,x_2,x;\ y_0,y_1,y_2,y]=\frac{(y-y_1)(x-x_2)}{\phi[x_0,x_1,x;\ y_0,y_1,y]-\phi[x_0,x_1,x_2;\ y_0,y_1,y_2]},$$

(4)

$$\phi[x_0,\ldots,x_3,x;\ y_0,\ldots,y_3,y]$$

$$= \frac{(y-y_2)(x-x_3)}{\phi[x_0,x_1,x_2,x;\ y_0,y_1,y_2,y]-\phi[x_0,\dots,x_3;\ y_0,\dots,y_3]},$$

(5)

$$\begin{aligned}&\phi[x_0,\dots,x_4,x;\ y_0,\dots,y_4,y]\\ &\quad= \frac{(y-y_3)(x-x_4)}{\phi[x_0,\dots,x_3,x;\ y_0,\dots,y_3,y]-\phi[x_0,\dots,x_4;\ y_0,\dots,y_4]},\end{aligned}$$

$(2n)$

$$\begin{aligned}&\phi[x_0,\dots,x_{2n-1},x;\ y_0,\dots,y_{2n-1},y]\\ &\quad= \frac{(y-y_{2n-2})(x-x_{2n-1})}{\phi[x_0,\dots,x_{2n-2},x;\ y_0,\dots,y_{2n-2},y]-\phi[x_0,\dots,x_{2n-1};\ y_0,\dots,y_{2n-1}]},\end{aligned}$$

$(2n+1)$

$$\begin{aligned}&\phi[x_0,\dots,x_{2n},x;\ y_0,\dots,y_{2n},y]\\ &\quad= \frac{(y-y_{2n-1})(x-x_{2n})}{\phi[x_0,\dots,x_{2n-1},x;\ y_0,\dots,y_{2n-1},y]-\phi[x_0,\dots,x_{2n};\ y_0,\dots,y_{2n}]},\end{aligned}$$

Up to $(2n+2)$

$$\begin{aligned}&\phi[x_0,\dots,x_{2n+1},x;\ y_0,\dots,y_{2n+1},y]\\ &\quad= \frac{(y-y_{2n})(x-x_{2n+1})}{\phi[x_0,\dots,x_{2n},x;\ y_0,\dots,y_{2n},y]-\phi[x_0,\dots,x_{2n+1};\ y_0,\dots,y_{2n+1}]}.\end{aligned}$$

Theorem 2.5 *By letting the interpolating nodes* (x_i,y_i), $i=0,1,2,\dots,2n$, *and using Definition 2.4, we have the equivalent identity*

$$f(x,y)=c_0+\frac{x-x_0\ \mid}{\mid\ c_1}+K_{i=0}^{2n-2}\frac{(y-y_i)(x-x_{i+1})\ \mid}{\mid\ c_{i+2}}+\frac{(y-y_{2n-1})(x-x_{2n})\ \mid}{\mid\ c_{2n+1}(x,y)}, \tag{2.14}$$

where the interpolating coefficients satisfy

$$c_i=\phi[x_0,\dots,x_i;\ y_0,\dots,y_i],\quad i=0,1,\dots,2n.$$

And we calculate the function $c_{2n+1}(x,y)$ *in the last term*

$$c_{2n+1}(x,y)=\phi[x_0,\dots,x_{2n},x;\quad y_0,\dots,y_{2n},y]. \tag{2.15}$$

Theorem 2.6 *Given the interpolating nodes* (x_i,y_i), $i=0,1,2,\dots,2n$, *we obtain the equivalent identity by means of Definition 2.4*

$$f(x,y)=c_0+\frac{x-x_0\ \mid}{\mid\ c_1}+K_{i=0}^{2n-1}\frac{(y-y_i)(x-x_{i+1})\ \mid}{\mid\ c_{i+2}}+\frac{(y-y_{2n})(x-x_{2n+1})\ \mid}{\mid\ c_{2n+2}(x,y)}, \tag{2.16}$$

where the interpolating coefficients satisfy

$$c_i=\phi[x_0,\dots,x_i;\ y_0,\dots,y_i],\quad i=0,1,\dots,2n+1,$$

and the function $c_{2n+2}(x,y)$ *in the last term satisfies*

$$c_{2n+2}(x,y)=\phi[x_0,\dots,x_{2n+1},x;\ y_0,\dots,y_{2n+1},y]. \tag{2.17}$$

3. Characterization theorem

In this section, one may see that the three-term recurrence relations for the bivariate continued fraction $R_{2n+1}(x,y)$ and $R_{2n+2}(x,y)$ play a vital role on determining the degrees of the numerator and the denominator, respectively. This will be done with sufficient preparation for them.

Denote by $P_{2n+1}(x,y)$ and $Q_{2n+1}(x,y)$ the numerator and the denominator of $R_{2n+1}(x,y)$ as defined in (2.1), respectively, the degrees of which are denoted by $\deg P_{2n+1}$ and $\deg Q_{2n+1}$, respectively. And we say the rational function $R_{2n+1}(x,y)$ is of type $(\deg P_{2n+1})/(\deg Q_{2n+1})$. Similarly, we denote by $P_{2n+2}(x,y)$ and $Q_{2n+2}(x,y)$ the numerator and the denominator of $R_{2n+2}(x,y)$ as defined in (2.2), respectively, with degrees being denoted by $\deg P_{2n+2}$ and $\deg Q_{2n+2}$, respectively. And we say the rational function $R_{2n+2}(x,y)$ is of type $(\deg P_{2n+2})/(\deg Q_{2n+2})$.

Lemma 3.1 ([9]) *For the continued fraction*

$$b_0 + \frac{a_1 \mid}{\mid b_1} + \frac{a_2 \mid}{\mid b_2} + \cdots = b_0 + K_{i=1}^{\infty} \frac{a_i}{b_i}, \tag{3.1}$$

and its nth convergent

$$R_n = b_0 + K_{i=1}^{n} \frac{a_i}{b_i} \equiv \frac{P_n}{Q_n}, \tag{3.2}$$

the three-term recurrence relations for the continued fraction are valid, i.e.,

$$P_n = b_n P_{n-1} + a_n P_{n-2},\ Q_n = b_n Q_{n-1} + a_n Q_{n-2}, \tag{3.3}$$

where $P_{-1} = 1,\ P_0 = b_0,\ Q_{-1} = 0,\ Q_0 = 1,\ n = 1, 2, \ldots$.

Lemma 3.2 ([9]) *Let*

$$R_n(x) = b_0(x) + K_{i=1}^{n} \frac{x - x_{i-1}}{b_i} = \frac{P_n(x)}{Q_n(x)}.$$

Then $\deg P_n(x) = [(n+1)/2], \deg Q_n(x) = [n/2]$. In other words, the continued fraction $R_n(x)$ is of type $[(n+1)/2]/[n/2]$, where $[n/2]$ means the entire part of $n/2$.

Besides these, throughout the paper we let $\mathbb{P}_{n,n}(x,y)$ and $\mathbb{P}_{n+1,n}(x,y)$ denote the bivariate tensor-product-type polynomial spaces

$$\mathbb{P}_{n,n} = \mathrm{span}\{1, x, \ldots, x^n\} \bigotimes \{1, y, \ldots, y^n\},$$
$$\mathbb{P}_{n+1,n} = \mathrm{span}\{1, x, \ldots, x^{n+1}\} \bigotimes \{1, y, \ldots, y^n\},$$

respectively.

Now by directly calculating, we can obtain the three-term recurrence relations for the bivariate continued fraction $R_{2n+1}(x,y)$ and $R_{2n+2}(x,y)$, respectively. And then we establish the corresponding algorithms for computing them.

Theorem 3.3 *By letting*

$$R_{2n+1}(x,y) \equiv \frac{P_{2n+1}(x,y)}{Q_{2n+1}(x,y)} \tag{3.4}$$

as defined in (2.1), we obtain the three-term recurrence relations

$$P_{2n+1}(x,y) = c_{2n} P_{2n}(x,y) + (y - y_{2n-2})(x - x_{2n-1}) P_{2n-1}(x,y),$$

$$Q_{2n+1}(x,y)=c_{2n}Q_{2n}(x,y)+(y-y_{2n-2})(x-x_{2n-1})Q_{2n-1}(x,y), \tag{3.5}$$

where $n=1,2,\ldots,$ *and*

$$P_1=c_0,\quad Q_1=1,\quad P_2=c_0c_1+x-x_0,\quad Q_2=c_1. \tag{3.6}$$

Theorem 3.4 *Suppose*

$$R_{2n+2}(x,y)\equiv\frac{P_{2n+2}(x,y)}{Q_{2n+2}(x,y)} \tag{3.7}$$

as defined in (2.2), then for $n=0,1,2,\ldots,$ *the three-term recurrence relations*

$$\begin{aligned}P_{2n+2}(x,y)&=c_{2n+1}P_{2n+1}(x,y)+(y-y_{2n-1})(x-x_{2n})P_{2n}(x,y),\\ Q_{2n+2}(x,y)&=c_{2n+1}Q_{2n+1}(x,y)+(y-y_{2n-1})(x-x_{2n})Q_{2n}(x,y),\end{aligned} \tag{3.8}$$

where

$$P_0=1,\quad P_1=c_0,\quad Q_0=0,\quad Q_1=1,\quad y-y_{-1}=1. \tag{3.9}$$

Algorithm 3.1 *Let* $2n+1$ *interpolating nodes* (x_i,y_i) *and the corresponding vertical coordinate* $f_i(i-0,1,\ldots,2n)$ *be given.*

1. *Initialization: For* $i=0,1,\ldots,2n$, *calculate the interpolating coefficients*

$$c_i=\phi[x_0,\ldots,x_i;y_0,\ldots,y_i].$$

2. *Recursive case:*
   For $k=1,2,\ldots,n$

$$\begin{aligned}&P_0=1;\quad Q_0=0;\\ &P_1=c_0;\quad Q_1=1;\\ &P_{2k}=c_{2k-1}P_{2k-1}+(y-y_{2k-3})(x-x_{2k-2})P_{2k-2};\\ &Q_{2k}=c_{2k-1}Q_{2k-1}+(y-y_{2k-3})(x-x_{2k-2})Q_{2k-2};\\ &P_{2k+1}=c_{2k}P_{2k}+(y-y_{2k-2})(x-x_{2k-1})P_{2k-1};\\ &Q_{2k+1}=c_{2k}Q_{2k}+(y-y_{2k-2})(x-x_{2k-1})Q_{2k-1};\end{aligned}$$

   end
3. *Result:* $P_{2n+1}(x,y),\quad Q_{2n+1}(x,y)\to R_{2n+1}(x,y)=\frac{P_{2n+1}(x,y)}{Q_{2n+1}(x,y)}$.

Algorithm 3.2 *Let* $2n+2$ *interpolating nodes* (x_i,y_i) *and the corresponding vertical coordinate* $f_i(i=0,1,\ldots,2n+1)$ *be given.*

1. *Initialization: For* $i=0,1,\ldots,2n+1$, *calculate the interpolating coefficients*

$$c_i=\phi[x_0,\ldots,x_i;y_0,\ldots,y_i].$$

2. *Recursive case:*
   For $k=1,2,\ldots,n$

$$P_1=c_0;\quad Q_1=1;$$

$$P_2 = c_0c_1 + x - x_0; \quad Q_2 = c_1;$$
$$P_{2k+1} = c_{2k}P_{2k} + (y - y_{2k-2})(x - x_{2k-1})P_{2k-1};$$
$$Q_{2k+1} = c_{2k}Q_{2k} + (y - y_{2k-2})(x - x_{2k-1})Q_{2k-1};$$
$$P_{2k+2} = c_{2k+1}P_{2k+1} + (y - y_{2k-1})(x - x_{2k})P_{2k};$$
$$Q_{2k+2} = c_{2k+1}Q_{2k+1} + (y - y_{2k-1})(x - x_{2k})Q_{2k};$$

end

3. Result: $P_{2n+2}(x,y), \quad Q_{2n+2}(x,y) \to R_{2n+2}(x,y) = \dfrac{P_{2n+2}(x,y)}{Q_{2n+2}(x,y)}.$

Based upon the three-term recurrence relations for the bivariate continued fraction $R_{2n+1}(x,y)$ in Theorem 3.3 and $R_{2n+2}(x,y)$ in Theorem 3.4, respectively, the degrees of the numerators $P_{2n+1}(x,y)$, $P_{2n+2}(x,y)$ and the denominators $Q_{2n+1}(x,y)$, $Q_{2n+2}(x,y)$ can be determined, which is called the characterization theorem.

Theorem 3.5 *(i) The bivariate continued fraction $R_{2n+1}(x,y)$ is of type $(2n)/(2n)$, and*

$$P_{2n+1}(x,y) \in \mathbb{P}_{n,n}, \; Q_{2n+1}(x,y) \in \mathbb{P}_{n,n}, \quad n = 0, 1, 2, \ldots.$$

(ii) The bivariate continued fraction $R_{2n+2}(x,y)$ is of type $(2n+1)/(2n)$, and

$$P_{2n+2}(x,y) \in \mathbb{P}_{n+1,n}, \; Q_{2n+2}(x,y) \in \mathbb{P}_{n,n}, \quad n = 0, 1, 2, \ldots.$$

Proof The proof of Theorem 3.5 is performed for n by induction.

For $n = 0$, with direct calculation, the continued fraction $R_2(x,y)$ over two interpolating nodes is of type $1/0$, and

$$P_2(x,y) \in \text{span}\{1, x\} \bigotimes \{1\} \equiv \mathbb{P}_{1,0}, \quad Q_2(x,y) \in \text{span}\{1\} \bigotimes \{1\} \equiv \mathbb{P}_{0,0}.$$

Moreover, for $n = 1$, the continued fraction $R_3(x,y)$ over three interpolating nodes is of type $2/2$, and

$$P_3(x,y) \in \text{span}\{1, x\} \bigotimes \{1, y\} \equiv \mathbb{P}_{1,1}, \quad Q_3(x,y) \in \text{span}\{1, x\} \bigotimes \{1, y\} \equiv \mathbb{P}_{1,1}.$$

For $n = 2$, the continued fraction $R_4(x,y)$ over four interpolating nodes is of type $3/2$, and

$$P_4(x,y) \in \text{span}\{1, x, x^2\} \bigotimes \{1, y\} \equiv \mathbb{P}_{2,1}, \quad Q_4(x,y) \in \text{span}\{1, x\} \bigotimes \{1, y\} \equiv \mathbb{P}_{1,1}.$$

Furthermore, by assuming that the conclusion holds for the interpolating continued fraction $R_{2n-1}(x,y)$ over Π_{2n-1} and $R_{2n}(x,y)$ over Π_{2n}, respectively, we shall show it valid for $R_{2n+1}(x,y)$ over Π_{2n+1}.

$$\begin{aligned}
\deg P_{2n+1}(x,y) &= \max\{c_{2n}P_{2n}(x,y), \; (y - y_{2n-2})(x - x_{2n-1})P_{2n-1}(x,y)\} \\
&= \max\{2n-1, 2n-2+2\} = 2n, \\
\deg Q_{2n+1}(x,y) &= \max\{c_{2n}Q_{2n}(x,y), \; (y - y_{2n-2})(x - x_{2n-1})Q_{2n-1}(x,y)\} \\
&= \max\{2n-2, 2n-2+2\} = 2n,
\end{aligned}$$

and

$$P_{2n-1}(x,y) \in \mathbb{P}_{n-1,n-1}, \quad P_{2n}(x,y) \in \mathbb{P}_{n,n} \Rightarrow \quad P_{2n+1}(x,y) \in \mathbb{P}_{n,n},$$

$$Q_{2n-1}(x,y)\in\mathbb{P}_{n-1,n-1},\quad Q_{2n}(x,y)\in\mathbb{P}_{n-1,n-1}\Rightarrow\quad Q_{2n+1}(x,y)\in\mathbb{P}_{n,n}.$$

Meanwhile, by using the above conclusion, we also prove the result for $R_{2n+2}(x,y)$ over Π_{2n+2}.

$$\begin{aligned}\deg P_{2n+2}(x,y)&=\max\{c_{2n+1}P_{2n+1}(x,y),\quad (y-y_{2n-1})(x-x_{2n})P_{2n}(x,y)\}\\&=\max\{2n,2n-1+2\}=2n+1,\\ \deg Q_{2n+2}(x,y)&=\max\{c_{2n+1}Q_{2n+1}(x,y),\quad (y-y_{2n-1})(x-x_{2n})Q_{2n}(x,y)\}\\&=\max\{2n,2n-2+2\}=2n,\end{aligned}$$

and

$$\begin{aligned}&P_{2n+1}(x,y)\in\mathbb{P}_{n,n},\quad P_{2n}(x,y)\in\mathbb{P}_{n,n-1}\Rightarrow\quad P_{2n+2}(x,y)\in\mathbb{P}_{n+1,n},\\&Q_{2n+1}(x,y)\in\mathbb{P}_{n,n},\quad Q_{2n}(x,y)\in\mathbb{P}_{n-1,n-1}\Rightarrow\quad Q_{2n+2}(x,y)\in\mathbb{P}_{n,n}.\end{aligned}$$

Thus, the proof of Theorem 3.5 is completed. □

Remark 3.6 For the sake of illustration, we list the number of the interpolating nodes, the type of the bivariate continued fractions, the bivariate polynomial spaces which the numerators and the denominators are contained in. All the results are shown in Table 2.

Scattered data Π_n	Type of $R_n(x,y)$	Polynomial Space for P_n	Space for Q_n
Π_2	$1/0$	$\mathbb{P}_{1,0}$	$\mathbb{P}_{0,0}$
Π_3	$2/2$	$\mathbb{P}_{1,1}$	$\mathbb{P}_{1,1}$
Π_4	$3/2$	$\mathbb{P}_{2,1}$	$\mathbb{P}_{1,1}$
Π_5	$4/4$	$\mathbb{P}_{2,2}$	$\mathbb{P}_{2,2}$
Π_6	$5/4$	$\mathbb{P}_{3,2}$	$\mathbb{P}_{2,2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
Π_{2n-1}	$(2n-2)/(2n-2)$	$\mathbb{P}_{n-1,n-1}$	$\mathbb{P}_{n-1,n-1}$
Π_{2n}	$(2n-1)/(2n-2)$	$\mathbb{P}_{n,n-1}$	$\mathbb{P}_{n-1,n-1}$
Π_{2n+1}	$(2n)/(2n)$	$\mathbb{P}_{n,n}$	$\mathbb{P}_{n,n}$
Π_{2n+2}	$(2n+1)/(2n)$	$\mathbb{P}_{n+1,n}$	$\mathbb{P}_{n,n}$

Table 2 The characterization of bivariate interpolating continued fractions over Π_i's

Remark 3.7 Bivariate Thiele-typed continued fraction interpolating scheme $\tilde{R}_{2n+1}(x,y)\equiv\dfrac{\tilde{P}_{2n+1}(x,y)}{\tilde{Q}_{2n+1}(x,y)}$ shall be constructed over the tensor-product-typed mesh $\tilde{\Pi}_{2n+1}=\{(x_i,y_j),i,j=0,\dots,2n\}$, where $x_i\neq x_j$, $y_i\neq y_j$, for $i\neq j$. Based upon Lemma 3.2, it follows that

$$\tilde{P}_{2n+1}(x,y)\in\mathbb{P}_{n+1,n+1},\ \tilde{Q}_{2n+1}(x,y)\in\mathbb{P}_{n,n},\quad n=0,1,2,\dots.$$

Obviously, the degree of $\tilde{P}_{2n+1}(x,y)$ is higher than that of $P_{2n+1}(x,y)$ in Theorem 3.5 (i).

Remark 3.8 One can also construct the bivariate Thiele-typed interpolating continued fraction

$\tilde{R}_{2n+2}(x,y) \equiv \dfrac{\tilde{P}_{2n+2}(x,y)}{\tilde{Q}_{2n+2}(x,y)}$ over $\tilde{\Pi}_{2n+2} = \{(x_i, y_j), i, j = 0, \dots, 2n+1\}$, where

$$\tilde{P}_{2n+2}(x,y) \in \mathbb{P}_{n+1,n+1}, \ \tilde{Q}_{2n+2}(x,y) \in \mathbb{P}_{n+1,n+1}, \quad n = 0, 1, 2, \dots.$$

Consequently, the degrees of $\tilde{P}_{2n+2}(x,y)$ and $\tilde{Q}_{2n+2}(x,y)$ are higher than those of $P_{2n+2}(x,y)$ and $Q_{2n+2}(x,y)$ in Theorem 3.5 (ii), respectively.

4. Numerical examples

By means of Algorithms 2.1, 2.2, 3.1 and 3.2, some bivariate continued fraction interpolations are determined over scattered data with some numerical examples in the section. On the one hand, we work out the representation of the numerators and the denominators. On the other hand, we will plot the figures of continued fractions and the corresponding error functions.

For the sake of illustration, we make some preparation with some denotations for the numerical examples. Denote the scattered data collection by $\Pi_{k+1} = \{(x_0, y_0), (x_1, y_1), \dots, (x_k, y_k)\}$, where $k = 2, 3, 4, 5$, and $x_i \neq x_j$, $y_i \neq y_j$, for $i \neq j$. Then by (2.1) and (2.2), we obtain the explicit representation of the interpolating continued fractions as follows

$$R_3(x,y) \equiv \frac{P_3(x,y)}{Q_3(x,y)} = c_0 + \frac{x - x_0 \ |}{| \ c_1} + \frac{(y-y_0)(x-x_1) \ |}{| \ c_2}, \tag{4.1}$$

$$\begin{aligned} R_4(x,y) \equiv & \frac{P_4(x,y)}{Q_4(x,y)} \\ = & c_0 + \frac{x - x_0 \ |}{| \ c_1} + \frac{(y-y_0)(x-x_1) \ |}{| \ c_2} + \frac{(y-y_1)(x-x_2) \ |}{| \ c_3}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} R_5(x,y) \equiv & \frac{P_5(x,y)}{Q_5(x,y)} \\ = & c_0 + \frac{x - x_0 \ |}{| \ c_1} + \frac{(y-y_0)(x-x_1) \ |}{| \ c_2} + \frac{(y-y_1)(x-x_2) \ |}{| \ c_3} + \\ & \frac{(y-y_2)(x-x_3) \ |}{| \ c_4}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} R_6(x,y) \equiv & \frac{P_6(x,y)}{Q_6(x,y)} \\ = & c_0 + \frac{x - x_0 \ |}{| \ c_1} + \frac{(y-y_0)(x-x_1) \ |}{| \ c_2} + \frac{(y-y_1)(x-x_2) \ |}{| \ c_3} + \\ & \frac{(y-y_2)(x-x_3) \ |}{| \ c_4} + \frac{(y-y_3)(x-x_4) \ |}{| \ c_5} \end{aligned} \tag{4.4}$$

Example 4.1 Suppose interpolating nodes $P_0(-7, -9.5)$, $P_1(-5, -4)$, $P_2(-3, -2)$, $P_3(0.2, -1)$, $P_4(4, 2)$, $P_5(7.8, 8)$, and the corresponding node collection $\Pi_{k+1}, k = 2, 3, 4, 5$ are given as shown in Figure 1.1. We also give the corresponding vertical coordinates, i.e., VCs, f_i at $P_i(x_i, y_i)$, $i = 0, \dots, 5$ in Table 3. Then we calculate the six interpolating coefficients c_i's in Table 4 by using Algorithms 2.1 and 2.2. Moreover, we determine the explicit representation of the interpolating continued fractions over Π_{k+1}, $k = 2, 3, 4, 5$ as the following, respectively. Actually, we obtain $R_3(x,y)$ and $R_5(x,y)$ by using Algorithm 3.1, and $R_4(x,y)$ and $R_6(x,y)$ by means of Algorithm 3.2. Finally, we plot the figures of the interpolating rational functions $R_i(x,y)$, $i=3,4,5,6$ in Figure 1.2 to 1.5, respectively. To be mentioned, we provide the programming procedure of calculating the interpolating coefficients, and testing the interpolating property of the rational function $R_6(x,y)$ at the interpolating nodes, and plotting Figure 1.5 in the appendix.

VCs	f_0	f_1	f_2	f_3	f_4	f_5
Values	-0.058745	0.018686	-0.124112	0.835460	-0.217184	-0.088092

Table 3 The vertical coordinates (VCs) f_i's in Example 4.1

c_0	c_1	c_2	c_3	c_4	c_5
-0.058745	25.829193	-0.172369	-4.148767	-0.363684	158.478514

Table 4 The coefficients of $R_i(x,y), i=2,3,4,5$, in Example 4.1

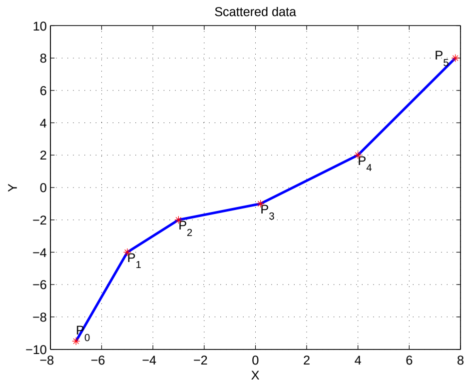

Figure 1.1 Scattered data Π_6

$$\begin{aligned}
P_3(x,y) =& -0.730450x-0.293727y-0.058745xy-3.735443,\\
Q_3(x,y) =& xy+5.0y+9.5x+43.047846,\\
R_3(x,y) =& \frac{P_3(x,y)}{Q_3(x,y)},\\
P_4(x,y) =& 36.961088x+17.666570y+8.726376xy+x^2y+4.0x^2+81.289351,\\
Q_4(x,y) =& 63.903487x+56.743745y+21.680426xy+131.354841,\\
R_4(x,y) =& \frac{P_4(x,y)}{Q_4(x,y)},\\
P_5(x,y) =& -0.058745x^2y^2-1.211624x^2y-2.915635x^2-0.281978xy^2-\\
& 7.326952xy-20.620862x+0.058745y^2-5.560469y-28.069457,\\
Q_5(x,y) =& x^2y^2+11.5x^2y+19.0x^2+4.8xy^2+42.863022xy+59.055017x-\\
& 1.0y^2-31.246360y-64.990789,\\
R_5(x,y) =& \frac{P_5(x,y)}{Q_5(x,y)},
\end{aligned}$$

$$P_6(x,y) = 3.155002x^3y^2 + 15.775009x^3y + 12.620007x^3 - 14.460938x^2y^2 - 524.7681670x^2y - 1391.685375x^2 - 195.377658xy^2 - 3927.845914xy - 10520.412261x - 193.579579y^2 - 4029.058999y - 15060.600628,$$

$$Q_6(x,y) = 568.401784x^2y^2 + 6020.017403x^2y + 9701.615619x^2 + 2305.419482xy^2 + 20944.892838xy + 29135.470886x - 1216.106476y^2 - 17996.985482y - 34153.093667,$$

$$R_6(x,y) = \frac{P_6(x,y)}{Q_6(x,y)}.$$

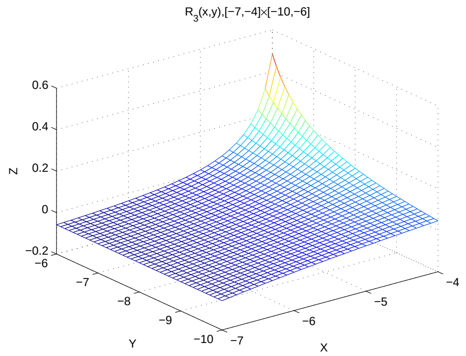

Figure 1.2 Part of the interpolating continued fraction $R_3(x,y)$ over Π_3

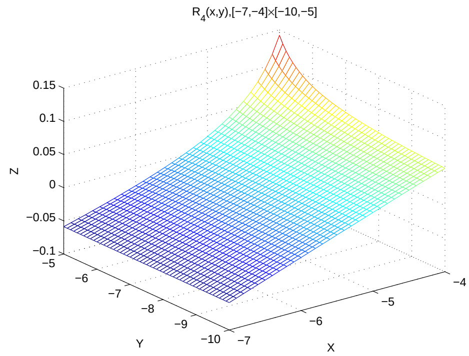

Figure 1.3 Part of the interpolating continued fraction $R_4(x,y)$ over Π_4

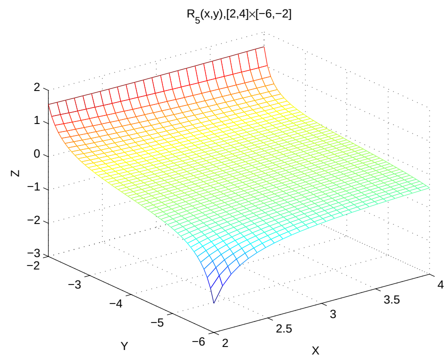

Figure 1.4 Part of the interpolating continued fraction $R_5(x,y)$ over Π_5

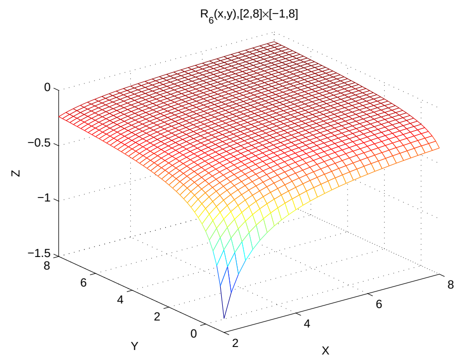

Figure 1.5 Part of the interpolating continued fraction $R_6(x,y)$ over Π_6

Example 4.2 Considering the scattered data $P_0(-0.9, 0.2)$, $P_1(-0.4, 0.6)$, $P_2(0.2, 0.8)$, $P_3(0.8, 0.4)$, $P_4(0.6, -0.4)$, $P_5(-0.6, -0.8)$, we denote by Π_{k+1}, $k = 2, 3, 4, 5$ the corresponding node collection as shown in Figure 2.1. By means of the given vertical coordinates z_i's at P_i's in Table 5, we calculate the six interpolating coefficients c_i's in Table 6. Consequently, we work out the representation of the bivariate interpolating continued fractions $R_i(x,y)$'s as the following. In fact, we compute $R_3(x,y)$ and $R_5(x,y)$ by Algorithm 3.1, and $R_4(x,y)$ and $R_6(x,y)$ by means of Algorithm 3.2. And then we plot the corresponding figures in Figure 2.2 to 2.5, respectively.

VCs	f_0	f_1	f_2	f_3	f_4	f_5
Values	1.576055	1.311592	1.432173	1.531926	1.311592	1.718282

Table 5 The vertical coordinates (VCs) f_i's in Example 4.2

c_0	c_1	c_2	c_3	c_4	c_5
1.576055	−1.890620	−0.062559	−2.142556	0.717394	−2.705955

Table 6 The coefficients of $R_i(x, y)$'s in Example 4.2

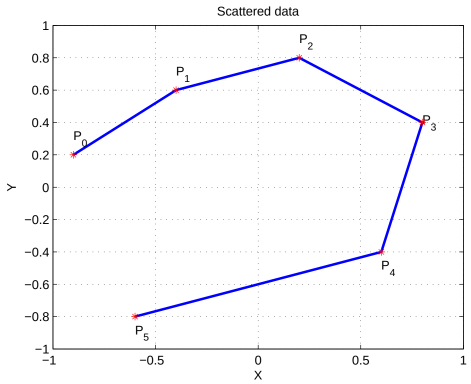

Figure 2.1 Scattered data Π_6

$$
\begin{aligned}
P_3(x,y) =& 0.630442y - 0.377770x + 1.576055xy + 0.004022,\\
Q_3(x,y) =& xy + 0.4y - 0.2x + 0.038276,\\
R_3(x,y) =& \frac{P_3(x,y)}{Q_3(x,y)},\\
P_4(x,y) =& 2.177227x - 0.934770y - 5.656508xy + x^2y - 0.600000x^2 - 0.258183,\\
Q_4(x,y) =& 1.562883x - 0.478898y - 4.033176xy - 0.308883,\\
R_4(x,y) =& \frac{P_4(x,y)}{Q_4(x,y)},\\
P_5(x,y) =& 1.576055x^2y^2 - 0.921220x^2y - 0.128220x^2 - 0.630422xy^2 -\\
& 3.247370xy + 1.316940x - 0.504338y^2 - 0.270346y - 0.182645,\\
Q_5(x,y) =& x^2y^2 - 1.0x^2y + 0.16x^2 - 0.400000xy^2 - 2.375101xy + 0.962582x -\\
& 0.320000y^2 - 0.118180y - 0.197094,\\
R_5(x,y) =& \frac{P_5(x,y)}{Q_5(x,y)},\\
P_6(x,y) =& 2.309720x^3y^2 - 2.309720x^3y + 0.554333x^3 - 24.301130x^2y^2 +\\
& 17.398228x^2y - 1.542739x^2 + 9.620051xy^2 + 14.410494xy -\\
& 6.785431x + 4.447545y^2 + 1.529287y + 0.998413,\\
Q_6(x,y) =& -15.565510x^2y^2 + 13.586028x^2y - 2.443929x^2 + 6.983185xy^2 +\\
& 10.171779xy - 4.864409x + 2.663673y^2 + 0.901214y + 1.060614,
\end{aligned}
$$

$$R_6(x,y) = \frac{P_6(x,y)}{Q_6(x,y)}.$$

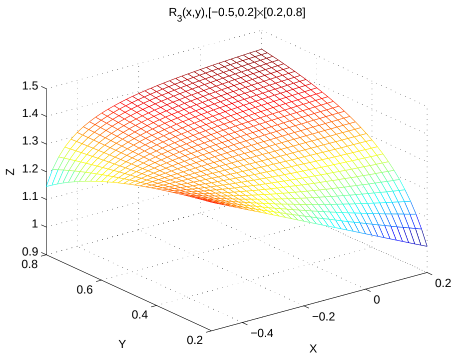

Figure 2.2 Part of the interpolating continued fraction $R_3(x,y)$ over Π_3

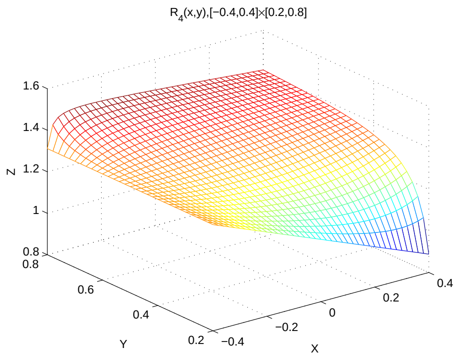

Figure 2.3 Part of the interpolating continued fraction $R_3(x,y)$ over Π_4

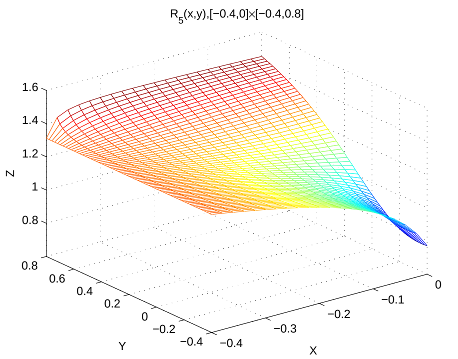

Figure 2.4 Part of the interpolating continued fraction $R_4(x,y)$ over Π_5

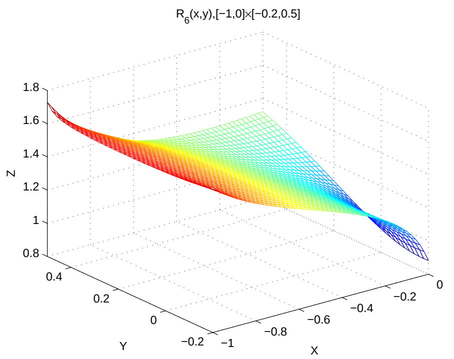

Figure 2.5 Part of the interpolating continued fraction $R_3(x,y)$ over Π_6

Example 4.3 Consider 33 points (x_i, y_i, z_i), $i = 0, 1, \ldots, 32$, where

$$x_i = -10 + \frac{20}{33}(i+1),\ \ y_i = \sin(i+1)\cdot x_i,\ \ z_i = y_i + \sin\sqrt{x_i^2 + y_i^2}\quad i = 0, 1, \ldots, 32.$$

Then we plot these interpolating nodes in Figure 3.1, and obtain the representation of the rational interpolating function as

$$\begin{aligned} R_{2n+1}(x,y) =& c_0 + \frac{x - x_0\ \ |}{|\ \ c_1} + \frac{(y-y_0)(x-x_1)\ |}{|\ \ c_2} + \frac{(y-y_1)(x-x_2)\ |}{|\ \ c_3} + \cdots + \\ & \frac{(y-y_{29})(x-x_{30})\ |}{|\ \ c_{31}} + \frac{(y-y_{30})(x-x_{31})\ |}{|\ \ c_{32}}. \end{aligned}$$

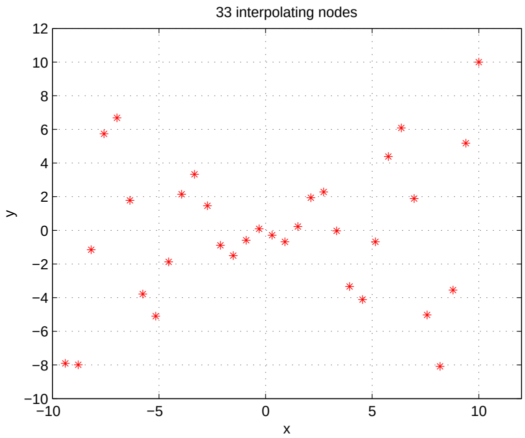

Figure 3.1 Scattered data Π_{33}

For the sake of illustration and convenience, we list the coordinates of the 33 interpolating nodes in Tables 7, 8 and 9, respectively.

x_0	x_1	x_2	x_3	x_4	x_5	x_6
−9.3939	−8.7879	−8.1818	−7.5758	−6.9697	−6.3636	−5.7576
x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
−5.1515	−4.5455	−3.9394	−3.3333	−2.7273	−2.1212	−1.5152
x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}
−0.9091	−0.3030	0.3030	0.9091	1.5152	2.1212	2.7273
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}
3.3333	3.9394	4.5455	5.1515	5.7576	6.3636	6.9697
x_{28}	x_{29}	x_{30}	x_{31}	x_{32}		
7.5758	8.1818	8.7879	9.3939	10.0000		

Table 7 The 33 x_i's in Example 4.3

y_0	y_1	y_2	y_3	y_4	y_5	y_6
−7.9047	−7.9908	−1.1546	5.7334	6.6834	1.7781	−3.7827
y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	y_{13}
−5.0967	−1.8733	2.1431	3.3333	1.4634	−0.8913	−1.5009
y_{14}	y_{15}	y_{16}	y_{17}	y_{18}	y_{19}	y_{20}
−0.5912	0.0872	−0.2913	−0.6827	0.2271	1.9366	2.2818
y_{21}	y_{22}	y_{23}	y_{24}	y_{25}	y_{26}	y_{27}
−0.0295	−3.3336	−4.1163	−0.6818	4.3905	6.0860	1.8881
y_{28}	y_{29}	y_{30}	y_{31}	y_{32}		
−5.0275	−8.0839	−3.5506	5.1801	9.9991		

Table 8 The 33 y_i's in Example 4.3

z_0	z_1	z_2	z_3	z_4	z_5	z_6
−8.1898	−8.6263	−0.2371	5.6575	6.4539	2.0966	−3.2132
z_7	z_8	z_9	z_{10}	z_{11}	z_{12}	z_{13}
−4.2755	−2.8525	1.1689	2.3333	1.5099	−0.1461	−0.6547
z_{14}	z_{15}	z_{16}	z_{17}	z_{18}	z_{19}	z_{20}
0.2929	0.3974	0.1168	0.2246	1.2263	2.2027	1.8792
z_{21}	z_{22}	z_{23}	z_{24}	z_{25}	z_{26}	z_{27}
−0.2202	−4.2348	−4.2666	−1.5669	5.2082	6.6665	2.6944
z_{28}	z_{29}	z_{30}	z_{31}	z_{32}		
−4.7011	−8.9585	−3.6039	4.2158	10.9991		

Table 9 The 33 z_i's in Example 4.3

And we also list the 33 interpolating coefficients in Table 10, which are calculated by means of Algorithm 2.1.

c_0	c_1	c_2	c_3	c_4	c_5	c_6
−8.1898	0.0625	16.7667	−0.4023	24.1140	0.3717	−52.5315
c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}
−0.8585	13.8632	5.3770	−3.2423	−1.7277	7.0914	4.8380
c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}
−0.1681	−6.8490	−0.4013	−3.3552	0.2729	4.4127	−23.0042
c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}
−0.1554	26.3680	321.7954	−0.0150	−360.5940	0.4162	8.5241
c_{28}	c_{29}	c_{30}	c_{31}	c_{32}		
−0.4977	168.3309	0.0034	−151.2422	4.4141		

Table 10 The 33 interpolating coefficients in Example 4.3

5. Complexity of the bivariate rational function interpolation

Now that we have learned the "how" of the bivariate non-tensor-product-typed continued fraction interpolation, here are a few words about "why." Since classic method for scattered data interpolation involves radial basis functions, we shall make analysis of the complexity of the rational interpolation and that of radial basis function interpolation.

As we know, every natural cubic spline $s(x)$ has a representation of the form [17]

$$s(x) = \sum_{j=1}^{N} \alpha_j \phi(|x - x_j|) + p(x), \quad x \in \mathbb{R}, \tag{5.1}$$

where $\phi(r) = r^3, r \geq 0$, and $p(x) \in \mathbb{P}_1(\mathbb{R})$, i.e., a univariate polynomial with degree one.

From the first result on radial basis functions, the resulting interpolation is up to a low-degree polynomial a linear combination of shifts of a radial function $\Phi = \phi(|\cdot|)$. Consequently, experts have generalized the idea to construct scattered data interpolation of the form over the interpolating nodes $\Pi = \{\mathbf{x}_0, \dots, \mathbf{x}_N\} \subset \mathbb{R}^2$

$$s(\mathbf{x}) = \sum_{j=0}^{N} \alpha_j \phi(\|\mathbf{x} - \mathbf{x}_j\|_2) + p(x), \quad \mathbf{x} = (x, y), \mathbf{x}_j = (x_j, y_j) \in \mathbb{R}^2, \tag{5.2}$$

where $\phi : [0, +\infty] \to \mathbb{R}$ is a univariate fixed function and $p(\mathbf{x}) \in \mathbb{P}_{m-1}(\mathbb{R}^2)$, i.e., a bivariate polynomial with degree $m-1$. And the additional conditions on the coefficients satisfy

$$\sum_{j=0}^{N} \alpha_j q(\mathbf{x}_j) = 0, \quad \forall q(\mathbf{x}) \in \mathbb{P}_{m-1}(\mathbb{R}^2). \tag{5.3}$$

In the particular case, one only considers the interpolation problems without the additional conditions (5.3), which boils down to the question whether the matrix $A_{\phi,\Pi} = (\phi(\|\mathbf{x}_k - \mathbf{x}_j\|_2))_{0 \le k,j \le N}$ is nonsingular. Astonishingly, there exist nonsingular matrix $A_{\phi,\Pi}$ in the case of the Gaussians $\phi(r) = e^{-\alpha r^2}$ $(\alpha > 0)$, the inverse multiquadric $\phi(r) = 1/\sqrt{c^2 + r^2}$, and the multiquadric $\phi(r) = \sqrt{c^2 + r^2}$ $(c > 0)$.

Hence, we can obtain the α_i's by solving the system of the linear equations.

$$s(\mathbf{x}_k) = \sum_{j=0}^{N} \alpha_j \phi(\|\mathbf{x}_k - \mathbf{x}_j\|_2), \quad k = 0, 1, \dots, N. \tag{5.4}$$

One can make a total count of operations for the elimination step of Gaussian elimination [18]. The elimination of each entry in the matrix $A_{\phi,\Pi}$ requires the operations of additions/substractions, multiplications, and divisions, which are $\frac{2}{3}(N+1)^3 + \frac{1}{2}(N+1)^2 - \frac{7}{6}(N+1)$, i.e., $O(N^3)$. Then, because of the triangular shape of the nonzero coefficients of the equations, one can start at the bottom and work your way up to the top equation. Counting operations of solving the system of the linear equations (5.4) yields $(N+1)^2$.

To be mentioned, however, the approximate number of operations with the rational interpolation presented in the paper is $O(N^2)$, which is smaller than that with radial basis function interpolation. Actually, we shall make an analysis of the operation count from two aspects.

On the one hand, based on $\Pi_{N+1} = \{\mathbf{x}_0, \dots, \mathbf{x}_N\}$, the computation of the bivariate non-tensor-product-typed partially inverse divided difference $\phi_{0\cdots,N}$ will require $\frac{5}{2}N^2 + \frac{1}{2}N$ $(N \ge 1)$ operations, including additions/substractions, multiplications, and divisions.

In fact, we can prove it by induction. By Definition 2.1, Algorithms 2.1 and 2.2, for ϕ_{01}, the numbers of the operations of additions/substractions, multiplications, and divisions are 2,0,1, respectively. Then for ϕ_{012}, the numbers of them are $2\times 2+3$, 1, $2+1$, respectively. Moreover, for ϕ_{0123}, the numbers of them are $2\times 3 + 3\times 2 + 3$, $2+1$, $3+2+1$, respectively. Furthermore, for $\phi_{0\cdots 4}$, they are $2\times 4 + 3\times 3 + 3\times 2 + 3$, $3+2+1$, $4+3+2+1$, respectively. Hence, by induction, for $\phi_{0\cdots,N}$, they are

$$2N + 3(N-1) + 3(N-2) + \cdots + 3\times 2 + 3\times 1 = \frac{3}{2}N^2 + \frac{1}{2}N,$$

$$(N-1) + (N-2) + \cdots + 2 + 1 = \frac{1}{2}N(N-1),$$

$$N + (N-1) + \cdots + 2 + 1 = \frac{1}{2}N(N+1),$$

respectively, which is totaled up as $\frac{5}{2}N^2 + \frac{1}{2}N$ $(N \geq 1)$.

On the other hand, similarly, for the rational interpolating function $R_{N+1}(x,y)$ over $\Pi_{N+1} = \{\mathbf{x}_0, \ldots, \mathbf{x}_N\}$, the numbers of operations of calculating $P_{N+1}(x,y)$ and $Q_{N+1}(x,y)$ are $3(N-1)+2$ and $3(N-1)+1$ based on Algorithms 3.1 and 3.2, respectively.

Consequently, we approximate a total count of operations for the computation of $R_{N+1}(x,y)$ as $O(N^2)$, which is smaller than that of radial basis function interpolation as $O(N^3)$.

Appendix: Programming procedure of calculating the results in Table 3, 4 and plotting Figure 1.5

```
clc
(i) To give the vertical coordinates f_i's in Table 3
    x=[-7,-5,-3,0.2,4,7.8];
    y=[-9.5,-4,-2,-1,2,8];
for i=1:6
    f(i)=sin(sqrt(x(i)∧2+y(i)∧2))/sqrt(x(i)∧2+y(i)∧2);
end
    f=vpa(f)
(ii) To calculate the coefficients in Table 4
for i=1:5
    phi1(i)=(x(i+1)-x(1))/(f(i+1)-f(1));
end
    phi1;
for i=1:4
    phi2(i)=(y(i+2)-y(1))*(x(i+2)-x(2))/(phi1(i+1)-phi1(1));
end
    phi2;
for i=1:3
    phi3(i)=(y(i+3)-y(2))*(x(i+3)-x(3))/(phi2(i+1)-phi2(1));
end
    phi3;
for i=1:2
     phi4(i)=(y(i+4)-y(3))*(x(i+4)-x(4))/(phi3(i+1)-phi3(1));
end
    phi4;
    phi5=(y(6)-y(4))*(x(6)-x(5))/(phi4(2)-phi4(1));
    c=[f(1),phi1(1),phi2(1),phi3(1),phi4(1),phi5];
    c=vpa(c)
```

(iii) To Simplify the representation of the bivariate interpolating continued fraction $R_6(x,y)$

```
syms u v
    ff5=c(5)+(v-y(4))*(u-x(5))/c(6);
    ff4=c(4)+(v-y(3))*(u-x(4))/ff5;
    ff3=c(3)+(v-y(2))*(u-x(3))/ff4;
    ff2=c(2)+(v-y(1))*(u-x(2))/ff3;
    ff1=c(1)+(u-x(1))/ff2;
  simplify(ff1);
```

(iv) To test the interpolating property of the rational function $R_6(x,y)$ at the interpolating nodes

```
for i=1:6
P6(i)=3.155002*x(i)∧3*y(i)∧2+15.775009*x(i)∧3*y(i)+12.620007*x(i)∧3-14.460938*x(i)∧2*y(i)∧2-
...
     524.7681670*x(i)∧2*y(i)-1391.685375*x(i)∧2-195.377658*x(i)*y(i)∧2-3927.845914*x(i)*y(i)-
...
      10520.412261*x(i)-193.579579*y(i)∧2-4029.058999*y(i)-15060.600628;
  Q6(i)=568.401784*x(i)∧2*y(i)∧2+6020.017403*x(i)∧2*y(i)+9701.615619*x(i)∧2+...
     2305.419482*x(i)*y(i)∧2+20944.892838*x(i)*y(i)+29135.470886*x(i)-1216.106476*y(i)∧2-
...
      17996.985482*y(i)-34153.093667;
  R6(i)=P6(i)/Q6(i);
  error6(i)=f(i)-R6(i);
end
   R6=vpa(R6)
   error6
```

(v) To plot the figure of $R_6(x,y)$, $[2,8]\times[-1,8]$ as shown in Fig. 1.5

```
[s,t]=meshgrid(2:0.2:8,-1:0.2:8);
 polynomialP6=3.155002*s.∧3.*t.∧2+15.775009*s.∧3.*t+12.620007*s.∧3-14.460938*s.∧2.*t.∧2-
...
   524.7681670*s.∧2.*t-1391.685375*s.∧2-195.377658*s.*t.∧2-3927.845914*s.*t-...
   10520.412261*s-193.579579*t.∧2-4029.058999*t-15060.600628;
 polynomialQ6=568.401784*s.∧2.*t.∧2+6020.017403*s.∧2.*t+9701.615619*s.∧2+...
   2305.419482*s.*t.∧2+20944.892838*s.*t+29135.470886*s-1216.106476*t.∧2-...
   17996.985482*t-34153.093667;
   CFR6=polynomialP6./polynomialQ6;
   mesh(s,t,CFR6)
   title('R_6(x,y),[2,8]×[−1,8]')
hold on
   axis([2,8,-1,8])
   xlabel('X')
   ylabel('Y')
```

zlabel('Z')

Acknowledgements The authors would like to thank the editors and reviewers for their valuable suggestions to improve the quality and readability of the paper.

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