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The Nullity of Bicyclic Graphs in Terms of Their Matching Number

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Abstract Let G be a graph with $n(G)$ vertices and $m(G)$ be its matching number. The nullity of G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero of adjacency matrix of G. It is well known that if G is a tree, then $\eta(G) = n(G) - 2m(G)$. Guo et al. [Jiming GUO, Weigen YAN, Yeongnan YEH. On the nullity and the matching number of unicyclic graphs. Linear Alg. Appl., 2009, 431: 1293–1301] proved that if G is a unicyclic graph, then $\eta(G)$ equals $n(G)-2m(G)-1$, $n(G)-2m(G)$, or $n(G)-2m(G)+2$. In this paper, we prove that if G is a bicyclic graph, then $\eta(G)$ equals $n(G) - 2m(G)$, $n(G) - 2m(G) \pm 1$, $n(G) - 2m(G) \pm 2$ or $n(G) - 2m(G) + 4$. We also give a characterization of these six types of bicyclic graphs corresponding to each nullity.

Keywords nullity; bicyclic graphs; matching number

MR(2010) Subject Classification 05C50

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. A matching in G is a set of pairwise nonadjacent edges. If M is a matching, each vertex incident with an edge of M is said to be covered by M. A perfect matching is the one which covers every vertex of G, and a maximum matching is the one which covers as many vertices as possible. We denote by $m(G)$ the matching number of G, i.e., the number of edges in a maximum matching of G. The adjacency matrix of G, denoted by $A(G) = (a_{ij})_{n\times n}$, is the $n \times n$ matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise, $i, j = 1, \ldots, n$. Clearly, $A(G)$ is a real symmetric matrix and all the eigenvalues of $A(G)$ are real. The multiplicity of the eigenvalue zero of $A(G)$ is called the nullity of G, which is denoted by $\eta(G)$. The graph G is called singular (or nonsingular) if $\eta(G) > 0$ (or $\eta(G = 0)$. Let $\mathcal{G}_n$ be the set of all graphs of order n, and let $[0, n] = \{0, 1, 2, \ldots, n\}$. A subset N of $[0, n]$ is said to be the nullity set of $\mathcal{G}_n$ provided that for any $k \in N$, there exists at least one graph $G \in \mathcal{G}_n$ such that $\eta(G) = k$.

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Collatz and Sinogowitz [1] first posed the problem of characterizing all singular or non-singular graphs. But, this problem has not been completely solved so far. At present, only some particular cases are known. On the other hand, this problem has very strong chemical background. Longuet-Higgins [2] pointed out that the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternate hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The problem is also of interest in mathematics itself, as it is closely related to the minimum rank problem of symmetric matrices whose patterns are described by graphs. The topics on the nullity of graphs include the computing nullity, the nullity distribution, bounds on nullity, characterization of a graph with special nullity, graph structure features reflected in nullity, and so on [3–19]. Among them, the study of the relationship between the nullity with other parameters of graphs, such as the number of vertices, size, matching number and maximum degree, is also an interesting problem. The following result gives a concise formula for the nullity of a tree T in terms of the matching number of T:

Theorem 1.1 ([3]) *Suppose T is a tree with n vertices and $m(T)$ is the matching number of T. Then $\eta(T) = n - 2m(T)$.*

The cycle and the path with n vertices are denoted by C_n and P_n, respectively. A unicyclic graph is a simple connected graph with equal number of vertices and edges. Guo et al. [4] investigated the nullity of the unicyclic graphs in terms of their matching number and proved the following interesting results:

Theorem 1.2 ([4]) *Suppose G is a unicyclic graph with n vertices and the unique cycle in G is C_l. Let E_1 be the set of edges of G between C_l and $G - C_l$ and E_2 the set of matchings of G with $m(G)$ edges. Then*

(1) $\eta(G) = n - 2m(G) - 1$ if $m(G) = \frac{l-1}{2} + m(G - C_l)$;

(2) $\eta(G) = n - 2m(G) + 2$ if G satisfies properties: $m(G) = \frac{l}{2} + m(G - C_l)$, $l = 0 \pmod 4$ and $E_1 \bigcap M = \emptyset$ for all $M \in E_2$;

(3) $\eta(G) = n - 2m(G)$ otherwise.

For convenience, hereafter in this paper we denote a unicyclic graph with nullity $n-2m(G)-1$, $n - 2m(G)$, $n - 2m(G) + 2$ by U_1, U_2, U_3-unicyclic graph, respectively. Motivated by above results on the nullity of graphs, in this paper, we prove that if G is a bicyclic graph, then $\eta(G)$ equals $n(G) - 2m(G)$, $n(G) - 2m(G) \pm 1$, $n(G) - 2m(G) \pm 2$ or $n(G) - 2m(G) + 4$. We also give a characterization of these six types of bicyclic graphs corresponding to each nullity.

2. Preliminaries

The following results are often cited in works related to nullity of graphs, and also play important roles in this paper.

Lemma 2.1 ([3]) *Let $G = G_1 \bigcup G_2 \cdots \bigcup G_t$, where $G_1, G_2, \ldots, G_t$ are connected components* *of* G*. Then*

$$\eta(G) = \sum_{i=1}^{t} \eta(G_i).$$

Lemma 2.2 ([3]) *Let* G *be a graph containing a pendant vertex, and let* H *be the induced subgraph of* G *obtained by deleting the pendant vertex together with the vertex adjacent to it. Then* $\eta(G) = \eta(H)$*.*

Lemma 2.3 ([3]) *A path with four vertices of degree* 2 *in a graph* G *can be replaced by an edge (see Figure 1) without changing the nullity of* G*.*

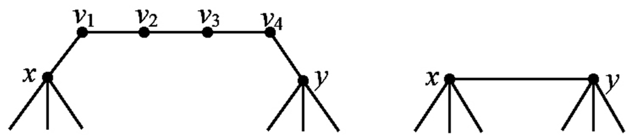

Figure 1 A path $xv_1v_2v_3v_4y$ is replaced by an edge xy

For a tree T on at least two vertices, a vertex $v \in V(T)$ is called mismatched vertex in T if there exists a maximum matching M of T that does not cover v; otherwise, v is called matched vertex in T. If a tree consists of only one vertex, then this vertex is considered mismatched vertex. $T(u) \odot^k G$ is a graph obtained from $T \bigcup G$ by joining u and arbitrary k vertices of G. In [5], the nullity of $T(u) \odot^k G$ was studied, as follows.

Lemma 2.4 ([5]) *Let* T *be a tree with a matched vertex* u *and let* G *be a graph of order* n*. Then for each integer* k $(1 \le k \le n)$*,*

$$\eta(T(u) \odot^k G) = \eta(T) + \eta(G).$$

Lemma 2.5 ([5]) *Let* T *be a tree with a mismatched vertex* u *and let* G *be a graph of order* n*. Then for each integer* k $(1 \le k \le n)$*,*

$$\eta(T(u) \odot^k G) = \eta(T - u) + \eta(G + u),$$

where $G + u$ *is the subgraph of* $T(u) \odot^k G$ *induced by* $V(G)$ *and* u*.*

For the matching number of $T(u) \odot^k G$, we have similar results, as follows.

Lemma 2.6 *Let* T *be a tree with a matched vertex* u *and let* G *be a graph of order* n*. Then for each integer* k $(1 \le k \le n)$*,*

$$m(T(u) \odot^k G) = m(T) + m(G).$$

Proof Let M be a maximum matching in $T(u) \odot^k G$. Then $|M| = |M \bigcap E(T)| + |M \bigcap E(G+u)|$. Clearly, $|M| \ge m(T) + m(G)$.

If M contains no edge which joins u and one of the vertices of G. Then we have $|M| = m(T) + m(G)$.

Now suppose that M contains an edge which joins u and one of the vertices of G. Since u is a matched vertex in T, $m(T-u)=m(T)-1$. Moreover, we have $|M\bigcap E(T)|\le m(T-u)=m(T)-1$. Therefore,

$$|M|=|M\bigcap E(T)|+|M\bigcap E(G+u)|\le m(T)-1+m(G+u)\le m(T)+m(G),$$

where the last inequality follows from $m(G+u)\le m(G)+1$. Then $|M|=m(T)+m(G)$. □

Lemma 2.7 *Let T be a tree with a mismatched vertex u and let G be a graph of order n. Then for each integer k $(1\le k\le n)$, $m(T(u)\odot^k G)=m(T-u)+m(G+u)$.*

Proof Since u is a mismatched vertex in T, then for its any neighbor v in T, v is a matched vertex in T, and it is also a matched vertex in the component of $T-u$ that contains v. For each neighbor v of u in T, we put it into Lemma 2.6, then complete the proof. □

A bicyclic graph is a simple connected graph in which the number of edges equals the number of its vertices plus one. Let C_l and C_k be two vertex-disjoint cycles. Suppose that v_1 is a vertex of C_l and v_x is a vertex of C_k. Joining v_1 and v_x by a path $v_1v_2\cdots v_x$ of length $x-1$, where $x\ge 1$ and $x=1$ means identifying v_1 with v_x; the resulting graph is shown in Figure 2, denoted by $B(l,x,k)$. Let P_{l+1}, P_{x+1} and P_{k+1} be three vertex-disjoint paths, where $l,x,k\ge 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph is shown in Figure 2, denoted by $\theta(l,x,k)$. All bicyclic graphs consists of two shapes of graphs: first shape is B-shape bicyclic graphs each of which contains $B(l,x,k)$ as its vertex induced subgraph; second shape is θ-shape bicyclic graphs each of which contains $\theta(l,x,k)$ as its vertex induced subgraph.

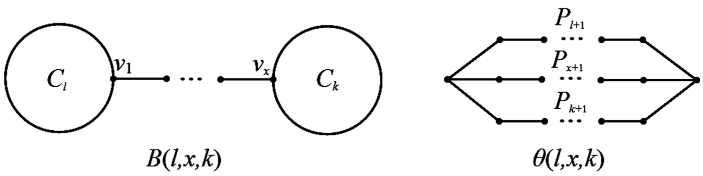

Figure 2 $B(l,x,k)$ and $\theta(l,x,k)$

To investigate the relation between nullity of bicyclic graph and its matching number, we first compute the nullity of $B(l,x,k)$ and $\theta(l,x,k)$ with respect to their matching number.

Lemma 2.8 *(1) $\eta(B(l,x,k))=n(B(l,x,k))-2m(B(l,x,k))-1$ if one of l,k is $2 \pmod 4$, the other one is odd, x is even or $l,x,k\equiv 1 \pmod 2$, $l\equiv k \pmod 4$;*

(2) $\eta(B(l,x,k))=n(B(l,x,k))-2m(B(l,x,k))+1$ if one of l,k is $0 \pmod 4$, the other one is odd, x is odd;

(3) $\eta(B(l,x,k))=n(B(l,x,k))-2m(B(l,x,k))+2$ if one of l,k is $0 \pmod 4$, the other one is $2 \pmod 4$, x is even or $l,k\equiv 0 \pmod 4$;

(4) $\eta(B(l,x,k))=n(B(l,x,k))-2m(B(l,x,k))$ otherwise.

Proof Denote $B(l,x,k)$ by B for short. Without loss of generality, we assume $l\le k$ in B. By

Lemma 2.3, it does not change the nullity of B if we replace a path with four vertices of degree 2 in B by an edge. By repeating the operation as many as possible, after a finite number of steps, we finally obtain a graph $B'(l,x,k)$ which is denoted by B', where $l,k=3,4,5,6$, $x=1,2,3,4,5$. Clearly, the number of such graph $B'(l,x,k)$ is 50.

Then we can calculate the nullity of B' by distinguishing the five cases in terms of x. When $x=1$, the nullity of $B'(l,1,k)$ shown in Table 1. can be calculated directly and represented by the formula $\eta(B'(l,1,k))=n(B'(l,1,k))-2m(B'(l,1,k))+a$, where a is a constant.

$\eta(B'(l,1,k))$	$k=3$	$k=4$	$k=5$	$k=6$
$l=3$	$0=5-2\times 2-1$	$1=6-2\times 3+1$	$1=7-2\times 3+0$	$0=8-2\times 4+0$
$l=4$		$3=7-2\times 3+2$	$1=8-2\times 4+1$	$1=9-2\times 4+0$
$l=5$			$0=9-2\times 4-1$	$0=10-2\times 5+0$
$l=6$				$1=11-2\times 5+0$

Table 1 Nullity of $B'(l,1,k)$ represented by $\eta(B'(l,1,k))=n(B'(l,1,k))-2m(B'(l,1,k))+a$

When $x=2,3,4$ or 5, we also can calculate the nullity of $B'(l,x,k)$ and represent in formula $\eta(B'(l,x,k))=n(B'(l,x,k))-2m(B'(l,x,k))+a$, in a similar way. From above representation of the nullity of B', we can find that the constant a is always -1, 0, 1 or 2. Therefore, 50 graphs of such B' shown in Table 2 can be divided into the following four groups corresponding to the value of a.

Formula of nullity	Graphs B' corresponding to the formula
$\eta(B')=n(B')-2m(B')-1$	$B'(3,1,3)$, $B'(3,3,3)$, $B'(3,5,3)$, $B'(3,2,6)$, $B'(3,4,6)$ $B'(5,1,5)$, $B'(5,3,5)$, $B'(5,5,5)$, $B'(5,2,6)$, $B'(5,4,6)$
$\eta(B')=n(B')-2m(B')+1$	$B'(3,1,4)$, $B'(3,3,4)$, $B'(3,5,4)$ $B'(4,1,5)$, $B'(4,3,5)$, $B'(4,5,5)$
$\eta(B')=n(B')-2m(B')+2$	$B'(4,1,4)$, $B'(4,3,4)$, $B'(4,5,4)$, $B'(4,2,4)$, $B'(4,4,4)$ $B'(4,2,6)$, $B'(4,4,6)$
$\eta(B')=n(B')-2m(B')$	$B'(3,1,5)$, $B'(3,3,5)$, $B'(3,5,5)$, $B'(3,2,3)$, $B'(3,4,3)$ $B'(3,1,6)$, $B'(3,3,6)$, $B'(3,5,6)$, $B'(3,2,4)$, $B'(3,4,4)$ $B'(4,1,6)$, $B'(4,3,6)$, $B'(4,5,6)$, $B'(3,2,5)$, $B'(3,4,5)$ $B'(5,1,6)$, $B'(5,3,6)$, $B'(5,5,6)$, $B'(4,2,5)$, $B'(4,4,5)$ $B'(6,1,6)$, $B'(6,3,6)$, $B'(6,5,6)$, $B'(5,2,5)$, $B'(5,4,5)$ $B'(6,2,6)$, $B'(6,4,6)$

Table 2 Formula of nullity of B' and the corresponding graphs

Next, we show that if B' is obtained from B by replacing a path with four vertices of degree 2 in B by an edge and $\eta(B') = n(B') - 2m(B') + a$, then $\eta(B) = n(B) - 2m(B) + a$, where $a = -1, 0, 1, 2$. In fact, suppose that B' is obtained from B by removing v_1, v_2, v_3, v_4 of a path $P = xv_1v_2v_3v_4y$ and then adding an edge xy, where v_i on P is the vertex of degree 2, $i = 1, 2, 3, 4$. Denote B' by $B - \{v_1, v_2, v_3, v_4\} + xy$. Then, we have $n(B) = n(B') + 4$ and $m(B) = m(B') + 2$. The first one is obvious. As for the second one, it is easy to see that any maximum matching of B contains two or three nonadjacent edges in P. Let M be a maximum matching of B. If M contains two nonadjacent edges of P, then one of x and y must be covered by M in $B - \{v_1, v_2, v_3, v_4\}$. Thus $xy \in E(B')$ is not in any maximum matching of B'. If M contains three nonadjacent edges of P, i.e., $\{xv_1, v_2v_3, v_4y\} \subset M$. Then both x and y are not covered by M in $B - \{v_1, v_2, v_3, v_4\}$. Hence, $xy \in E(B')$ is in every maximum matching of B'. Therefore, $\eta(B') = n(B') - 2m(B') + a = [n(B) - 4] - 2[m(B) - 2] + a = n(B) - 2m(B) + a$. By Lemma 2.3, we know that $\eta(B') = \eta(B)$. Thus the result follows immediately from above. Combining with above classification of the nullity (Table 2), the lemma thus follows. $\square$

Lemma 2.9 *(1) $\eta(\theta(l,x,k)) = n(\theta(l,x,k)) - 2m(\theta(l,x,k)) - 1$ if two of l, x, k are odd and they congruence with modulus 4;*

(2) $\eta(\theta(l,x,k)) = n(\theta(l,x,k)) - 2m(\theta(l,x,k)) + 1$ if two of l, x, k are even and they congruence with modulus 4;

(3) $\eta(\theta(l,x,k)) = n(\theta(l,x,k)) - 2m(\theta(l,x,k)) + 2$ if $l, x, k \equiv 0 \pmod 2$, $l \equiv x \equiv k \pmod 4$;

(4) $\eta(\theta(l,x,k)) = n(\theta(l,x,k)) - 2m(\theta(l,x,k))$ otherwise.

Proof Denote $\theta(l,x,k)$ by θ for short. Without loss of generality, we assume $l \le x \le k$ in θ. Similar to the proof in Lemma 2.8, we finally obtain a graph $\theta'(l,x,k)$ denoted by θ' from θ, after as many steps as possible replacing a path with four vertices of degree 2 in θ by an edge, where $l, x, k = 1, 2, 3, 4$. We have the formula of nullity of θ' shown in Table 3 and divide these 16 graphs of such θ' into the following four groups.

Formula of nullity	Graphs θ' corresponding to the formula
$\eta(\theta') = n(\theta') - 2m(\theta') - 1$	$\theta'(2,3,3)$, $\theta'(3,3,4)$
$\eta(\theta') = n(\theta') - 2m(\theta') + 1$	$\theta'(1,2,2)$, $\theta'(2,2,3)$, $\theta'(3,4,4)$ $\theta'(1,4,4)$
$\eta(\theta') = n(\theta') - 2m(\theta') + 2$	$\theta'(2,2,2)$, $\theta'(4,4,4)$
$\eta(\theta') = n(\theta') - 2m(\theta')$	$\theta'(1,2,3)$, $\theta'(2,2,4)$, $\theta'(3,3,3)$ $\theta'(1,2,4)$, $\theta'(2,3,4)$ $\theta'(1,3,3)$, $\theta'(2,4,4)$ $\theta'(1,3,4)$

Table 3 Formula of nullity of θ' and the corresponding graphs

Similarly to the proof in Lemma 2.8, since the formula of nullity of θ keeps unchanged in each step, according to classification of the nullity (Table 3), the lemma thus follows. $\square$

3. Main results

In what follows we first introduce a classification of bicyclic graphs and then provide our main results with their complete proofs. A bicyclic graph can be regarded as the graph which is obtained from $B(l,x,k)$ (or $\theta(l,x,k)$) by attaching some trees to some vertices of $B(l,x,k)$ (or $\theta(l,x,k)$). For $v \in V(B(l,x,k))$ (or $v \in V(\theta(l,x,k))$), $G_B\{v\}$ (or $G_\theta\{v\}$) denote an induced connected subgraph of G with maximum possible of vertices, which contains the vertex v and contains no other vertices of $B(l,x,k)$ (or $\theta(l,x,k)$). We say that $G_B\{v\}$ (or $G_\theta\{v\}$) is a tree attached on v. Now according to whether v is a matched vertex in $G_B\{v\}$ (or $G_\theta\{v\}$) or not, we divide all bicyclic graphs of order n into three types:

Type I, denote by $\mathcal{B}_n^1$ (or θ_n^1) the set of those graphs each of which is B-shape bicyclic graph (or θ-shape bicyclic graph) and there exists a vertex v on $B(l,x,k)$ (or $\theta(l,x,k)$) which is a matched vertex in $G_B\{v\}$ (or $G_\theta\{v\}$) and $v \in V(C_l)\bigcup V(C_k)$ (or v is one endpoint of P_{l+1}, P_{x+1} and P_{k+1}).

Type II, denote by $\mathcal{B}_n^2$ (or θ_n^2) the set of those graphs each of which is B-shape bicyclic graph (or θ-shape bicyclic graph) and there exists a vertex on $B(l,x,k)$ (or $\theta(l,x,k)$) which is a matched vertex in $G_B\{v\}$ (or $G_\theta\{v\}$) but $v \notin V(C_l)\bigcup V(C_k)$ (or v is not endpoint of P_{l+1}, P_{x+1} and P_{k+1}).

Type III, denote by $\mathcal{B}_n^3$ (or θ_n^3) the set of those B-shape bicyclic graphs (or θ-shape bicyclic graphs) each of which belongs to neither Type I nor Type II.

Theorem 3.1 *Let G be a B-shape bicyclic graph and $B(l,x,k)$ as its induced subgraph. Then*

(1) $\eta(G) = n(G) - 2m(G) - 2$ if $G \in \mathcal{B}_n^2$ and $G - G_B\{v\}$ is disjoint union of two U_1-unicyclic graphs.

(2) $\eta(G) = n(G) - 2m(G) - 1$ if G satisfies one of following conditions:

(i) $G \in \mathcal{B}_n^1$ and $x \neq 1$ and $G - G_B\{v\}$ is disjoint union of a tree and U_1-unicyclic graph;

(ii) $G \in \mathcal{B}_n^2$ and $G - G_B\{v\}$ is disjoint union of U_1-unicyclic graph and U_2-unicyclic graph;

(iii) $G \in \mathcal{B}_n^3$ and $\eta(B(l,x,k)) = n(B(l,x,k)) - 2m(B(l,x,k)) - 1$.

(3) $\eta(G) = n(G) - 2m(G)$ if G satisfies one of following conditions:

(i) $G \in \mathcal{B}_n^1$ and $x = 1$;

(ii) $G \in \mathcal{B}_n^1$ $x \neq 1$ and $G - G_B\{v\}$ is disjoint union of a tree and U_2-unicyclic graph;

(iii) $G \in \mathcal{B}_n^2$ and $G - G_B\{v\}$ is disjoint union of two U_2-unicyclic graphs;

(iv) $G \in \mathcal{B}_n^3$ and $\eta(B(l,x,k)) = n(B(l,x,k)) - 2m(B(l,x,k))$.

(4) $\eta(G) = n(G) - 2m(G) + 1$ if G satisfies one of following conditions:

(i) $G \in \mathcal{B}_n^2$ and $G - G_B\{v\}$ is disjoint union of U_1-unicyclic graph and U_3-unicyclic graph;

(ii) $G \in \mathcal{B}_n^3$ and $\eta(B(l,x,k)) = n(B(l,x,k)) - 2m(B(l,x,k)) + 1$.

(5) $\eta(G) = n(G) - 2m(G) + 2$ if G satisfies one of following conditions:

(i) $G \in \mathcal{B}_n^1$ and $x \neq 1$ and $G - G_B\{v\}$ is disjoint union of a tree and U_3-unicyclic graph;

(ii) $G \in \mathcal{B}_n^2$ *and* $G - G_B\{v\}$ *is disjoint union of* U_2*-unicyclic graph and* U_3*-unicyclic graph;*

(iii) $G \in \mathcal{B}_n^3$ *and* $\eta(B(l,x,k)) = n(B(l,x,k)) - 2m(B(l,x,k)) + 2$.

(6) $\eta(G) = n(G) - 2m(G) + 4$ *if* $G \in \mathcal{B}_n^2$ *and* $G - G_B\{v\}$ *is disjoint union of two* U_3*-unicyclic graphs.*

Proof If G belongs to Type I, then there exists a vertex $v \in V(C_l) \bigcup V(C_k)$ on $B(l,x,k)$ such that v is a matched vertex in $G_B\{v\}$. Let $n(G_B\{v\}) = n_1$, $m(G_B\{v\}) = m_1$.

Case 1 $v \in V(C_l) \setminus v_1$ or $v \in V(C_k) \setminus v_x$.

Then $G - G_B\{v\}$ is a unicyclic graph. Clearly, $G = G_B\{v\}(v) \odot^2 (G - G_B\{v\})$. Assume that $G - G_B\{v\}$ is of order n_2 and its matching number is m_2. By Lemma 2.4, we have $\eta(G) = \eta(G_B\{v\}) + \eta(G - G_B\{v\})$. It follows from Lemmas 1.1, 1.2 and 2.6, we have

$$\begin{aligned}\eta(G) &= \eta(G_B\{v\}) + \eta(G - G_B\{v\}) \\ &= n_1 - 2m_1 + \begin{cases} n_2 - 2m_2 - 1, & G - G_B\{v\} \text{ is } U_1\text{-unicyclic graph;} \\ n_2 - 2m_2, & G - G_B\{v\} \text{ is } U_2\text{-unicyclic graph;} \\ n_2 - 2m_2 + 2, & G - G_B\{v\} \text{ is } U_3\text{-unicyclic graph,} \end{cases} \\ &= \begin{cases} n(G) - 2m(G) - 1; \\ n(G) - 2m(G); \\ n(G) - 2m(G) + 2. \end{cases}\end{aligned}$$

Case 2 $v = v_1$ or v_x.

If $v_1 \neq v_x$, then $G - G_B\{v\}$ has two components a tree and a unicyclic graph, say G_2 and G_3, respectively. Let $n(G_i) = n_i$ and $m(G_i) = m_i$, $i = 1, 2$. Clearly, $G = G_B\{v\}(v) \odot^3 (G - G_B\{v\})$. Similar to Case 1, we have

$$\begin{aligned}\eta(G) &= \eta(G_B\{v\}) + \eta(G - G_B\{v\}) \\ &= n_1 - 2m_1 + n_2 - 2m_2 + \begin{cases} n_3 - 2m_3 - 1, & G_3 \text{ is } U_1\text{-unicyclic graph;} \\ n_3 - 2m_3, & G_3 \text{ is } U_2\text{-unicyclic graph;} \\ n_3 - 2m_3 + 2, & G_3 \text{ is } U_3\text{-unicyclic graph,} \end{cases} \\ &= \begin{cases} n(G) - 2m(G) - 1; \\ n(G) - 2m(G); \\ n(G) - 2m(G) + 2. \end{cases}\end{aligned}$$

If $v_1 = v_x$, then $G - G_B\{v\}$ has two components which both are trees. Let n_2, n_3 be their vertex numbers, respectively, and m_2, m_3 be their corresponding matching numbers, respectively. Clearly, $G = G_B\{v\}(v) \odot^4 (G - G_B\{v\})$. Similarly to Case 1, we have

$$\eta(G) = \eta(G_B\{v\}) + \eta(G - G_B\{v\}) = n_1 - 2m_1 + n_2 - 2m_2 + n_3 - 2m_3 = n(G) - 2m(G).$$

If G belongs to Type II, then there exists a vertex $v \notin V(C_l) \bigcup V(C_k)$ on $B(l,x,k)$ such that v is a matched vertex in $G_B\{v\}$. Then $v \in V(P_x) \setminus \{v_1, v_x\}$, where $P_x = v_1 v_2 \cdots v_x$. Thus, $G - G_B\{v\}$ has two components, say G_2 and G_3, which both are unicyclic graphs. Let $n(G_i) = n_i$

and $m(G_i) = m_i$, $i = 1, 2$. Clearly, $G = G_B\{v\}(v) \odot^2 (G - G_B\{v\})$. By Lemmas 1.1 and 1.2, we have

$$\eta(G_B\{v\}) = n_1 - 2m_1,$$

$$\eta(G_1) = \begin{cases} n_2 - 2m_2 - 1, & G_2 \text{ is } U_1\text{-unicyclic graph;} \\ n_2 - 2m_2, & G_2 \text{ is } U_2\text{-unicyclic graph;} \\ n_2 - 2m_2 + 2, & G_2 \text{ is } U_3\text{-unicyclic graph,} \end{cases}$$

$$\eta(G_2) = \begin{cases} n_3 - 2m_3 - 1, & G_3 \text{ is } U_1\text{-unicyclic graph;} \\ n_3 - 2m_3, & G_3 \text{ is } U_2\text{-unicyclic graph;} \\ n_3 - 2m_3 + 2, & G_3 \text{ is } U_3\text{-unicyclic graph.} \end{cases}$$

Moreover,

$$\begin{aligned} \eta(G - G_B\{v\}) &= \eta(G_1) + \eta(G_2) \\ &= \begin{cases} (n_2+n_3) - 2(m_2+m_3) - 2, & \text{both } G_2 \text{ and } G_3 \text{ are } U_1\text{-unicyclic graphs;} \\ (n_2+n_3) - 2(m_2+m_3) - 1, & \text{one of } G_2, G_3 \text{ is } U_1\text{-unicyclic graph} \\ & \text{the other is } U_2\text{-unicyclic graph;} \\ (n_2+n_3) - 2(m_2+m_3), & \text{both } G_2 \text{ and } G_3 \text{ are } U_2\text{-unicyclic graph;} \\ (n_2+n_3) - 2(m_2+m_3) + 1, & \text{one of } G_2, G_3 \text{ is } U_1\text{-unicyclic graph} \\ & \text{the other is } U_3\text{-unicyclic graph;} \\ (n_2+n_3) - 2(m_2+m_3) + 2, & \text{one of } G_2, G_3 \text{ is } U_2\text{-unicyclic graph} \\ & \text{the other is } U_3\text{-unicyclic graph;} \\ (n_2+n_3) - 2(m_2+m_3) + 4, & \text{both } G_2 \text{ and } G_3 \text{ are } U_3\text{-unicyclic graph.} \end{cases} \end{aligned}$$

And then by Lemmas 2.4 and 2.6, we have

$$\eta(G) = \eta(G_B\{v\}) + \eta(G - G_B\{v\}) = \begin{cases} n(G) - 2m(G) - 2; \\ n(G) - 2m(G) - 1; \\ n(G) - 2m(G); \\ n(G) - 2m(G) + 1; \\ n(G) - 2m(G) + 2; \\ n(G) - 2m(G) + 4. \end{cases}$$

If G belongs to Type III, by Lemma 2.5, we have $\eta(G) = \eta(G - B(l,x,k)) + \eta(B(l,x,k))$. Let $n(B(l,x,k)) = n_1$, $m(B(l,x,k)) = m_1$, $n(G - B(l,x,k)) = n_2$, $m(G - B(l,x,k)) = m_2$. By Lemmas 1.1 and 2.8, we have

$$\begin{aligned} \eta(G) &= \eta(G - B(l,x,k)) + \eta(B(l,x,k)) = n_2 - 2m_2 + \eta(B(l,x,k)) \\ &= n_2 - 2m_2 + \begin{cases} n_1 - 2m_1 - 1; \\ n_1 - 2m_1; \\ n_1 - 2m_1 + 1; \\ n_1 - 2m_1 + 2. \end{cases} = \begin{cases} n(G) - 2m(G) - 1; \\ n(G) - 2m(G); \\ n(G) - 2m(G) + 1; \\ n(G) - 2m(G) + 2. \end{cases} \end{aligned}$$

By above discussion, the proof of Theorem 3.1 is completed. □

Theorem 3.2 *Let G be a θ-shape bicyclic graph and $\theta(l,x,k)$ as its an induced subgraph. Then*

(1) $\eta(G)=n(G)-2m(G)-1$ if G satisfies one of following conditions:

(i) $G\in\theta_n^2$ and $G-G_\theta\{v\}$ is U_1-unicyclic graph;

(ii) $G\in\theta_n^3$ and $\eta(\theta(l,x,k))=n(\theta(l,x,k))-2m(\theta(l,x,k))-1$.

(2) $\eta(G)=n(G)-2m(G)$ if G satisfies one of following conditions:

(i) $G\in\theta_n^1$;

(ii) $G\in\theta_n^2$ and $G-G_\theta\{v\}$ is U_2-unicyclic graph;

(iii) $G\in\theta_n^3$ and $\eta(\theta(l,x,k))=n(\theta(l,x,k))-2m(\theta(l,x,k))$.

(3) $\eta(G)=n(G)-2m(G)+1$ if $G\in\theta_n^3$ and $\eta(\theta(l,x,k))=n(\theta(l,x,k))-2m(\theta(l,x,k))+1$.

(4) $\eta(G)=n(G)-2m(G)+2$ if G satisfies one of following conditions:

(i) $G\in\theta_n^2$ and $G-G_\theta\{v\}$ is U_3-unicyclic graph;

(ii) $G\in\theta_n^3$ and $\eta(\theta(l,x,k))=n(\theta(l,x,k))-2m(\theta(l,x,k))+2$.

Proof If G belongs to Type I, then there exists a vertex v on $\theta(l,x,k)$ which is one endpoint of P_{l+1}, P_{x+1} and P_{k+1} such that v is a matched vertex in $G_\theta\{v\}$. Let $n(G_\theta\{v\})=n_1$, $m(G_\theta\{v\})=m_1$. Then $G-G_\theta\{v\}$ is a tree. Assume that the order of $G-G_\theta\{v\}$ is n_2 and its matching number is m_2. Clearly, $G=G_\theta\{v\}(v)\odot^3(G-G_\theta\{v\})$. Similarly to the proof in Theorem 3.1, we have

$$\eta(G)=\eta(G_\theta\{v\})+\eta(G-G_\theta\{v\})=n_1-2m_1+n_2-2m_2=n(G)-2m(G).$$

If G belongs to Type II, then there exists a vertex v on $\theta(l,x,k)$ which is not endpoint of P_{l+1}, P_{x+1} and P_{k+1} such that v is a matched vertex in $G_\theta\{v\}$. Then v is internal vertex of P_{l+1}, P_{x+1} or P_{k+1}. Therefore, $G-G_\theta\{v\}$ is a unicyclic graph. Clearly, $G=G_\theta\{v\}(v)\odot^2(G-G_\theta\{v\})$. Assume that order of $G-G_\theta\{v\}$ is n_2 and its matching number is m_2. Similarly, we have

$$\begin{aligned}\eta(G)&=\eta(G_\theta\{v\})+\eta(G-G_\theta\{v\})\\&=n_1-2m_1+\begin{cases}n_2-2m_2-1; & G-G_B\{v\}\text{ is }U_1\text{-unicyclic graph;}\\ n_2-2m_2; & G-G_B\{v\}\text{ is }U_2\text{-unicyclic graph;}\\ n_2-2m_2+2; & G-G_B\{v\}\text{ is }U_3\text{-unicyclic graph,}\end{cases}\\&=\begin{cases}n(G)-2m(G)-1;\\ n(G)-2m(G);\\ n(G)-2m(G)+2.\end{cases}\end{aligned}$$

If G belongs to Type III, by Lemma 2.5, we have $\eta(G)=\eta(G-\theta(l,x,k))+\eta(\theta(l,x,k))$. Let $n(\theta(l,x,k))=n_1$, $m(\theta(l,x,k))=m_1$, $n(G-\theta(l,x,k))=n_2$, $m(G-\theta(l,x,k))=m_2$. It follows from Lemmas 1.1, 2.7 and 2.9, we have

$$\eta(G)=\eta(G-\theta(l,x,k))+\eta(\theta(l,x,k))=n_2-2m_2+\eta(\theta(l,x,k))$$

$$= n_2 - 2m_2 + \begin{cases} n_1 - 2m_1 - 1; \\ n_1 - 2m_1; \\ n_1 - 2m_1 + 1; \\ n_1 - 2m_1 + 2. \end{cases} = \begin{cases} n(G) - 2m(G) - 1; \\ n(G) - 2m(G); \\ n(G) - 2m(G) + 1; \\ n(G) - 2m(G) + 2. \end{cases}$$

By above discussion, the proof of Theorem 3.2 is completed. □

Remark 3.3 From Theorems 3.1 and 3.2, we immediately have that the nullity set of bicyclic graphs with order n is $[0, n-4]$. In fact, when we want to characterize a bicyclic graph with certain nullity, it suffices to determine its matching number. Moreover, a bicyclic graph with extreme nullity can be given by Theorems 3.1 and 3.2.

Corollary 3.4 ([6]) *The nullity set of bicyclic graphs with order n ($n \geq 6$) is $[0, n-4]$.*

Wang and Wong [7] have proved that for every graph G, $|V(G)| - 2m(G) - c(G) \leq \eta(G) \leq |V(G)| - 2m(G) + 2c(G)$, where $c(G) = |E(G)| - |V(G)| + \omega(G)$, $\omega(G)$ is the number of connected components of G. By Theorem 1.1, we know that for a tree T, its nullity cannot be $|V(T)| - 2m(T) - 1$ while $2c(T) - 1 = -1$. By Theorem 1.2, there is no unicyclic graph G with nullity $|V(G)| - 2m(G) + 1$ while $2c(G) - 1 = 1$. By Theorems 3.1 and 3.2, there is no bicyclic graph G with nullity $|V(G)| - 2m(G) + 3$ while $2c(G) - 1 = 3$. Up to now, we have not found any graph G with nullity $|V(G)| - 2m(G) + 2c(G) - 1$. Then, at the end of this paper, we propose the following question:

Question: Is there any graph G with nullity $|V(G)| - 2m(G) + 2c(G) - 1$, where $c(G) = |E(G)| - |V(G)| + \omega(G)$, $\omega(G)$ is the number of connected components of G ?

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