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1-Planar Graphs with Girth at Least 7 are $(1,1,1,0)$-Colorable

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Abstract A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is shown that 1-planar graphs with girth at least 7 are $(1,1,1,0)$-colorable.

Keywords 1-planar; improper coloring; discharging; important 4-vertex

MR(2010) Subject Classification 05C15

1. Introduction

We consider only finite, simple and undirected graphs in this paper. Any undefined notation and terminology follows that of Bondy and Murty [1].

Let $d_1,\ldots,d_k$ be k nonnegative integers. A graph $G=(V,E)$ is called improperly $(d_1,\ldots,d_k)$-colorable, or just $(d_1,\ldots,d_k)$-colorable, if the vertex set V can be partitioned into subsets $V_1,\ldots,V_k$, such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1\le i\le k$. This notion generalizes those of proper k-coloring (when $d_1=\cdots=d_k=0$).

Improper coloring of planar graphs has been studied extensively. By Four-Color Theorem, every plane graph is $(0,0,0,0)$-colorable, but there exist non-$(1,1,1)$-colorable plane graphs [2]. Motivated by Steinberg's conjecture, many known results are obtained, for example, every planar graph with neither 4-cycles nor 5-cycles is $(1,1,1)$-colorable [3]. In [4–6], some results about $(1,1,0)$-coloring of planar graphs were given.

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar graphs was introduced by Ringel, and he conjectured that each 1-planar graph is 6-colorable [7]. It was confirmed by Borodin [8] in 1986, and in [9] a new simpler proof was given. Since there exists a 7-regular 1-planar graph, the bound 6 is sharp. Borodin et al. [10] also proved that each 1-planar graph is acyclically 20-colorable.

In this paper, we will show the following result.

Theorem 1.1 *1-Planar graphs with girth at least 7 are* $(1,1,1,0)$*-colorable.*

2. Preliminaries

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The vertex set, edge set, face set and minimum degree of a graph G are denoted by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. For a vertex $v \in V$, let $d(v)$ and $N(v)$ denote the degree and neighborhood of v in G, respectively. Call v a k-vertex, a k^+-vertex or a k^--vertex, if $d(v) = k$, $d(v) \geq k$ or $d(v) \leq k$, respectively. For a face $f \in F$, the number of edges of f, denoted by $d(f)$, is called the degree of f. The k-face, k^+-face and k^--face can be defined similarly. The girth of a graph is the length of a shortest cycle.

For any 1-planar graph G, we assume that G has been embedded on a plane such that every edge is crossed by at most one other edge. The associated plane graph G^* of a 1-planar graph G is the plane graph obtained from G by turning each crossing of G into a new 4-vertex, called a crossing vertex.

Some definitions of non-crossing vertex are as follows:

(1) Important 4-vertex: a 4-vertex which is incident with two 3-faces, one 4-face and one 6^+-face (Figure 1-1).

(2) Special 4-vertex: a 4-vertex which is incident with two 4-faces, one 3-face and one 5^+-face (Figure 1-2).

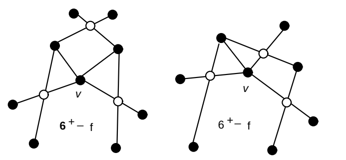


Figure 1-1
Important 4−vertex

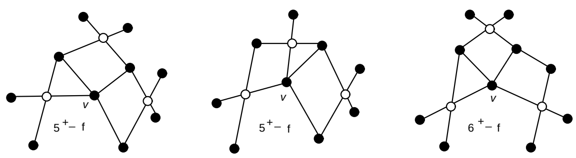


Figure 1-2
Special 4−vertex

We can get the following observation:

(1) Non-important 4-vertex which is incident with two 3-faces can be seen in Figure 1-3.

(2) Non-special 4-vertex which is incident with one 3-face can be seen in Figure 1-4.

(3) 5-vertex which is incident with three 3-faces can be seen in Figure 1-5.

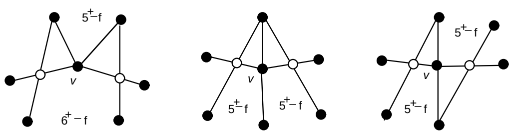


Figure 1-3
Non−important 4−vertex with two 3−faces

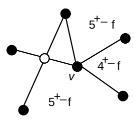


Figure 1-4
Non special 4−vertex
with with one 3−face

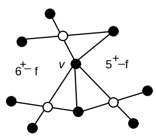


Figure 1-5
5−vertex with three 3−faces

(The white vertices represent crossing vertices in Figures 1-1 up to 1-5.)

3. Structural properties

In the sequel, let $c = \{1, 2, 3, 4\}$ denote the color set with four colors. The proof of Theorem 1.1 is by contradiction. Let G be a counterexample with the least number of vertices and edges which is a 1-planar graph and has no $(1, 1, 1, 0)$-coloring. Thus, G is connected. Moreover, every subgraph G' of G with fewer vertices and edges has a $(1,1,1,0)$ -coloring by using color set c. In other words, $V(G')$ is partitioned into four subsets V_1, V_2, V_3 and V_4, such that $\Delta(G[V_1]) \leq 1$, $\Delta(G[V_2]) \leq 1$, $\Delta(G[V_3]) \leq 1$ and $\Delta(G[V_4]) = 0$. As usual, to properly color a vertex v means to assign v a color such that v has no neighbor of that color. Now suppose that the vertices in $G[V_i]$ are colored with i $(i = 1,2,3,4)$.

Claim 1 The minimum degree $\delta(G)$ is at least 4.

Proof Suppose to the contrary that G contains a 3^--vertex v. Let $G' = G - v$. By the minimality of G, G' has a $(1,1,1,0)$-coloring φ by using color set c. We may easily extend φ to G by properly coloring v. This contradicts the choice of G, which is a contradiction. $\square$

Claim 2 Every 4-vertex is adjacent to at most one 4-vertex.

Proof Suppose to the contrary that a 4-vertex v is adjacent to two 4-vertices x and y. Let z and w denote other neighbors of v. Let $G' = G - \{v,x,y\}$. Clearly, G' is $(1,1,1,0)$ -colorable by the minimality of G. Let φ denote a $(1,1,1,0)$ -coloring of G' by using c. First, properly color x and y. If $\{\varphi(x), \varphi(y), \varphi(z), \varphi(w)\} \neq c$, we may color v with a color in $c \setminus \{\varphi(x), \varphi(y), \varphi(z), \varphi(w)\}$. Otherwise, we assign a color in $\{1,2,3\} \setminus \{\varphi(z), \varphi(w)\}$ to v. It is easy to check that in each case the obtained coloring of G is a $(1,1,1,0)$ -coloring, which is a contradiction. $\square$

Claim 3 ([11]) Let G be a 1-planar graph and G^* the associated plane graph of G. Then for any two crossing vertices u and v in G^*, $uv \notin E(G^*)$.

Since the girth of G is at least 7, we can easily get Claims 4 and 5.

Claim 4 Every v in G is incident with at most $(2 \times \lfloor \frac{d}{3} \rfloor + 1)$ 3-faces.

Claim 5 The graph G does not contain the following subgraphs (Figure 1-6), where white vertices represent crossing vertices.

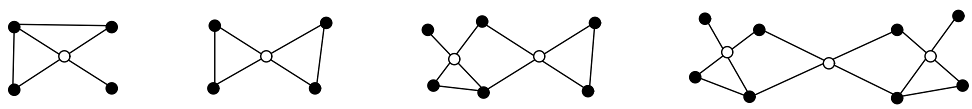


Figure 1-6

Four kinds of non−existence cycle

4. Proof of Theorem 1.1

Now we complete the proof of Theorem 1.1 by the discharging method. Define an initial charge μ on $V(G^*) \cup F(G^*)$ by letting $\mu(x) = d(x) - 4$, for every $x \in V(G^*) \cup F(G^*)$. Note that G^* is a planar graph, so by Euler's formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$ and the relation $\sum_{v \in V(G^*)} d(v) = \sum_{f \in F(G^*)} d(f) = 2|E(G^*)|$, we can easily deduce that

$$\sum_{v \in V(G^*)} (d(v) - 4) + \sum_{f \in F(G^*)} (d(f) - 4) = -8.$$

Since any discharging procedure preserves the total charge of G^*, we shall define a suitable discharging rules to change the initial charge μ to the final charge μ^* for every $x \in V(G^*) \cup F(G^*)$ such that

$$-8 = \sum_{x \in V(G^*) \cup F(G^*)} \mu(x) = \sum_{x \in V(G^*) \cup F(G^*)} \mu^*(x) \geq 0.$$

This will be a contradiction.

Our discharging rules are defined as follows.

R1 Every non-crossing vertex sends $\frac{1}{2}$ to every incident 3-face.

R2 Charge from a 5-face.

R2.1 Every 5-face sends $\frac{1}{2}$ to every incident 4-vertex which is incident with two 3-faces.

R2.2 Every 5-face sends $\frac{1}{2}$ to every incident special 4-vertex.

R2.3 Every 5-face sends $\frac{1}{4}$ to every incident 4-vertex which is incident with one 3-face.

R3 Charge from a 6^+-face.

R3.1 Every 6^+-face sends 1 to every incident important 4-vertex.

R3.2 Every 6^+-face sends $\frac{1}{2}$ to every incident special 4-vertex.

R3.3 Every 6^+-face sends $\frac{1}{2}$ to every incident 4-vertex which is incident with two 3-faces.

R3.4 Every 6^+-face sends $\frac{1}{2}$ to every incident 5-vertex which is incident with three 3-faces.

R3.5 Every 6^+-face sends $\frac{1}{4}$ to every incident 4-vertex which is incident with one 3-face.

In the following, we will prove that $\mu^*(x) \geq 0$ for all $x \in V(G^*) \cup F(G^*)$.

First we consider vertices.

By Claim 1, $\delta \geq 4$.

(1) $d(v) = 4$.

If v is a crossing vertex, then $\mu^*(v) = d(v) - 4 = 0$; If v is a non-crossing vertex, then it is incident with at most two 3-faces.

Case 1 If v is incident with two 3-faces, then

Case 1.1 v is an important 4-vertex: $\mu^*(v) = d(v) - 4 - 2 \times \frac{1}{2} + 1 = 0$ by R1, R3.1 (Figure 1-1).

Case 1.2 v is not an important 4-vertex: $\mu^*(v) = d(v) - 4 - 2 \times \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$ by R1, R2.1, R3.3 and observation(1) (Figure 1-3).

Case 2 If v is incident with one 3-face, then

Case 2.1 v is a special 4-vertex: $\mu^*(v) = d(v) - 4 - \frac{1}{2} + \frac{1}{2} = 0$ by R1, R2.2, R3.2 (Figure 1-2).

Case 2.2 v is not a special 4-vertex: $\mu^*(v) \geq d(v) - 4 - \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by R1, R2.3, R3.5 and observation(2) (Figure 1-4).

Case 3 If v is not incident with 3-faces, then $\mu^*(v) = d(v) - 4 = 0$.

(2) $d(v) = 5$.

By Claim 4, v is incident with at most three 3-faces.

Case 1 v is incident with three 3-faces. Then $\mu^*(v) = d(v) - 4 - 3 \times \frac{1}{2} + \frac{1}{2} = 0$ by R1, R3.4

and observation(3) (Figure 1-5).

Case 2 v is incident with at most two 3-faces. Then $\mu^*(v) \geq d(v) - 4 - 2 \times \frac{1}{2} = 0$ by R1.

(3) $d(v) \geq 6$.

By Claim 4 and G has no 3-cycles, v is incident with at most $\lfloor \frac{2d}{3} \rfloor$ 3-faces. Thus,

$$\mu^*(v) \geq d(v) - 4 - \frac{1}{2} \times \lfloor \frac{2d}{3} \rfloor \geq \frac{2d}{3} - 4 \geq 0$$

by R1.

Next we consider faces.

(1) $d(f) = 3$. Since G has no 3-cycles, it must be incident with a crossing vertex. Thus, $\mu^*(f) = d(f) - 4 + 2 \times \frac{1}{2} = 0$ by R1.

(2) $d(f) = 4$. $\mu^*(f) = d(f) - 4 = 0$.

(3) $d(f) = 5$. Since G has no 5-cycles, f is incident with at least one crossing vertex. Furthermore, by the definition of important 4-vertex, it is obvious that f has no important 4-vertex.

Case 1 f is incident with two crossing vertices. Then f is incident with at most one special 4-vertex, and one 4-vertex which is incident with two 3-faces. Furthermore, they cannot exist at the same time. In fact, let $v_1, \ldots, v_5$ be the five vertices of f, with edges $v_i v_{i+1}$ (i mod 5). By Claim 3, we may assume that v_1 and v_4 are crossing vertices. Suppose that v_5 is a special 4-vertex. There are two ways to place the 3-face and 4-face incident with v_5, but in any way it is easy to see that the face of $G^* \setminus v_5$ whose interior contains v_5 will correspond to a cycle of length at most 6 in G, a contradiction. So v_5 is not a special 4-vertex. By a similar argument, at most one of v_2, v_3 and v_5 can be a special vertex or a 4-vertex which is incident with two 3-faces. (Figure 2-1(a)(b) v is a special 4-vertex; (c)(d)(e)v' is a 4-vertex which is incident with two 3-faces.) Thus,

$$\mu^*(f) \geq d(f) - 4 - \frac{1}{2} - 2 \times \frac{1}{4} = 0$$

by R2.

Case 2 f is incident with one crossing vertex. By the definition of special 4-vertex, it is obvious that f has no special 4-vertex (Figure 2-2 (a)(b)). By Claim 2, f is incident with at most three 4-vertices. Moreover, if v and v' are 4-vertices which are incident with two 3-faces, then the other two vertices u and u' are 5-vertices (Figure 2-2 (c)). Thus,

$$\mu^*(f) = d(f) - 4 - 2 \times \frac{1}{2} = 0$$

or

$$\mu^*(f) \geq d(f) - 4 - \frac{1}{2} - 2 \times \frac{1}{4} = 0$$

by R2.1 and R2.3.

(4) $d(f) = 6$. Since G has no 6-cycles, f is incident with at least one crossing vertex.

Case 1 f is incident with three crossing vertices. Then f is incident with at most one important

4-vertex. If v is an important 4-vertex, then u and w cannot be special 4-vertex, 4-vertex which is incident with two 3-faces, and 5-vertex which is incident with three 3-faces (Figure 3-1). Thus,

$$\mu^*(f) \geq d(f) - 4 - 1 - 2 \times \frac{1}{4} = \frac{1}{2} > 0$$

by R3.1 and R3.5.

Case 2 f is incident with two crossing vertices. Then there is no important 4-vertex, and f is incident with at most four 4-vertices (Figure 3-2). Thus,

$$\mu^*(f) \geq d(f) - 4 - 4 \times \frac{1}{2} = 0$$

by R3.2–R3.5.

Case 3 f is incident with one crossing vertex. Then there is no important 4-vertex, and 5-vertex which is incident with three 3-faces. f is incident with at most four 4-vertices (Figure 3-3). Thus,

$$\mu^*(f) \geq d(f) - 4 - 4 \times \frac{1}{2} = 0$$

by R3.2–R3.5.

(5) $d(f) = 7$.

Case 1 f is incident with three crossing vertices. Then there is at most one important 4-vertex (Figure 4-1). Thus,

$$\mu^*(f) \geq d(f) - 4 - 1 - 3 \times \frac{1}{2} = \frac{1}{2} > 0$$

by R3.1–R3.5.

Case 2 f is incident with two crossing vertices. Then f is incident with at most four 4-vertices. There is at most one important 4-vertex (Figure 4-2). Thus,

$$\mu^*(f) \geq d(f) - 4 - 1 - 4 \times \frac{1}{2} = 0$$

by R3.

Case 3 f is incident with one crossing vertex. Then there is no important 4-vertex. Moreover, f is incident with at most four 4-vertices, and one 5-vertex v which is incident with three 3-faces (Figure 4-3). Thus,

$$\mu^*(f) \geq d(f) - 4 - 4 \times \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0$$

by R3.

Case 4 f is not incident with any crossing vertices. Then f is incident with at most four 4-vertices. There is no important 4-vertex, special 4-vertex, and 5-vertex which is incident with three 3-faces. (Figure 4-4 v_1, v_2, v_3 and v_4 are 4-vertices which are incident with two 3-faces.) Thus, $\mu^*(f) \geq d(f) - 4 - 4 \times \frac{1}{2} = 1 > 0$ by R3.2–R3.5.

(6) $d(f) \geq 8$.

Case 1 f is incident with at least one crossing vertex. Since the girth of G is at least 7, and by the definition of important 4-vertex, it is obvious that f is incident with at most $\lfloor \frac{d}{4} \rfloor$ important 4-vertices, and the other non-crossing vertices are at most $(d - \lfloor \frac{d}{4} \rfloor - 2 \times \lfloor \frac{d}{4} \rfloor)$. Thus,

$$\mu^*(f) \geq d(f) - 4 - 1 \times \lfloor \frac{d}{4} \rfloor - \frac{1}{2} \times (d - \lfloor \frac{d}{4} \rfloor - 2 \times \lfloor \frac{d}{4} \rfloor) > 0$$

by R3.

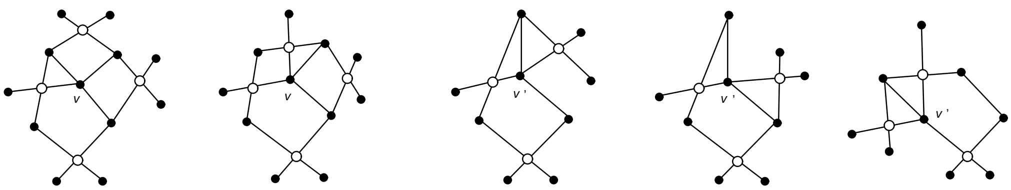
Case 2 f is not incident with any crossing vertices. Then by Claim 2, f is incident with at most $\lfloor \frac{2d}{3} \rfloor$ 4-vertices. Thus,

$$\mu^*(f) \geq d(f) - 4 - \frac{1}{2} \times \lfloor \frac{2d}{3} \rfloor \geq \frac{2d}{3} - 4 > 0$$

by R3.

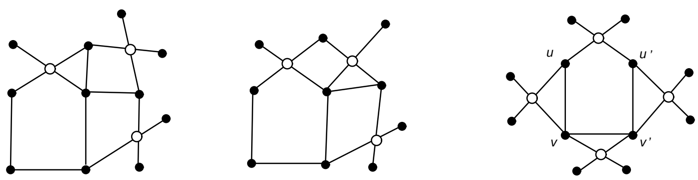
The proof of Theorem 1.1 is completed. □

The pictures in the proof of Theorem 1.1 are as follows.



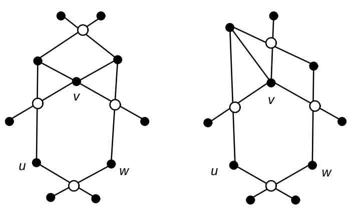
(a) (b) (c) (d) (e)

Figure 2-1
5−face with two crossing vertices



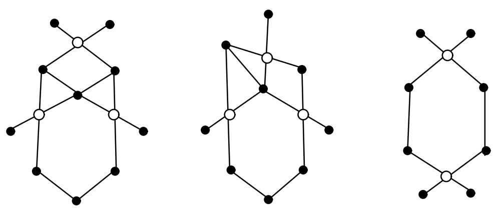
(a) 5−cycle (b) 6−cycle (c)

Figure 2-2
5−face with one crossing vertex



(a) (b)

Figure 3-1
6−face with three crossing vertices



(a) 6−cycle (b) 5−cycle (c)

Figure 3-2
6−face with two crossing vertices

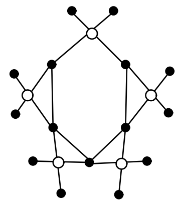


Figure 3-3
6−face with one crossing vertex

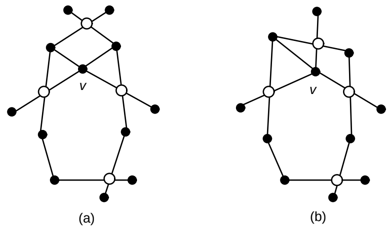


Figure 4-1
7−face with three crossing vertices

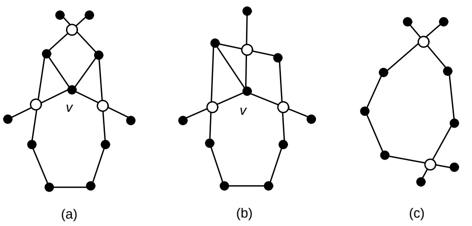


Figure 4-2

7−face with two crossing vertices

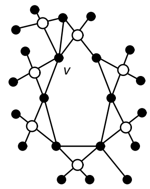


Figure 4-3

7−face with one crossing vertex

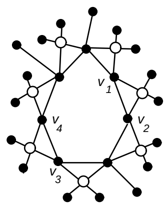


Figure 4-4
7−face with no crossing vertex

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References

[1] J. A. BONDY, U. S. R. MURTY. *Graph Theory with Applications*. North-Holland, New York, 1976.
[2] L. J. COWEN, R. H. COWEN, D. R. WOODALL. *Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency*. J. Graph Theory, 1986, **10**(2): 187–195.
[3] K. W. LIH, Weifan WANG, Zengmin SONG, et al. *A note on list improper coloring planar graphs*. Appl. Math. Lett., 2001, **14**(3): 269–273.
[4] Yuehua BU, Caixia FU. *(1,1,0)-coloring of planar graphs without cycles of length 4 and 6*. Discrete Math., 2013, **313**(23): 2737–2741.
[5] Runrun LIU, Xiangwen LI, Gexin YU. *Planar graphs without 5-cycles and intersecting triangles are* $(1,1,0)$-*colorable*. Eprint Arxiv, 2014.
[6] O. HILL, Gexin YU. *A relaxation of Steinberg's conjecture*. SIAM J. Discrete Math., 2013, **27**(1): 584–596.
[7] G. RINGEL. *Ein Sechsfarbenproblem auf der Kugel*. Abh. Math. Sem. Univ. Hamburg, 1965, **29**: 107–117. (in German)
[8] O. V. BORODIN. *Solution of Ringel's problems on the vertex-face coloring of plane graphs and on the coloring of 1-planar graphs*. Diskret. Analiz, 1984, **41**: 12–26. (in Russian)
[9] O. V. BORODIN. *A new proof of six color theorem*. J. Graph Theory, 1995, **19**: 507–521.
[10] O. V. BORODIN, A. V. KOSTOCHKA, A. RASPAUD, et al. *Acyclic colouring of 1-planar graphs*. Discrete Appl. Math., 2001, **114**(1-3): 29–41.
[11] Xin ZHANG, Jinliang WU. *On edge coloring of 1-planar graphs. Inform.* Process. Lett., 2011, **111**(3): 124–128.