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On Minimal Asymptotic Basis of Order 4

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Abstract Let $\mathbb{N}$ denote the set of all nonnegative integers and A be a subset of $\mathbb{N}$. Let W be a nonempty subset of $\mathbb{N}$. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W. Fix integer $g \geq 2$, let $A_g(W)$ be the set of all numbers of the form $\sum_{f\in F} a_f g^f$ where $F \in \mathcal{F}^*(W)$ and $1 \leq a_f \leq g-1$. For $i = 0, 1, 2, 3$, let $W_i = \{n \in \mathbb{N} \mid n \equiv i \pmod 4\}$. In this paper, we show that the set $A = \bigcup_{i=0}^{3} A_g(W_i)$ is a minimal asymptotic basis of order four.

Keywords minimal asymptotic basis; g-adic representation

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1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers and A be a subset of $\mathbb{N}$. Let $h \geq 2$ be an integer, and let hA be the set of all numbers n of the form $n = a_1 + \cdots + a_h$ where $a_1, \ldots, a_h$ are elements of A and are not necessarily distinct. Let W be an nonempty subset of $\mathbb{N}$. Denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W. For integer $g \geq 2$, let $A_g(W)$ be the set of all numbers of the form $\sum_{f\in F} a_f g^f$ where $F \in \mathcal{F}^*(W)$ and $1 \leq a_f \leq g-1$. For $i = 0, \ldots, h-1$, let $W_i = \{n \in \mathbb{N} \mid n \equiv i \pmod h\}$. The set A is called an asymptotic basis of order h if hA contains all sufficiently large integers. An asymptotic basis A of order h is minimal if no proper subset of A is an asymptotic basis of order h.

In 1988, based on the properties of powers of 2, Nathanson [1] proved the following result:

Theorem 1.1 ([1]) *Let $h \geq 2$. For $i = 0, 1, \ldots, h-1$, let $W_i = \{n \in \mathbb{N} \mid n \equiv i \pmod h\}$. Let $A = A_2(W_0) \cup \cdots \cup A_2(W_{h-1})$. Then A is a minimal asymptotic basis of order h.*

It is hard to extend Nathanson's method to all $g \geq 3$. In 1996, Jia [2] considered the g-adic minimal asymptotic bases of order h.

Theorem 1.2 ([2, Corollary 2]) *Let π be any partition of nonnegative integers into h pairwise disjoint infinite subsets $W_0, W_1, \ldots, W_{h-1}$. Then for any $g \geq h+1$, $A_g(\pi) = A_g(W_0) \cup \cdots \cup A_g(W_{h-1})$ is a minimal asymptotic basis of order h.*

It is natural to consider the following problem:

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Problem 1.3 Let $g,h\ge 2$ be integers. For $i=0,\dots,h-1$, let $W_i=\{n\in\mathbb{N}\mid n\equiv i \pmod h\}$. Is $A=A_g(W_0)\cup\cdots\cup A_g(W_{h-1})$ a minimal asymptotic basis of order h?

Recently, Ling and Tang (by private communication) have proved that for $h=3$, the answer to Problem 1.3 is affirmative. For related problems we refer to [3–6]. In this paper, we prove the following result:

Theorem 1.4 *For $i=0,1,2,3$, let $W_i=\{n\in\mathbb{N}\mid n\equiv i \pmod 4\}$. Then for any $g\ge 2$, $A=A_g(W_0)\cup A_g(W_1)\cup A_g(W_2)\cup A_g(W_3)$ is a minimal asymptotic basis of order* 4.

2. Proof of Theorem 1.4

To prove Theorem 1.4, we need the following Lemma:

Lemma 2.1 ([7, Lemma 1]) *Let $g\ge 2$ be any integer.*

(a) If W_1 and W_2 are disjoint subsets of $\mathbb{N}$, then $A_g(W_1)\cap A_g(W_2)=\emptyset$.

(b) If $W\subseteq\mathbb{N}$ and $W(x)=\theta x+O(1)$ for some $\theta\in(0,1]$, then there exist positive constants c_1 and c_2 such that

$$c_1x^\theta < A_g(W)(x) < c_2x^\theta$$

for all x sufficiently large.

(c) Let $\mathbb{N}=W_0\cup\cdots\cup W_{h-1}$, where $W_i\ne\emptyset$ for $i=0,1,\dots,h-1$. Then $A=A_g(W_0)\cup\cdots\cup A_g(W_{h-1})$ is an asymptotic basis of order h.

By Theorems 1.1 and 1.2, it is sufficient to prove that the theorem holds for $g=3,4$. Now we suppose that $g\in\{3,4\}$. Let $a\in A_g(W_u)$ for some $u\in\{0,1,2,3\}$, and so a has a unique g-adic representation in the form

$$a=a_ng^{4n+u}+\sum_{s\in S}a_sg^{4s+u},$$

where $n\ge 0$, and S is a finite, possibly empty, set of integers greater than n, $1\le a_n,a_s\le g-1$ for all $s\in S$. For any finite set T of integers greater than n, let

$$m=a_0g^u+\sum_{s\in S}a_sg^{4s+u}+(g-1)\sum_{\substack{i\ne u\\ 0\le i\le 3}}g^i+\sum_{t\in T}g^{4t+u+1},\quad \text{if}\ \ n=0. \tag{1}$$

$$m=a_ng^{4n+u}+\sum_{s\in S}a_sg^{4s+u}+(g-1)\sum_{t=u}^{u+2}g^{4n-3+t}+\sum_{t\in T}g^{4t+u+1},\quad \text{if}\ \ n>0. \tag{2}$$

By Lemma 2.1(c), we know that for each $i\in\{0,1,2,3\}$ there exists $j_i\in\{0,1,2,3\}$ such that $m_i\in A_g(W_{j_i})$ and

$$m=m_0+m_1+m_2+m_3. \tag{3}$$

For $i=0,1,2,3$, let $c_i^{(k)}$ be the least nonnegative residue of m_i modulo g^k. Write $M=\{m_0,m_1,m_2,m_3\}$. For fixed $j_i\in\{0,1,2,3\}$, let

$$I_{j_i}=\sharp\{i: m_i\in A_g(W_{j_i}), i=0,1,2,3\}.$$

We shall show that for any $j \in \{0,1,2,3\}$,

$$M \not\subseteq \bigcup_{i\in\{0,1,2,3\}\setminus\{j\}} A_g(W_i). \tag{4}$$

It is equivalent to prove the following four statements.

(a) $M \not\subseteq A_g(W_1) \cup A_g(W_2) \cup A_g(W_3)$; (b) $M \not\subseteq A_g(W_0) \cup A_g(W_2) \cup A_g(W_3)$;

(c) $M \not\subseteq A_g(W_0) \cup A_g(W_1) \cup A_g(W_3)$; (d) $M \not\subseteq A_g(W_0) \cup A_g(W_1) \cup A_g(W_2)$.

(I) We shall show that (a)–(d) hold for $n = 0$.

Proof of (a) Suppose that $M \subseteq \bigcup_{i\in\{1,2,3\}} A_g(W_i)$, then $m_i \equiv 0 \pmod g$, $i = 0,1,2,3$, thus by (3) we have $m \equiv 0 \pmod g$. On the other hand, by (1) we have $m \equiv a_0$ or $g-1 \pmod g$, a contradiction.

Proof of (b) Suppose that $M \subseteq \bigcup_{i\in\{0,2,3\}} A_g(W_i)$. By (a) we know $I_0 > 0$, thus we have the following observations:

(b_1) If $I_0 = 4$, then $\sum_{i=0}^{3} c_i^{(3)} \le 4(g-1)$; If $I_0 \ne 4$, then $\sum_{i=0}^{3} c_i^{(2)} \le 3(g-1)$;

(b_2) If $I_0 \ge 3$, then $\sum_{i=0}^{3} c_i^{(4)} \le g^4 - g^3 + 3g - 3$; If $I_0 < 3$, then $\sum_{i=0}^{3} c_i^{(2)} \le 2(g-1)$.

If $u = 0,2,3$, then by (1) we have $m \equiv a_0 + g^2 - g$ or $g^2 - 1 \pmod{g^2}$ and $m \equiv a_0 + g^3 - g, g^2a_0 + g^2 - 1$ or $g^3 - 1 \pmod{g^3}$ which contradicts the fact (b_1).

If $u = 1$, then by (1) we have

$$m \equiv \sum_{i=0}^{3} c_i^{(2)} \equiv ga_0 + g - 1 \pmod{g^2}, \tag{5}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4)} \equiv ga_0 + g^4 - g^2 + g - 1 \pmod{g^4}. \tag{6}$$

By (b_2), we have (5), (6) cannot hold.

Proof of (c) Suppose that $M \subseteq \bigcup_{i\in\{0,1,3\}} A_g(W_i)$. By (a), (b) we know $I_0, I_1 > 0$, thus we have the following facts:

(c_1) $\sum_{i=0}^{3} c_i^{(3)} \le 3g^2 - 2g - 1$; ($c_2$) If $I_3 = 0$, then $\sum_{i=0}^{3} c_i^{(4)} \le 3g^2 - 2g - 1$; If $I_3 > 0$, then $\sum_{i=0}^{3} c_i^{(3)} \le 2g^2 - g - 1$.

If $u = 0,1,3$, then by (1) we have $m \equiv a_0 + g^3 - g, ga_0 + g^3 - g^2 + g - 1$ or $g^3 - 1 \pmod{g^3}$, which contradicts the fact (c_1). If $u = 2$, then by (1) we have

$$m \equiv g^2a_0 + g^4 - g^3 + g^2 - 1 \pmod{g^4}, \quad m \equiv g^2a_0 + g^2 - 1 \pmod{g^3}. \tag{7}$$

By (c_2), we have (7) cannot hold.

Proof of (d) Suppose that $M \subseteq \bigcup_{i\in\{0,1,2\}} A_g(W_i)$. By (a)–(c) we know $I_0, I_1, I_2 > 0$, thus $\sum_{i=0}^{3} c_i^{(4)} \le 2g^3 - g^2 - 1$. If $u = 0,1,2,3$, then by (1) we have

$$m \equiv a_0 + g^4 - g,\ ga_0 + g^4 - g^2 + g - 1,\ g^2a_0 + g^4 - g^3 + g^2 - 1 \text{ or } g^3a_0 + g^3 - 1 \pmod{g^4},$$

which contradicts $m \equiv \sum_{i=0}^{3} c_i^{(4)} \pmod{g^4}$.

(II) We shall show that (a)–(d) hold for $n > 0$.

Case 1 $u=0$. By (2) we have

$$m \equiv \sum_{i=0}^{3} c_i^{(4n-2)} \equiv (g-1)g^{4n-3} \pmod{g^{4n-2}}, \tag{8}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n-1)} \equiv (g-1)g^{4n-2} + (g-1)g^{4n-3} \pmod{g^{4n-1}}, \tag{9}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n)} \equiv (g-1)g^{4n-1} + (g-1)g^{4n-2} + (g-1)g^{4n-3} \pmod{g^{4n}}, \tag{10}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+1)} \equiv a_n g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2} + (g-1)g^{4n-3} \pmod{g^{4n+1}}. \tag{11}$$

Proof of (a) Suppose that $M \subseteq \bigcup_{i\in\{1,2,3\}} A_g(W_i)$. If $I_3 > 2$, then

$$\sum_{i=0}^{3} c_i^{(4n-1)} \le 3(g-1)\sum_{i=0}^{n-2} g^{4i+3} + (g-1)\sum_{i=0}^{n-1} g^{4i+2} < (g-1)g^{4n-2} + (g-1)g^{4n-3},$$

which contradicts (9). If $I_3 \le 2$, then

$$\begin{aligned}\sum_{i=0}^{3} c_i^{(4n+1)} &\le 2(g-1)\sum_{i=0}^{n-1} g^{4i+3} + 2(g-1)\sum_{i=0}^{n-1} g^{4i+2}\\ &< a_n g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2} + (g-1)g^{4n-3},\end{aligned}$$

which contradicts (11).

Proof of (b) Suppose that $M \subseteq \bigcup_{i\in\{0,2,3\}} A_g(W_i)$. By (a) we know $I_0 > 0$. If $I_0 \ge 3$, then

$$\sum_{i=0}^{3} c_i^{(4n)} \le (g-1)\sum_{i=0}^{n-1} g^{4i+3} + 3(g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1} + (g-1)g^{4n-2} + (g-1)g^{4n-3},$$

which contradicts (10). If $I_0 < 3$, then

$$\sum_{i=0}^{3} c_i^{(4n-2)} \le 2(g-1)\sum_{i=0}^{n-2} g^{4i+3} + 2(g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-3},$$

which contradicts (8).

Proof of (c) Suppose that $M \subseteq \bigcup_{i\in\{0,1,3\}} A_g(W_i)$. By (a), (b) we know $I_0, I_1 > 0$, thus

$$\sum_{i=0}^{3} c_i^{(4n-1)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-2} + (g-1)g^{4n-3},$$

which contradicts (9).

Proof of (d) Suppose that $M \subseteq \bigcup_{i\in\{0,1,2\}} A_g(W_i)$. By (a)–(c) we know $I_0, I_1, I_2 > 0$, thus

$$\sum_{i=0}^{3} c_i^{(4n)} \le 2(g-1)\sum_{i=0}^{n-1} g^{4i+2} + (g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i}$$

$$< (g-1)g^{4n-1} + (g-1)g^{4n-2} + (g-1)g^{4n-3},$$

which contradicts (10).

Case 2 $u = 1$. By (2) we have

$$m \equiv \sum_{i=0}^{3} c_i^{(4n-1)} \equiv (g-1)g^{4n-2} \pmod{g^{4n-1}}, \tag{12}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n)} \equiv (g-1)g^{4n-1} + (g-1)g^{4n-2} \pmod{g^{4n}}, \tag{13}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+1)} \equiv (g-1)g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2} \pmod{g^{4n+1}}, \tag{14}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+2)} \equiv a_n g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2} \pmod{g^{4n+2}}. \tag{15}$$

Proof of (a) Suppose that $M \subseteq \bigcup_{i\in\{1,2,3\}} A_g(W_i)$. We have

$$\sum_{i=0}^{3} c_i^{(4n+1)} \le 4(g-1)\sum_{i=0}^{n-1} g^{4i+3} < (g-1)g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2},$$

which contradicts (14).

Proof of (b) Suppose that $M \subseteq \bigcup_{i\in\{0,2,3\}} A_g(W_i)$. By (a) we know $I_0 > 0$.

If $I_3 = 0$, then

$$\sum_{i=0}^{3} c_i^{(4n)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+2} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1} + (g-1)g^{4n-2},$$

which contradicts (13). If $I_3 > 0, I_2 = 0$, then

$$\sum_{i=0}^{3} c_i^{(4n-1)} \le (g-1)\sum_{i=0}^{n-2} g^{4i+3} + 3(g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-2},$$

which contradicts (12). If $I_3 > 0, I_2 > 0$, then

$$\begin{aligned}\sum_{i=0}^{3} c_i^{(4n+2)} &\le (g-1)\sum_{i=0}^{n-1} g^{4i+3} + (g-1)\sum_{i=0}^{n-1} g^{4i+2} + 2(g-1)\sum_{i=0}^{n} g^{4i} \\ &< a_n g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1} + (g-1)g^{4n-2},\end{aligned}$$

which contradicts (15).

Proof of (c) Suppose that $M \subseteq \bigcup_{i\in\{0,1,3\}} A_g(W_i)$. By (a), (b) we know $I_0, I_1 > 0$. If $I_3 > 0$, then

$$\sum_{i=0}^{3} c_i^{(4n-1)} \le (g-1)\sum_{i=0}^{n-2} g^{4i+3} + 2(g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-2},$$

which contradicts (12). If $I_3 = 0$, then

$$\sum_{i=0}^{3} c_i^{(4n)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1} + (g-1)g^{4n-2},$$

which contradicts (13).

Proof of (d) Suppose that $M \subseteq \bigcup_{i\in\{0,1,2\}} A_g(W_i)$. By $(a)-(c)$ we know $I_0, I_1, I_2 > 0$, thus

$$\sum_{i=0}^{3} c_i^{(4n)} \le 2(g-1)\sum_{i=0}^{n-1} g^{4i+2} + (g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1} + (g-1)g^{4n-2},$$

which contradicts (13).

Case 3 $u = 2$. By (2) we have

$$m \equiv \sum_{i=0}^{3} c_i^{(4n)} \equiv (g-1)g^{4n-1} \pmod{g^{4n}}, \tag{16}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+1)} \equiv (g-1)g^{4n} + (g-1)g^{4n-1} \pmod{g^{4n+1}}, \tag{17}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+2)} \equiv (g-1)g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1} \pmod{g^{4n+2}}, \tag{18}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+3)} \equiv a_n g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1} \pmod{g^{4n+3}}. \tag{19}$$

Proof of (a) Suppose that $M \subseteq \bigcup_{i\in\{1,2,3\}} A_g(W_i)$. If $I_3 = 4$, then

$$\sum_{i=0}^{3} c_i^{(4n+3)} \le 4(g-1)\sum_{i=0}^{n-1} g^{4i+3} < a_n g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1},$$

which contradicts (19). If $I_3 < 4$, then

$$\sum_{i=0}^{3} c_i^{(4n+1)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+3} + (g-1)\sum_{i=0}^{n-1} g^{4i+2} < (g-1)g^{4n} + (g-1)g^{4n-1},$$

which contradicts (17).

Proof of (b) Suppose that $M \subseteq \bigcup_{i\in\{0,2,3\}} A_g(W_i)$. By (a) we know $I_0 > 0$. We have

$$\sum_{i=0}^{3} c_i^{(4n+2)} \le 4(g-1)\sum_{i=0}^{n} g^{4i} < (g-1)g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1},$$

which contradicts (18).

Proof of (c) Suppose that $M \subseteq \bigcup_{i\in\{0,1,3\}} A_g(W_i)$. By (a), (b) we know $I_0, I_1 > 0$. If $I_3 = 0$, then

$$\sum_{i=0}^{3} c_i^{(4n)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1},$$

which contradicts (16). If $I_3 > 0$, then

$$\sum_{i=0}^{3} c_i^{(4n+3)} \le (g-1)\sum_{i=0}^{n-1} g^{4i+3} + 2(g-1)\sum_{i=0}^{n} g^{4i+1} + (g-1)\sum_{i=0}^{n} g^{4i}$$
$$< a_n g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n} + (g-1)g^{4n-1},$$

which contradicts (19).

Proof of (d) Suppose that $M \subseteq \bigcup_{i\in\{0,1,2\}} A_g(W_i)$. By (a)–(c) we know $I_0, I_1, I_2 > 0$, thus

$$\sum_{i=0}^{3} c_i^{(4n)} \le 2(g-1)\sum_{i=0}^{n-1} g^{4i+2} + (g-1)\sum_{i=0}^{n-1} g^{4i+1} + (g-1)\sum_{i=0}^{n-1} g^{4i} < (g-1)g^{4n-1},$$

which contradicts (16).

Case 4 $u = 3$. By (2) we have

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+1)} \equiv (g-1)g^{4n} \pmod{g^{4n+1}}, \tag{20}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+2)} \equiv (g-1)g^{4n+1} + (g-1)g^{4n} \pmod{g^{4n+2}}, \tag{21}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+3)} \equiv (g-1)g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n} \pmod{g^{4n+3}}, \tag{22}$$

$$m \equiv \sum_{i=0}^{3} c_i^{(4n+4)} \equiv a_n g^{4n+3} + (g-1)g^{4n+2} + (g-1)g^{4n} + (g-1)g^{4n+1} \pmod{g^{4n+4}}. \tag{23}$$

Proof of (a) Suppose that $M \subseteq \bigcup_{i\in\{1,2,3\}} A_g(W_i)$. If $I_3 = 4$, then

$$\sum_{i=0}^{3} c_i^{(4n+2)} \le 4(g-1)\sum_{i=0}^{n-1} g^{4i+3} < (g-1)g^{4n+1} + (g-1)g^{4n},$$

which contradicts (21). If $I_3 = 3$, then

$$\sum_{i=0}^{3} c_i^{(4n+3)} \le 3(g-1)\sum_{i=0}^{n-1} g^{4i+3} + (g-1)\sum_{i=0}^{n} g^{4i+2} < (g-1)g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n},$$

which contradicts (22). If $I_3 < 3$, then

$$\sum_{i=0}^{3} c_i^{(4n+1)} \le 2(g-1)\sum_{i=0}^{n-1} g^{4i+3} + 2(g-1)\sum_{i=0}^{n-1} g^{4i+2} < (g-1)g^{4n},$$

which contradicts (20).

Proof of (b) Suppose that $M \subseteq \bigcup_{i\in\{0,2,3\}} A_g(W_i)$. By (a) we know $I_0 > 0$. If $I_2 = 0$, then

$$\sum_{i=0}^{3} c_i^{(4n+3)} \le 4(g-1)\sum_{i=0}^{n} g^{4i} < (g-1)g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n},$$

which contradicts (22). If $I_2 > 0$, then

$$\sum_{i=0}^{3} c_i^{(4n+2)} \le (g-1)\sum_{i=0}^{n-1} g^{4i+2} + 3(g-1)\sum_{i=0}^{n} g^{4i} < (g-1)g^{4n+1} + (g-1)g^{4n},$$

which contradicts (21).

Proof of (c) Suppose that $M \subseteq \bigcup_{i\in\{0,1,3\}} A_g(W_i)$. By (a), (b) we know $I_0, I_1 > 0$, thus

$$\sum_{i=0}^{3} c_i^{(4n+3)} \le 3(g-1)\sum_{i=0}^{n} g^{4i+1} + (g-1)\sum_{i=0}^{n} g^{4i} < (g-1)g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n},$$

which contradicts (22).

Proof of (d) Suppose that $M \subseteq \bigcup_{i\in\{0,1,2\}} A_g(W_i)$. By (a)–(c) we know $I_0, I_1, I_2 > 0$, thus

$$\begin{aligned}\sum_{i=0}^{3} c_i^{(4n+4)} &\le 2(g-1)\sum_{i=0}^{n} g^{4i+2} + (g-1)\sum_{i=0}^{n} g^{4i+1} + (g-1)\sum_{i=0}^{n} g^{4i}\\ &< a_n g^{4n+3} + (g-1)g^{4n+2} + (g-1)g^{4n+1} + (g-1)g^{4n},\end{aligned}$$

which contradicts (23).

By (I) and (II), we show that for any $j \in \{0,1,2,3\}$, $M \not\subseteq \bigcup_{i\in\{0,1,2,3\}\setminus\{j\}} A_g(W_i)$. After suitable renumbering we have $m_i \in A_g(W_i)$, $i = 0,1,2,3$. Moreover, the g-adic representation of m is unique. Hence $m \notin 4(A \setminus \{a\})$.

This completes the proof of Theorem 1.4. $\Box$

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